CHAPTER 3

ON ITERATED ENTIRE FUNCTIONS WITH (p,q)-th ORDER
Chapter 3

ON ITERATED ENTIRE FUNCTIONS WITH \((p,q)\)-th ORDER

3.1 Introductory Remarks.

Let \( f \) be an entire function defined in the open complex plane \( \mathbb{C} \). The maximum term \( \mu(r,f) \), the maximum modulus \( M(r,f) \) and Nevanlinna’s characteristic function \( T(r,f) \) of \( f = \sum_{n=0}^{\infty} a_n z^n \) on \( |z| = r \) are respectively defined as \( \mu(r,f) = \max(|a_n| r^n) \), \( M(r,f) = \max |f(z)| \) and \( T(r,f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta \) where \( \log^+ x = \max(\log x, 0) \) for all \( x \geq 0 \).

In this chapter we would like to investigate some growth properties of iterated entire functions on the basis of their maximum terms, \((p,q)\)-th order and \((p,q)\)-th lower order where \( p,q \) are positive integers with \( p \geq q \). The existing literature, relevant definitions and examples of this chapter have already been discussed in Chapter One and Chapter Two respectively. Therefore, we need not mention those here again.

3.2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

**Lemma 3.2.1** [6] If \( f \) and \( g \) are any two entire functions then for all sufficiently large values of \( r \),

\[
M(r, f \circ g) \leq M(M(r, g), f).
\]

The results of this chapter have been published in Journal of Mathematics, see[23].
Lemma 3.2.2 Let $f$ and $g$ be any two entire functions such that $\rho_f(p, q) < \infty$ and $\rho_g(a, b) < \infty$ where $a, b, p$ and $q$ are any four positive integers with $a \geq b$ and $p \geq q$. Then for any even number $n$ and for all sufficiently large values of $r$, 

(i) \[ \log^{\left[ p + \frac{n-2}{2} (a-q) + \frac{n-2}{2} (p-b) \right]} \mu(r, f_n) \leq (\rho_f(p, q) + \varepsilon) \log^{[q]} M(r, g) + O(1) \]
when $q < a$ and $b < p$,

(ii) \[ \log^{\left[ p + \frac{n-2}{2} (a-q) \right]} \mu(r, f_n) \leq (\rho_f(p, q) + \varepsilon) (\rho_g(a, b) + \varepsilon) \log^{[q]} M(r, g) + O(1) \]
when $p = b \geq q$, $a > q$ and $n > 2$,

(iii) \[ \log^{\left[ p + \frac{n-2}{2} (p-b) \right]} \mu(r, f_n) \leq (\rho_f(p, q) + \varepsilon) \log^{[q]} M(r, g) + O(1) \]
when $p \geq b$ and $q = a$,

(iv) \[ \log^{[p]} \mu(r, f_n) \leq (\rho_f(p, q) + \varepsilon) \left( \rho_g(a, b) + \varepsilon \right)^{\frac{n-2}{2}} \log^{[q]} M(r, g) \]
when $a = b = p = q$,

(v) \[ \log^{\left[ p + (n-2) (p-q) \right]} \mu(r, f_n) \leq (\rho_f(p, q) + \varepsilon) \log^{[q]} M(r, g) + O(1) \]
when $p = a > q = b$,

(vi) \[ \log^{[p+a-q]} \mu(r, f_n) \leq \left( \rho_g(a, b) + \varepsilon \right)^{\frac{n-2}{2}} \log^{[q]} M(r, g) + O(1) \]
when $p < b, q < a$ and $b - p = a - q$,

(vii) \[ \log^{\left[ p + a + \frac{n-2}{2} (a-b) - q \right]} \mu(r, f_n) \leq \log^{[q]} M(r, g) + O(1) \]
when $p < b, q < a$ and $b - p < a - q$,

