CHAPTER 7

MAXIMUM MODULUS AND MAXIMUM TERMS-RELATED GROWTH PROPERTIES OF ENTIRE FUNCTIONS BASED ON RELATIVE TYPE AND RELATIVE WEAK TYPE
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7.1 Introduction, Definitions and Notations.

Let $\mathbb{C}$ be the set of all finite complex numbers. Also let $f$ be an entire function defined in the open complex plane $\mathbb{C}$. The maximum term $\mu(r, f)$ of $f = \sum_{n=0}^{\infty} a_n z^n$ on $|z| = r$ is defined by $\mu(r, f) = \max_{n \geq 0} (|a_n| r^n)$ and the maximum modulus $M(r, f)$ of $f = \sum_{n=0}^{\infty} a_n z^n$ on $|z| = r$ is defined by $M(r, f) = \max_{|z|=r} |f(z)|$. In the sequel we use the following notation:

$$\log^k x = \log \left( \log^{k-1} x \right) \text{ for } k = 1, 2, 3, \ldots \text{ and } \log^0 x = x.$$

To start our chapter we just recall the following definitions:

**Definition 7.1.1** The order $\rho_f$ and lower order $\lambda_f$ of an entire function $f$ are defined as

$$\rho_f = \limsup_{r \to \infty} \frac{\log^2 M_f(r)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log^2 M_f(r)}{\log r}.$$

The results of this chapter have been published in the *International Journal of Statistika and Mathematica*, see [24]
Using the inequalities $\mu(r, f) \leq M(r, f) \leq \frac{R}{r} \mu(R, f)$ \cite{[50]}, for $0 \leq r < R$ one may verify that
\[
\rho_f = \limsup_{r \to \infty} \frac{\log [\mu_f(r)]}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log [\mu_f(r)]}{\log r}.
\]

**Definition 7.1.2** The type $\sigma_f$ of an entire function $f$ is defined as
\[
\sigma_f = \limsup_{r \to \infty} \frac{\log M_f(r)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.
\]

Datta and Jha \cite{[11]} introduced the definition of weak type of a meromorphic function of finite positive lower order in the following way:

**Definition 7.1.3** \cite{[11]} The weak type $\tau_f$ of an entire function $f$ of finite positive lower order $\lambda_f$ is defined by
\[
\tau_f = \liminf_{r \to \infty} \frac{\log M_f(r)}{r^{\lambda_f}}.
\]

If an entire function $g$ is non-constant then $M_g(r)$ is strictly increasing and continuous and its inverse $M_g^{-1} : ([f(0)], \infty) \to (0, \infty)$ exists and is such that $\lim_{s \to \infty} M_g^{-1}(s) = \infty$.

Bernal \cite{[1]} introduced the definition of relative order of an entire function $f$ with respect to an entire function $g$, denoted by $\rho_g(f)$ as follows:
\[
\rho_g(f) = \inf \{ \mu > 0 : M_f(r) < M_g(r^\mu) \text{ for all } r > r_0(\mu) > 0 \} = \limsup_{r \to \infty} \frac{\log M_g^{-1}M_f(r)}{\log r}.
\]

The definition coincides with the classical one \cite{[52]} if $g(z) = \exp z$.

Similarly, one can define the relative lower order of an entire function $f$ with respect to an entire function $g$ denoted by $\lambda_g(f)$ as follows:
\[
\lambda_g(f) = \liminf_{r \to \infty} \frac{\log M_g^{-1}M_f(r)}{\log r}.
\]

Datta and Maji \cite{[14]} gave an alternative definition of relative order and relative lower order of an entire with respect to another entire in the following way:

**Definition 7.1.4** \cite{[14]} The relative order $\rho_g(f)$ and relative lower order $\lambda_g(f)$ of an entire function $f$ with respect to an entire function $g$ are defined as follows:
\[
\rho_g(f) = \limsup_{r \to \infty} \frac{\log \mu_g^{-1}\mu_f(r)}{\log r} \quad \text{and} \quad \lambda_g(f) = \liminf_{r \to \infty} \frac{\log \mu_g^{-1}\mu_f(r)}{\log r}.
\]

Recently Roy \cite{[43]} introduced the notion of relative type of two entire functions in the following manner:
**Definition 7.1.5** [43] Let \( f \) and \( g \) be any two entire functions such that \( 0 < \rho_g(f) < \infty \). Then the relative type \( \sigma_g(f) \) of \( f \) with respect to \( g \) is defined as:

\[
\sigma_g(f) = \inf \{ k > 0 : M_f(r) < M_g(kr^{\rho_g(f)}) \text{ for all sufficiently large values of } r \}
\]

\[
= \limsup_{r \to \infty} \frac{M_f^{-1}(r)}{r^{\rho_g(f)}}.
\]

Analogously to determine the relative growth of two entire functions having the same non-zero finite relative lower order with respect to another entire function, one may introduce the definition of relative weak type (in the notion of Datta and Jha [11]) of an entire function \( f \) with respect to another entire function \( g \) of finite positive relative lower order \( \lambda_g(f) \) in the following way:

**Definition 7.1.6** The relative weak type \( \tau_g(f) \) of an entire function \( f \) with respect to another entire function \( g \) having finite positive relative lower order \( \lambda_g(f) \) is defined as:

\[
\tau_g(f) = \liminf_{r \to \infty} \frac{M_f^{-1}(r)}{r^{\lambda_g(f)}}.
\]

Considering \( g = \exp z \) one may easily verify that Definition 7.1.5 and Definition 7.1.6 coincide with the Definition 7.1.2 and Definition 7.1.3 respectively.