(viii) \[ \log^{[p]} \mu(r, f_n) \leq (\rho_f(p, q) + \varepsilon)^{\frac{n}{2}} \log^{[q]} M(r, g) + O(1) \]
when $p > b, q > a$ and $q - a = p - b$,
(ix) \[ \log^{[p+a-b-q]} \mu (r, f_n) \leq (\rho_f (p, q) + \varepsilon) \log^{[q]} M (r, g) + O(1) \] when \( p > b, q > a \) and \( q - a < p - b \),

and for any odd number \( n (\neq 1) \) and for all sufficiently large values of \( r \),

(x) \[ \log^{[p+a-b-q]} \mu (r, f_n) \leq (\rho_g (a, b) + \varepsilon) \log^{[b]} M (r, f) + O(1) \] when \( q < a \) and \( b > p \),

(xi) \[ \log^{[(p+1)(a-q)+\frac{n-3}{2}(p-b)]} \mu (r, f_n) \leq (\rho_g (a, b) + \varepsilon) \log^{[b]} M (r, f) + O(1) \] when \( p = b \geq q \) and \( a > q \),

(xii) \[ \log^{[p+a-b-q]} \mu (r, f_n) \leq (\rho_f (p, q) + \varepsilon) (\rho_g (a, b) + \varepsilon) \log^{[b]} M (r, f) + O(1) \] when \( p \geq b \) and \( q = a \),

(xiii) \[ \log^{[p+a-b-q]} \mu (r, f_n) \leq [(\rho_f (p, q) + \varepsilon) (\rho_g (a, b) + \varepsilon)]^{\frac{n-1}{2}} \log^{[b]} M (r, f) \] when \( a = b = p = q \),

(xiv) \[ \log^{[p+(n-2)(p-q)]} \mu (r, f_n) \leq (\rho_g (p, q) + \varepsilon) \log^{[q]} M (r, f) + O(1) \] when \( p = a > q = b \),

(xv) \[ \log^{[p+a-b-q]} \mu (r, f_n) \leq (\rho_g (a, b) + \varepsilon)^{\frac{n-1}{2}} \log^{[b]} M (r, f) + O(1) \] when \( p < b, q < a \) and \( b - p = a - q \),

(xvi) \[ \log^{[p+a-b-q]} \mu (r, f_n) \leq (\rho_g (a, b) + \varepsilon) \log^{[b]} M (r, f) + O(1) \] when \( p < b, q < a \) and \( b - p < a - q \),

(xvii) \[ \log^{[p+a-b-q]} \mu (r, f_n) \leq (\rho_f (p, q) + \varepsilon)^{\frac{n-1}{2}} \log^{[p]} M (r, f) + O(1) \] when \( p > b, q > a \) and \( q - a = p - b \),
and (xviii) \( \log^{[p+\frac{a-1}{2}(p+a-b-q)]} \mu(r, f_n) \leq \log^{[p]} M(r, f) + O(1) \)

when \( p > b, q > a \) and \( q - a < p - b \)

where \( \varepsilon > 0 \) is any arbitrary number.

**Proof.** Let us consider \( n \) to be an even number.

Then in view of Lemma 3.2.1 and the inequality \( \mu(r, f) \leq M(r, f) \) \( \{c.f. \ [57] \} \) we get for all sufficiently large values of \( r \) that

\[
\mu(r, f_n) \leq M(r, f_n) \\
i.e., \log^{[p]} \mu(r, f_n) \leq \log^{[p]} M(M(r, g_{n-1}), f) \\
i.e., \log^{[p]} \mu(r, f_n) \leq (\rho_f (p, q) + \varepsilon) \log^{[q]} M(r, g_{n-1}).
\]

(3.1)

**Case I.** Let \( q < a \) and \( b < p \).

We obtain from (3.1) for all sufficiently large values of \( r \) that

\[
\log^{[p+a-q]} \mu(r, f_n) \leq \log^{[a]} M(r, g_{n-1}) + O(1) \\
i.e., \log^{[p+a-q]} \mu(r, f_n) \leq \log^{[a]} M(M(r, f_{n-2}), g) + O(1) \\
i.e., \log^{[p+a-q]} \mu(r, f_n) \leq (\rho_g (a, b) + \varepsilon) \log^{[b]} M(r, f_{n-2}) + O(1) \\
i.e., \log^{[p+a-q+p-b]} \mu(r, f_n) \leq \log^{[p]} M(M(r, g_{n-3}), f) + O(1).
\]

Therefore,

\[
\log^{[p+\frac{a-1}{2}(a-q)+\frac{a-2}{2}(p-b)]} \mu(r, f_n) \leq (\rho_f (p, q) + \varepsilon) \log^{[q]} M(r, g) + O(1)
\]

when \( n \) is even.

Similarly,

\[
\log^{[p+\frac{a-1}{2}(a-q)+\frac{a-2}{2}(p-b)]} \mu(r, f_n) \leq (\rho_g (a, b) + \varepsilon) \log^{[b]} M(r, f) + O(1)
\]

when \( n \) is odd and \( n \neq 1 \).

This establishes the first and the tenth part of the lemma respectively.

**Case II.** Let \( p = b \geq q \) and \( a > q \).

Now we get from (3.2) for all sufficiently large values of \( r \) that
\[
\log^{[p+a-q]} \mu (r, f_n) \leq (\rho_g (a, b) + \varepsilon) \log^{[p]} M (r, f_{n-2}) + O(1)
\]
i.e., \[
\log^{[p+a-q]} \mu (r, f_n) \leq (\rho_g (a, b) + \varepsilon) \log^{[p]} M (r, g_{n-3}, f) + O(1)
\]

Thus,
\[
\log^{[p+\frac{n-2}{2}(a-q)]} \mu (r, f_n) \leq (\rho_f (p, q) + \varepsilon) (\rho_g (a, b) + \varepsilon) \log^{[q]} M (r, g) + O(1)
\]
when \( n \) is even and \( n > 2 \)

Analogously,
\[
\log^{[p+\frac{n-1}{2}(a-q)]} \mu (r, f_n) \leq (\rho_g (a, b) + \varepsilon) \log^{[b]} M (r, f) + O(1)
\]
when \( n \) is odd and \( n \neq 1 \),

from which the second and the eleventh part of the lemma follows.

**Case III.** Let \( p \geq b \) and \( q = a \).

Therefore it follows from \([3.1]\) for all sufficiently large values of \( r \) that
\[
\log^{[p]} \mu (r, f_n) \leq (\rho_f (p, q) + \varepsilon) \log^{[a]} M (r, g_{n-1})
\]
i.e., \[
\log^{[p]} \mu (r, f_n) \leq (\rho_f (p, q) + \varepsilon) \log^{[a]} M (r, f_{n-2}, g)
\]
i.e., \[
\log^{[p]} \mu (r, f_n) \leq (\rho_f (p, q) + \varepsilon) (\rho_g (a, b) + \varepsilon) \log^{[b]} M (r, f_{n-2})
\]
i.e., \[
\log^{[p+p-b]} \mu (r, f_n) \leq \log^{[p]} M (r, g_{n-3}, f) + O(1)
\]

Therefore,
\[
\log^{[p+\frac{n-2}{2}(p-b)]} \mu (r, f_n) \leq (\rho_f (p, q) + \varepsilon) \log^{[q]} M (r, g) + O(1) \text{ when } n \text{ is even.}
\]

Similarly,
\[
\log^{[p+\frac{n-3}{2}(p-b)]} \mu (r, f_n) \leq (\rho_f (p, q) + \varepsilon) (\rho_g (a, b) + \varepsilon) \log^{[b]} M (r, f) + O(1)
\]
when \( n \) is odd and \( n \neq 1 \).

This proves the third and the twelfth part of the lemma respectively.
Case IV. Let $a = b = p = q$.