In the chapter we study some relative growth properties of maximum terms and maximum moduli of composition of entire functions with respect to another entire function on the basis of relative order, relative type and relative weak type.

### 7.2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

**Lemma 7.2.1** [5] If \( f \) and \( g \) are two entire functions then for all sufficiently large values of \( r \),

\[
M_{f \circ g}(r) \leq M_f(M_g(r)).
\]

**Lemma 7.2.2** [49] Let \( f \) and \( g \) be any two entire functions. Then for every \( \alpha > 1 \) and \( 0 < r < R \),

\[
\mu_{f \circ g}(r) \leq \frac{\alpha}{\alpha - 1} \mu_f \left( \frac{\alpha R}{R - r} \mu_g(R) \right).
\]

**Lemma 7.2.3** [1] Suppose \( f \) is an entire function and \( \alpha > 1, 0 < \beta < \alpha \), then for all sufficiently large \( r \). Then for all sufficiently large \( r \),

\[
M_f(\alpha r) \geq \beta M_f(r).
\]

**Lemma 7.2.4** [14] If \( f \) is entire and \( \alpha > 1, 0 < \beta < \alpha \), then for all sufficiently large \( r \),

\[
\mu_f(\alpha r) \geq \beta \mu_f(r).
\]
Lemma 7.2.5 Let $f$ and $g$ be any two entire functions. Then for any $\alpha > 1$,

(i) \( M_h^{-1} M_f (r) \leq \mu_h^{-1} \left( \frac{\alpha}{(\alpha-1)} \mu_f (\alpha r) \right) \) and

(ii) \( \mu_h^{-1} \mu_f (r) \leq \alpha M_h^{-1} \left( \frac{\alpha}{(\alpha-1)} M_f (r) \right) \).

Proof. Taking \( R = \alpha r \) in the inequalities \( \mu_h (r) \leq M_h (r) \leq \frac{R}{R-r} \mu_h (R) \) \{cf. [50] \}, for \( 0 \leq r < R \) we obtain that

\[ M_h^{-1} (r) \leq \mu_h^{-1} (r) \]

and

\[ \mu_h^{-1} (r) \leq \alpha M_h^{-1} \left( \frac{\alpha r}{(\alpha-1)} \right) . \]

Since \( M_h^{-1} (r) \) and \( \mu_h^{-1} (r) \) are increasing functions of \( r \), the lemma follows from the above and the inequalities \( \mu_f (r) \leq M_f (r) \leq \frac{\alpha}{\alpha-1} \mu_f (\alpha r) \) \{cf. [50] \}.

This proves the lemma. \( \square \)

7.3 Theorems.

In this section we present the main results of the chapter.

Theorem 7.3.1 Let $f$, $g$ and $h$ be any three entire functions such that $0 < \lambda_h (f) \leq \rho_h (f) < \infty$ and $\sigma_g < \infty$. Then for any $\beta > 1$,

\[ \limsup \frac{\log \mu_h^{-1} \mu_{fog} (r)}{\log \mu_h^{-1} \mu_f (\exp (\beta r)^\rho_g)} \leq \frac{\sigma_g \cdot \rho_h (f)}{\lambda_h (f)} . \]

Proof. Taking \( R = \beta r \) in Lemma 7.2.2 and in view of Lemma 7.2.4 we have for all sufficiently large values of \( r \) that

\[ \mu_{fog} (r) \leq \left( \frac{\alpha}{\alpha-1} \right) \mu_f \left( \frac{\alpha \beta}{(\beta-1)} \mu_g (\beta r) \right) \]

i.e., \( \mu_{fog} (r) \leq \mu_f \left( \frac{(2\alpha - 1) \alpha \beta}{(\alpha - 1) (\beta - 1)} \mu_g (\beta r) \right) . \)

Since \( \mu_h^{-1} (r) \) is an increasing function \( r \), it follows from above for all sufficiently large values of \( r \) that

\[ \mu_h^{-1} \mu_{fog} (r) \leq \mu_h^{-1} \mu_f \left( \frac{(2\alpha - 1) \alpha \beta}{(\alpha - 1) (\beta - 1)} \mu_g (\beta r) \right) \]

i.e., \( \log \mu_h^{-1} \mu_{fog} (r) \leq \log \mu_h^{-1} \mu_f \left( \frac{(2\alpha - 1) \alpha \beta}{(\alpha - 1) (\beta - 1)} \mu_g (\beta r) \right) \) \hspace{1cm} (7.3.1)
Thus the theorem is established. ■

In the line of Theorem 7.3.1 the following theorem can be proved:

**Theorem 7.3.2** Let \( f, g \) and \( h \) be any three entire functions with \( \lambda_h (g) > 0, \rho_h (f) < \infty \) and \( \sigma_g < \infty \). Then for any \( \beta > 1, \)

\[
\limsup_{r \to \infty} \frac{\log \mu_h^{-1} f g (r)}{\log \mu_h^{-1} f (\exp (\beta r)^{\rho_g})} \leq \frac{\sigma_g \cdot \rho_h (f)}{\lambda_h (g)}.
\]

The proof is omitted.

With the help of Lemma 7.2.1 and in the line of Theorem 7.3.1 and Theorem 7.3.2 the following two theorems may be proved:

**Theorem 7.3.3** Let \( f, g \) and \( h \) be any three entire functions such that \( 0 < \lambda_h (f) \leq \rho_h (f) < \infty \) and \( \sigma_g < \infty \). Then

\[
\limsup_{r \to \infty} \frac{\log M_h^{-1} f g (r)}{\log M_h^{-1} f (\exp (r)^{\rho_g})} \leq \frac{\sigma_g \cdot \rho_h (f)}{\lambda_h (f)}.
\]

**Theorem 7.3.4** Let \( f, g \) and \( h \) be any three entire functions with \( \lambda_h (g) > 0, \rho_h (f) < \infty \) and \( \sigma_g < \infty \). Then

\[
\limsup_{r \to \infty} \frac{\log M_h^{-1} f g (r)}{\log M_h^{-1} g (\exp (r)^{\rho_g})} \leq \frac{\sigma_g \cdot \rho_h (f)}{\lambda_h (g)}.
\]
Using the notion of weak type, we may state the following two theorems without their proofs because those can be carried out in the line of Theorem 7.3.1 and Theorem 7.3.3 respectively.

**Theorem 7.3.5** Let \( f, g \) and \( h \) be any three entire functions satisfying \( 0 < \lambda_h(f) = \rho_h(f) < \infty \) and \( \tau_g < \infty \). Then for any \( \beta > 1 \),

\[
\liminf_{r \to \infty} \frac{\log \mu_h^{-1} \mu_f (r)}{\log \mu_h^{-1} \mu_f \left( \exp (\beta r)^{\lambda_g} \right)} \leq \tau_g.
\]

**Theorem 7.3.6** Let \( f, g \) and \( h \) be any three entire functions with \( 0 < \lambda_h(f) = \rho_h(f) < \infty \) and \( \tau_g < \infty \). Then

\[
\liminf_{r \to \infty} \frac{\log M_h^{-1} M_f (r)}{\log M_h^{-1} M_f \left( \exp (\beta r)^{\lambda_g} \right)} \leq \tau_g.
\]

**Theorem 7.3.7** Let \( f, g \) and \( h \) be any three entire functions such that (i) \( 0 < \rho_h(f) < \infty \), (ii) \( \rho_h(f) = \rho_g \), (iii) \( \sigma_g < \infty \), and (iv) \( 0 < \sigma_h(f) < \infty \). Then for any \( \alpha, \beta > 1 \) and \( \gamma > 0 \),

\[
\liminf_{r \to \infty} \frac{\log \mu_h^{-1} \mu_f (r)}{\mu_h^{-1} \mu_f (r)} \leq \left[ \frac{(\alpha + \gamma \alpha - \gamma) \alpha \beta}{(\alpha - 1)} \right]^{\rho_h(f)} \frac{\rho_h(f) \sigma_g}{\sigma_h(f)}.
\]

**Proof.** From (7.3.3) and the inequality \( \mu(r, f) \leq M(r, f) \) (cf. [50]), we get for all sufficiently large values of \( r \) that

\[
\log \mu_h^{-1} \mu_f (r) \leq (\rho_h(f) + \varepsilon) \{ \log \mu_g(\beta r) + O(1) \}
\]

i.e.,

\[
\log \mu_h^{-1} \mu_f (r) \leq (\rho_h(f) + \varepsilon) \{ \log M_g(\beta r) + O(1) \}.
\]

Using the definition of type, we obtain from (7.3.3) for all sufficiently large values of \( r \) that

\[
\log \mu_h^{-1} \mu_f (r) \leq (\rho_h(f) + \varepsilon) (\sigma_g + \varepsilon) \{ \beta r \}^{\rho_g} + O(1) .
\]

Now in view of condition (ii), we obtain from (7.3.4) for all sufficiently large values of \( r \) that

\[
\log \mu_h^{-1} \mu_f (r) \leq (\rho_h(f) + \varepsilon) (\sigma_g + \varepsilon) \{ \beta r \}^{\rho_h(f)} + O(1) .
\]

Again in view of Lemma 7.2.4, Lemma 7.2.5 and the definition of relative type we get for a sequence of values of \( r \) tending to infinity that

\[
\mu_h^{-1} \left[ \frac{\alpha}{(\alpha - 1) \mu_f (ar)} \right] \geq M_h^{-1} M_f (r)
\]

i.e.,

\[
\mu_h^{-1} \left[ \frac{(\alpha + \gamma \alpha - \gamma) ar}{(\alpha - 1)} \right] \geq M_h^{-1} M_f (r)
\]

i.e.,

\[
\mu_h^{-1} \mu_f (r) \geq M_h^{-1} M_f \left( \frac{(\alpha - 1)}{(\alpha + \gamma \alpha - \gamma) \alpha} \right)
\]
\[ i.e., \mu_h^{-1} \mu_f (r) \geq \left( \sigma_h (f) - \varepsilon \right) \left\{ \frac{(\alpha - 1)}{(\alpha + \gamma \alpha - \gamma)} \right\} \rho_h (f) . \tag{7.3.6} \]

Now from (7.3.5) and (7.3.6) it follows for a sequence of values of \( r \) tending to infinity that

\[ \log \frac{\mu_h^{-1} \mu_{fog} (r)}{\mu_h^{-1} \mu_f (r)} \leq \left( \rho_h (f) + \varepsilon \right) \left( \sigma_g + \varepsilon \right) \left\{ \beta r \right\} \rho_h (f) + O(1) \frac{\sigma_h (f) - \varepsilon}{(\alpha + \gamma \alpha - \gamma)} \right\} \rho_h (f) . \]

Since \( \varepsilon (> 0) \) is arbitrary, it follows from above that

\[ \lim_{r \to \infty} \frac{\log \mu_h^{-1} \mu_{fog} (r)}{\mu_h^{-1} \mu_f (r)} \leq \left[ \frac{(\alpha + \gamma \alpha - \gamma) \alpha \beta}{(\alpha - 1)} \right] \frac{\rho_h (f) \cdot \sigma_g}{\sigma_h (f)} . \]

This proves the theorem. \( \Box \)

Using the notion of weak type and relative weak type, we may state the following theorem without proof as it can be carried out in the line of Theorem 7.3.7:

**Theorem 7.3.8** Let \( f, g \) and \( h \) be any three entire functions satisfying (i) \( 0 < \rho_h (f) < \infty \), (ii) \( \lambda_h (f) = \lambda_g \), (iii) \( \tau_g < \infty \) and (iv) \( 0 < \tau_h (f) < \infty \). Then for any \( \alpha, \beta > 1 \) and \( \gamma > 0 \),

\[ \lim_{r \to \infty} \frac{\log \mu_h^{-1} \mu_{fog} (r)}{\mu_h^{-1} \mu_f (r)} \leq \left[ \frac{(\alpha + \gamma \alpha - \gamma) \alpha \beta}{(\alpha - 1)} \right] \frac{\rho_h (f) \cdot \sigma_g}{\tau_h (f)} . \]

Similarly, using the notion of type and relative weak type one may state the following two theorems without their proofs because those can also be carried out in the line of Theorem 7.3.7:

**Theorem 7.3.9** Let \( f, g \) and \( h \) be any three entire functions such that (i) \( \lambda_h (f) = \rho_g \), (ii) \( \sigma_g < \infty \) and (iii) \( 0 < \tau_h (f) < \infty \). Then for any \( \alpha, \beta > 1 \) and \( \gamma > 0 \),

\[ \lim_{r \to \infty} \frac{\log \mu_h^{-1} \mu_{fog} (r)}{\mu_h^{-1} \mu_f (r)} \leq \left[ \frac{(\alpha + \gamma \alpha - \gamma) \alpha \beta}{(\alpha - 1)} \right] \frac{\lambda_h (f) \cdot \sigma_g}{\tau_h (f)} . \]

**Theorem 7.3.10** Let \( f, g \) and \( h \) be any three entire functions with (i) \( 0 < \rho_h (f) < \infty \), (ii) \( \lambda_h (f) = \rho_g \), (iii) \( \sigma_g < \infty \) and (iv) \( 0 < \tau_h (f) < \infty \). Then for any \( \alpha, \beta > 1 \) and \( \gamma > 0 \),

\[ \lim_{r \to \infty} \frac{\log \mu_h^{-1} \mu_{fog} (r)}{\mu_h^{-1} \mu_f (r)} \leq \left[ \frac{(\alpha + \gamma \alpha - \gamma) \alpha \beta}{(\alpha - 1)} \right] \frac{\rho_h (f) \cdot \sigma_g}{\tau_h (f)} . \]

**Theorem 7.3.11** Let \( f, g \) and \( h \) be any three entire functions satisfying (i) \( 0 < \rho_h (f) < \infty \), (ii) \( \rho_h (f) = \rho_g \), (iii) \( \sigma_g < \infty \) and (iv) \( 0 < \sigma_h (f) < \infty \). Then

\[ \lim_{r \to \infty} \frac{\log M_h^{-1} M_{fog} (r)}{M_h^{-1} M_f (r)} \leq \frac{\rho_h (f) \cdot \sigma_g}{\sigma_h (f)} . \]
Theorem 7.3.12 Let \( f, g \) and \( h \) be any three entire functions such that (i) \( 0 < \rho_h (f) < \infty \), (ii) \( \lambda_h (f) = \lambda_g \), (iii) \( \tau_g < \infty \) and (iv) \( 0 < \tau_h (f) < \infty \). Then

\[
\liminf_{r \to \infty} \frac{\log M_{f \circ g}^{-1} (r)}{M_f^{-1} (r)} \leq \frac{\rho_h (f) \cdot \tau_g}{\tau_h (f)}.
\]

Theorem 7.3.13 Let \( f, g \) and \( h \) be any three entire functions with (i) \( \lambda_h (f) = \rho_g \), (ii) \( \sigma_g < \infty \) and (iii) \( 0 < \tau_h (f) < \infty \). Then

\[
\liminf_{r \to \infty} \frac{\log M_{f \circ g}^{-1} (r)}{M_f^{-1} (r)} \leq \frac{\lambda_h (f) \cdot \sigma_g}{\tau_h (f)}.
\]

Theorem 7.3.14 Let \( f, g \) and \( h \) be any three entire functions such that (i) \( 0 < \rho_h (f) < \infty \), (ii) \( \lambda_h (f) = \rho_g \), (iii) \( \sigma_g < \infty \) and (iv) \( 0 < \tau_h (f) < \infty \). Then

\[
\limsup_{r \to \infty} \frac{\log M_{f \circ g}^{-1} (r)}{M_f^{-1} (r)} \leq \frac{\rho_h (f) \cdot \sigma_g}{\tau_h (f)}.
\]

The proofs of Theorem 7.3.11, Theorem 7.3.12, Theorem 7.3.13 and Theorem 7.3.14 are omitted as those can be carried out in view of Lemma 7.2.1 and in the line of Theorem 7.3.7, Theorem 7.3.8, Theorem 7.3.9 and Theorem 7.3.10 respectively.

Theorem 7.3.15 Let \( f, g \) and \( h \) be any three entire functions satisfying (i) \( 0 < \rho_h (f) < \infty \), (ii) \( 0 < \sigma_h (f) < \infty \), (iii) \( \rho_h (f \circ g) = \rho_h (f) \) and (iv) \( \sigma_h (f \circ g) < \infty \). Then for any \( \alpha > 1 \) and \( \gamma > 0 \),

\[
\liminf_{r \to \infty} \frac{\mu_{f \circ g}^{-1} (r)}{\mu_f^{-1} (r)} \leq \frac{(\alpha + \gamma \alpha - \gamma)^{2\rho_h (f) \cdot \alpha^{\rho_h (f) + 1}}}{(\alpha - 1)^{2\rho_h (f)}} \cdot \frac{\sigma_h (f \circ g)}{\sigma_h (f)}
\]

and

\[
\frac{(\alpha - 1)^{2\rho_h (f)}}{(\alpha + \gamma \alpha - \gamma)^{2\rho_h (f)}} \cdot \frac{\sigma_h (f \circ g)}{\sigma_h (f)} \leq \limsup_{r \to \infty} \frac{\mu_{f \circ g}^{-1} (r)}{\mu_f^{-1} (r)}.
\]

**Proof.** From the definition of relative type and in view of Lemma 7.2.3 and Lemma 7.2.5 we obtain for all sufficiently large values of \( r \) that

\[
\mu_{f \circ g}^{-1} (r) \leq \alpha M_{f \circ g}^{-1} \left[ \frac{\alpha}{(\alpha - 1)} M_f^{-1} (r) \right] \leq \alpha M_h^{-1} \left[ M_f \left( \left( \frac{\alpha + \gamma \alpha - \gamma}{\alpha - 1} r \right)^{\rho_h (f \circ g)} \right) \right]
\]

i.e.,

\[
\mu_{f \circ g}^{-1} (r) \leq \alpha \left( \sigma_h (f \circ g) + \varepsilon \right) \left( \left( \frac{\alpha + \gamma \alpha - \gamma}{\alpha - 1} r \right)^{\rho_h (f \circ g)} \right).
\]

and

\[
\mu_f^{-1} (r) \leq \alpha \left( \sigma_h (f) + \varepsilon \right) \left( \left( \frac{\alpha + \gamma \alpha - \gamma}{\alpha - 1} r \right)^{\rho_h (f)} \right).
\]
Also in view of Lemma 7.2.4 and Lemma 7.2.5 we obtain for a sequence of values of \( r \) tending to infinity that
\[
\mu_h^{-1} \mu_{f g} (r) \geq M_h^{-1} M_{f g} \left( \frac{(\alpha - 1)}{(\alpha + \gamma \alpha - \gamma) \alpha} r \right)
\]
i.e., \( \mu_h^{-1} \mu_{f g} (r) \geq (\sigma_h (f \circ g) - \varepsilon) \left\{ \left( \frac{(\alpha - 1) r}{(\alpha + \gamma \alpha - \gamma) \alpha} \right) \right\} r_{\rho_h(fg)} \)
i.e., \( \mu_h^{-1} \mu_{f g} (r) \geq \left( \frac{(\alpha - 1)}{(\alpha + \gamma \alpha - \gamma) \alpha} \right) r_{\rho_h(fg)} \cdot (\sigma_h (f \circ g) - \varepsilon) \cdot r_{\rho_h(fg)} \) \hspace{1cm} (7.3.9)

and
\[
\mu_h^{-1} \mu_f (r) \geq \left( \frac{(\alpha - 1)}{(\alpha + \gamma \alpha - \gamma) \alpha} \right) r_{\rho_h(f)} \cdot (\sigma_h (f) - \varepsilon) \cdot r_{\rho_h(f)}. \hspace{1cm} (7.3.10)
\]

Now from (7.3.7) and (7.3.10) it follows for a sequence of values of \( r \) tending to infinity that
\[
\frac{\mu_h^{-1} \mu_{f g} (r)}{\mu_h^{-1} \mu_f (r)} \leq \frac{\alpha (\sigma_h (f \circ g) + \varepsilon)}{(\alpha - 1)} \left( \frac{\alpha + \gamma \alpha - \gamma}{\alpha} \right) r_{\rho_h(f)} \cdot (\sigma_h (f) - \varepsilon) \cdot r_{\rho_h(f)}. \hspace{1cm} (7.3.11)
\]

In view of the condition (\( iii \)), we get from (7.3.11) that
\[
\liminf_{r \to \infty} \frac{\mu_h^{-1} \mu_{f g} (r)}{\mu_h^{-1} \mu_f (r)} \leq \frac{\alpha (\sigma_h (f \circ g) + \varepsilon)}{(\alpha - 1)} \left( \frac{\alpha + \gamma \alpha - \gamma}{\alpha} \right) r_{\rho_h(f)} \cdot (\sigma_h (f) - \varepsilon).
\]

As \( \varepsilon (> 0) \) is arbitrary, it follows from above that
\[
\liminf_{r \to \infty} \frac{\mu_h^{-1} \mu_{f g} (r)}{\mu_h^{-1} \mu_f (r)} \leq \frac{\alpha (\sigma_h (f \circ g) + \varepsilon)}{(\alpha - 1)} \left( \frac{\alpha + \gamma \alpha - \gamma}{\alpha} \right) r_{\rho_h(f)} \cdot (\sigma_h (f) - \varepsilon). \hspace{1cm} (7.3.12)
\]

Again from (7.3.8) and (7.3.9) we get for a sequence of values of \( r \) tending to infinity that
\[
\frac{\mu_h^{-1} \mu_{f g} (r)}{\mu_h^{-1} \mu_f (r)} \geq \left( \frac{\alpha - 1)}{(\alpha + \gamma \alpha - \gamma) \alpha} \right) r_{\rho_h(f)} \cdot (\sigma_h (f \circ g) - \varepsilon) \cdot r_{\rho_h(f)} \)
\hspace{1cm} (7.3.13)
\]
\[
\limsup_{r \to \infty} \frac{\mu_h^{-1} \mu_{f g} (r)}{\mu_h^{-1} \mu_f (r)} \geq \left( \frac{\alpha - 1)}{(\alpha + \gamma \alpha - \gamma) \alpha} \right) r_{\rho_h(f)} \cdot (\sigma_h (f \circ g) - \varepsilon) \hspace{1cm} (7.3.14)
\]

As \( \varepsilon (> 0) \) is arbitrary, it follows from above that
\[
\limsup_{r \to \infty} \frac{\mu_h^{-1} \mu_{f g} (r)}{\mu_h^{-1} \mu_f (r)} \geq \frac{\alpha (\sigma_h (f \circ g) + \varepsilon)}{(\alpha + \gamma \alpha - \gamma)} \left( \frac{\alpha - 1)}{(\alpha + \gamma \alpha - \gamma) \alpha} \right) r_{\rho_h(f)} \cdot (\sigma_h (f) - \varepsilon).
\]

Thus the theorem follows from (7.3.12) and (7.3.14). 

In the line of Theorem 7.3.15 we may state the following theorem without their proofs:
Theorem 7.3.16  Let \( f, g \) and \( h \) be any three entire functions with (i) \( 0 < \rho_h(g) < \infty \), (ii) \( 0 < \sigma_h(g) < \infty \), (iii) \( \rho_h(f \circ g) = \rho_h(g) \) and (iv) \( \sigma_h(f \circ g) < \infty \). Then for any \( \alpha > 1 \) and \( \gamma > 0 \),

\[
\liminf_{r \to \infty} \frac{\mu_h^{-1} \mu_{fog}(r)}{\mu_h^{-1} \mu_g(r)} \leq \frac{(\alpha + \gamma \alpha - \gamma)2^{\rho_h(g)} \cdot \alpha^{\rho_h(g)+1}}{(\alpha - 1)2^{\rho_h(g)}} \cdot \frac{\sigma_{f \circ g}}{\sigma_g(r)} \leq \limsup_{r \to \infty} \frac{\mu_h^{-1} \mu_{fog}(r)}{\mu_h^{-1} \mu_{f}(r)} .
\]

Using the notion of relative weak type, we may state the following two theorems without proof because those may be carried out with the help of Lemma 7.2.3 and Lemma 7.2.5 and in the line of Theorem 7.3.15 and Theorem 7.3.16 respectively:

Theorem 7.3.17  Let \( f, g \) and \( h \) be any three entire functions such that (i) \( 0 < \lambda_h(f) < \infty \), (ii) \( 0 < \tau_h(f) < \infty \), (iii) \( \lambda_h(f \circ g) = \lambda_h(f) \) and (iv) \( \tau_h(f \circ g) < \infty \). Then for any \( \alpha > 1 \) and \( \gamma > 0 \),

\[
\liminf_{r \to \infty} \frac{\mu_h^{-1} \mu_{fog}(r)}{\mu_h^{-1} \mu_{f}(r)} \leq \frac{(\alpha + \gamma \alpha - \gamma)2^{\lambda_h(f)} \cdot \alpha^{\lambda_h(f)+1}}{(\alpha - 1)2^{\lambda_h(f)}} \cdot \frac{\tau_{f \circ g}}{\tau_f(r)} \leq \limsup_{r \to \infty} \frac{\mu_h^{-1} \mu_{fog}(r)}{\mu_h^{-1} \mu_{f}(r)} .
\]

Theorem 7.3.18  Let \( f, g \) and \( h \) be any three entire functions with (i) \( 0 < \lambda_h(g) < \infty \), (ii) \( 0 < \tau_h(g) < \infty \), (iii) \( \lambda_h(f \circ g) = \lambda_h(g) \) and (iv) \( \tau_h(f \circ g) < \infty \). Then for any \( \alpha > 1 \) and \( \gamma > 0 \),

\[
\liminf_{r \to \infty} \frac{\mu_h^{-1} \mu_{fog}(r)}{\mu_h^{-1} \mu_g(r)} \leq \frac{(\alpha + \gamma \alpha - \gamma)2^{\lambda_h(g)} \cdot \alpha^{\lambda_h(g)+1}}{(\alpha - 1)2^{\lambda_h(g)}} \cdot \frac{\tau_{f \circ g}}{\tau_g(r)} \leq \limsup_{r \to \infty} \frac{\mu_h^{-1} \mu_{fog}(r)}{\mu_h^{-1} \mu_{g}(r)} .
\]

Similarly, one may state the following four theorems without their proofs on the basis of relative type and relative weak type of entire functions with respect to another entire functions in terms of their maximum moduli :

Theorem 7.3.19  Let \( f, g \) and \( h \) be any three entire functions satisfying (i) \( 0 < \rho_h^*(f) < \infty \), (ii) \( 0 < \sigma_h^*(f) < \infty \), (iii) \( \rho_h^*(f \circ g) = \rho_h^*(f) \) and (iv) \( \sigma_h^*(f \circ g) < \infty \). Then

\[
\liminf_{r \to \infty} \frac{M_h^{-1} M_{fog}(r)}{M_h^{-1} M_f(r)} \leq \frac{\sigma_{f \circ g}}{\sigma_{h}^*(f)} \leq \limsup_{r \to \infty} \frac{M_h^{-1} M_{fog}(r)}{M_h^{-1} M_f(r)} .
\]

Theorem 7.3.20  Let \( f, g \) and \( h \) be any three entire functions such that (i) \( 0 < \rho_h^*(g) < \infty \), (ii) \( 0 < \sigma_h^*(g) < \infty \), (iii) \( \rho_h^*(f \circ g) = \rho_h^*(g) \) and (iv) \( \sigma_h^*(f \circ g) < \infty \). Then

\[
\liminf_{r \to \infty} \frac{M_h^{-1} M_{fog}(r)}{M_h^{-1} M_g(r)} \leq \frac{\sigma_{f \circ g}}{\sigma_{h}^*(g)} \leq \limsup_{r \to \infty} \frac{M_h^{-1} M_{fog}(r)}{M_h^{-1} M_g(r)} .
\]
Theorem 7.3.21 Let \( f, g \) and \( h \) be any three entire functions with (i) \( 0 < \lambda^+_h (f) < \infty \), (ii) \( 0 < \tau^+_h (f) < \infty \), (iii) \( \lambda^+_h (f \circ g) = \lambda^+_h (f) \) and (iv) \( \tau^+_h (f \circ g) < \infty \). Then

\[
\liminf_{r \to \infty} \frac{M^{-1}_h M_{f \circ g} (r)}{M^{-1}_h M_f (r)} \leq \frac{\tau^+_h (f \circ g)}{\tau^+_h (f)} \leq \limsup_{r \to \infty} \frac{M^{-1}_h M_{f \circ g} (r)}{M^{-1}_h M_f (r)} .
\]

Proof. From the definition of \( \tau_h (f \circ g) \) and in view of Lemma 7.2.4 and Lemma 7.2.5 we obtain for all sufficiently large values of \( r \) that

\[
\mu^{-1}_h M_{f \circ g} (r) \geq M^{-1}_h M_{f \circ g} \left( \frac{(\alpha - 1) \rho_h (f)}{\alpha + \gamma \alpha - \gamma} \right)^{\lambda_h (f \circ g)}
\]

i.e.,

\[
\mu^{-1}_h M_{f \circ g} (r) \geq \left( \frac{(\alpha - 1) \rho_h (f)}{\alpha + \gamma \alpha - \gamma} \right)^{\lambda_h (f \circ g)} \left( \tau_h (f \circ g) - \varepsilon \right) \cdot \mu_h (f \circ g) .
\]

Thus from (7.3.1) and (7.3.15) we get for all sufficiently large values of \( r \) that

\[
\mu^{-1}_h M_{f \circ g} (r) \geq \frac{(\alpha - 1) \rho_h (f)}{\alpha (\sigma_h (f) + \varepsilon)} \cdot \left( \tau_h (f \circ g) - \varepsilon \right) \cdot \mu_h (f \circ g) .
\]

Since \( \lambda_h (f \circ g) = \rho_h (f) \), we obtain from (7.3.16) that

\[
\liminf_{r \to \infty} \frac{\mu^{-1}_h M_{f \circ g} (r)}{\mu^{-1}_h M_f (r)} \geq \frac{(\alpha - 1) \rho_h (f)}{\alpha (\sigma_h (f) + \varepsilon)} \cdot \left( \tau_h (f \circ g) - \varepsilon \right) .
\]

As \( \varepsilon (> 0) \) is arbitrary, it follows from above that

\[
\liminf_{r \to \infty} \frac{\mu^{-1}_h M_{f \circ g} (r)}{\mu^{-1}_h M_f (r)} \geq \frac{(\alpha - 1) \rho_h (f)}{\alpha (\sigma_h (f) + \varepsilon)} \cdot \tau_h (f \circ g) .
\]

Thus the theorem is established. \( \blacksquare \)
Theorem 7.3.24 Let $f$, $g$ and $h$ be any three entire functions such that
(i) $0 < \lambda_h(f) < \infty$, (ii) $\tau_h(f) < \infty$, (iii) $\rho_h(f \circ g) = \lambda_h(f)$ and (iv) $\sigma_h(f \circ g) > 0$. Then for any $\alpha > 1$ and $\gamma > 0$,
\[ \limsup_{r \to \infty} \frac{\mu_h^{-1} \mu_{f \circ g}(r)}{\mu_h^{-1} \mu_f(r)} \leq \frac{(\alpha + \gamma \alpha - \gamma)^{2\lambda_h(f)} \cdot \alpha^\lambda_h(f) + 1}{(\alpha - 1)^{2\lambda_h(f)}} \cdot \frac{\sigma_h(f \circ g)}{\tau_h(f)}. \]

Proof. From the definition of $\tau_h(f)$ and in view of Lemma 7.2.4 and Lemma 7.2.5 we obtain for all sufficiently large values of $r$ that
\[ \mu_h^{-1} \mu_{f \circ g}(r) \geq M_h^{-1} \mu_f \left( \frac{(\alpha - 1)}{(\alpha + \gamma \alpha - \gamma) \alpha} r \right) \]
i.e., \[ \mu_h^{-1} \mu_f(r) \geq (\tau_h(f) - \epsilon) \left\{ \left( \frac{(\alpha - 1) r}{(\alpha + \gamma \alpha - \gamma) \alpha} \right)^{\lambda_h(f)} \right\} \]
i.e., \[ \mu_h^{-1} \mu_f(r) \geq \left( \frac{(\alpha - 1)}{(\alpha + \gamma \alpha - \gamma) \alpha} \right)^{\lambda_h(f)} \cdot (\tau_h(f) - \epsilon) \cdot r^{\lambda_h(f)}. \quad (7.3.17) \]
Thus from (7.3.7) and (7.3.17) we get for all sufficiently large values of $r$ that
\[ \frac{\mu_h^{-1} \mu_{f \circ g}(r)}{\mu_h^{-1} \mu_f(r)} \leq \frac{\alpha (\sigma_h(f \circ g) + \epsilon) \left[ \left( \frac{\alpha + \gamma \alpha - \gamma}{\alpha - 1} \right) r^{\rho_h(f \circ g)} \right]^{\lambda_h(f)}}{(\alpha - 1)^{\lambda_h(f)} \cdot (\tau_h(f) - \epsilon) \cdot r^{\lambda_h(f)}}. \quad (7.3.18) \]
In view of the condition (iii), we get from (7.3.18) that
\[ \limsup_{r \to \infty} \frac{\mu_h^{-1} \mu_{f \circ g}(r)}{\mu_h^{-1} \mu_f(r)} \leq \frac{\alpha (\sigma_h(f \circ g) + \epsilon) \left( \frac{\alpha + \gamma \alpha - \gamma}{\alpha - 1} \right)^{\lambda_h(f)}}{(\alpha - 1)^{\lambda_h(f)} \cdot (\tau_h(f) - \epsilon)}. \]
As $\epsilon (> 0)$ is arbitrary, it follows from above that
\[ \limsup_{r \to \infty} \frac{\mu_h^{-1} \mu_{f \circ g}(r)}{\mu_h^{-1} \mu_f(r)} \leq \frac{(\alpha + \gamma \alpha - \gamma)^{2\lambda_h(f)} \cdot \alpha^{\lambda_h(f) + 1}}{(\alpha - 1)^{2\lambda_h(f)}} \cdot \frac{\sigma_h(f \circ g)}{\tau_h(f)}. \]
Thus the theorem follows from above. $\blacksquare$

In the line of Theorem 7.3.23 and Theorem 7.3.24, we may state the following two theorems without their proofs:

Theorem 7.3.25 Let $f$, $g$ and $h$ be any three entire functions satisfying (i) $0 < \rho_h(g) < \infty$, (ii) $\sigma_h(g) < \infty$, (iii) $\lambda_h(f \circ g) = \rho_h(g)$ and (iv) $\tau_h(f \circ g) > 0$. Then for any $\alpha > 1$ and $\gamma > 0$,
\[ \liminf_{r \to \infty} \frac{\mu_h^{-1} \mu_{f \circ g}(r)}{\mu_h^{-1} \mu_g(r)} \geq \frac{(\alpha - 1)^{2\rho_h(g)}}{(\alpha + \gamma \alpha - \gamma)^{2\rho_h(g)} \cdot \alpha^{\rho_h(g) + 1}} \cdot \frac{\tau_h(f \circ g)}{\sigma_h(g)}. \]
Theorem 7.3.26 Let \( f, g \) and \( h \) be any three entire functions with (i) \( 0 < \lambda_h(g) < \infty \), (ii) \( \tau_h(g) < \infty \), (iii) \( \rho_h(f \circ g) = \lambda_h(g) \) and (iv) \( \sigma_h(f \circ g) > 0 \). Then for any \( \alpha > 1 \) and \( \gamma > 0 \),
\[
\limsup_{r \to \infty} \frac{\mu_h^{-1} \mu_{f \circ g}(r)}{\mu_h^{-1} \mu_g(r)} \leq \frac{(\alpha + \gamma \alpha - \gamma)^{2 \lambda_h(g)} \cdot \alpha^{\lambda_h(g)+1}}{(\alpha - 1)^{2 \lambda_h(g)}} \cdot \frac{\sigma_h(f \circ g)}{\tau_h(g)}.
\]

Analogously we may also state the following four theorems without their proofs on the basis of relative type and relative weak type of entire functions with respect to another entire functions in terms of their maximum modulus:

Theorem 7.3.27 Let \( f, g \) and \( h \) be any three entire functions such that (i) \( 0 < \rho_h(f) < \infty \), (ii) \( \sigma_h(f) < \infty \), (iii) \( \lambda_h(f \circ g) = \rho_h(f) \) and (iv) \( \tau_h(f \circ g) > 0 \). Then
\[
\liminf_{r \to \infty} \frac{M_h^{-1} M_{f \circ g}(r)}{M_h^{-1} M_f(r)} \geq \frac{\tau_h(f \circ g)}{\sigma_h(f)}.
\]

Theorem 7.3.28 Let \( f, g \) and \( h \) be any three entire functions with (i) \( 0 < \lambda_h(f) < \infty \), (ii) \( \tau_h(f) < \infty \), (iii) \( \rho_h(f \circ g) = \lambda_h(f) \) and (iv) \( \tau_h(f \circ g) > 0 \). Then
\[
\limsup_{r \to \infty} \frac{M_h^{-1} M_{f \circ g}(r)}{M_h^{-1} M_f(r)} \leq \frac{\sigma_h(f \circ g)}{\tau_h(f)}.
\]

Theorem 7.3.29 Let \( f, g \) and \( h \) be any three entire functions satisfying (i) \( 0 < \rho_h(g) < \infty \), (ii) \( \sigma_h(g) < \infty \), (iii) \( \lambda_h(f \circ g) = \rho_h(g) \) and (iv) \( \tau_h(f \circ g) > 0 \). Then
\[
\liminf_{r \to \infty} \frac{M_h^{-1} M_{f \circ g}(r)}{M_h^{-1} M_g(r)} \geq \frac{\tau_h(f \circ g)}{\sigma_h(g)}.
\]

Theorem 7.3.30 Let \( f, g \) and \( h \) be any three entire functions with (i) \( 0 < \lambda_h(g) < \infty \), (ii) \( \tau_h(g) < \infty \), (iii) \( \rho_h(f \circ g) = \lambda_h(g) \) and (iv) \( \sigma_h(f \circ g) > 0 \). Then
\[
\limsup_{r \to \infty} \frac{M_h^{-1} M_{f \circ g}(r)}{M_h^{-1} M_g(r)} \leq \frac{\sigma_h(f \circ g)}{\tau_h(g)}.
\]

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