Now we have from \((3.1)\) for all sufficiently large values of $r$ that

\[
\log^{[p]} \mu (r, f_n) \leq (\rho_f (p, q) + \varepsilon) \log^{[a]} M (r, g_{n-1})
\]

\[\text{i.e., } \log^{[p]} \mu (r, f_n) \leq (\rho_f (p, q) + \varepsilon) \log^{[a]} M (M (r, f_{n-2}), g) + O(1)
\]

\[\text{i.e., } \log^{[p]} \mu (r, f_n) \leq (\rho_f (p, q) + \varepsilon) (\rho_g (a, b) + \varepsilon) \log^{[b]} M (M (r, g_{n-3}), f)
\]

\[\text{i.e., } \log^{[p]} \mu (r, f_n) \leq (\rho_f (p, q) + \varepsilon) \log^{[a]} M (M (r, g_{n-3}), f)
\]

\[\text{Case IV. Case IV. Case IV. Case IV. Case IV.}
\]

Thus,

\[
\log^{[p]} \mu (r, f_n) \leq (\rho_f (p, q) + \varepsilon) \log^{[a]} M (r, g)
\]

when $n$ is even.

Analogously,

\[
\log^{[p]} \mu (r, f_n) \leq [(\rho_f (p, q) + \varepsilon) (\rho_g (a, b) + \varepsilon)]^{\frac{a-2}{a}} \log^{[b]} M (r, f)
\]

when $n$ is odd and $n \neq 1$.

This establishes the fourth and the thirteenth part of the lemma respectively.

Case V. Let $p = a > q = b$.

We obtain from \((3.1)\) for all sufficiently large values of $r$ that

\[
\log^{[p+p-q]} \mu (r, f_n) \leq \log^{[p]} M (r, g_{n-1}) + O(1)
\]

\[\text{i.e., } \log^{[p+p-q]} \mu (r, f_n) \leq \log^{[p]} M (M (r, f_{n-2}), g) + O(1)
\]

\[\text{i.e., } \log^{[p+p-q]} \mu (r, f_n) \leq (\rho_g (p, q) + \varepsilon) \log^{[b]} M (M (r, g_{n-3}), f) + O(1)
\]

\[\text{i.e., } \log^{[p+2(p-q)]} \mu (r, f_n) \leq \log^{[p]} M (M (r, g_{n-3}), f) + O(1)
\]

\[\text{Case V. Case V. Case V. Case V. Case V.}
\]

Therefore,

\[
\log^{[p+(n-2)(p-q)]} \mu (r, f_n) \leq (\rho_f (p, q) + \varepsilon) \log^{[q]} M (r, g) + O(1)
\]

when $n$ is even.

Similarly,

\[
\log^{[p+(n-2)(p-q)]} \mu (r, f_n) \leq (\rho_g (p, q) + \varepsilon) \log^{[q]} M (r, f) + O(1)
\]

when $n$ is odd and $n \neq 1$,

from which the fifth and the fourteenth part of the lemma follows.
**Case VI.** Let \( p < b, q < a, b - p = a - q \).

We get from (3.2), for all sufficiently large values of \( r \) that

\[
\log^{p+a-q} \mu(r, f_n) \leq (\rho_g(a, b) + \varepsilon) \log^{b-p} \left( \log^{[g]} M(r, f_{n-2}) \right) + O(1)
\]

\[\text{i.e., } \log^{p+a-q} \mu(r, f_n) \leq (\rho_g(a, b) + \varepsilon) \log^{b-p} \left( \log^{[a]} M(r, g_{n-3}) \right) + O(1) \quad (3.3)\]

\[\text{i.e., } \log^{p+a-q} \mu(r, f_n) \leq (\rho_g(a, b) + \varepsilon) \log^{[a]} M(r, g_{n-4}) + O(1)\]

Thus,

\[
\log^{p+a-q} \mu(r, f_n) \leq (\rho_g(a, b) + \varepsilon) \frac{n^2}{2} \log^{[a]} M(r, g) + O(1) \text{ when } n \text{ is even.}
\]

Analogously,

\[
\log^{p+a-q} \mu(r, f_n) \leq (\rho_g(a, b) + \varepsilon) \frac{n+1}{2} \log^{[b]} M(r, f) + O(1)\]

when \( n \) is odd and \( n \neq 1 \),

these proves the sixth and the fifteenth part of the lemma respectively.

**Case VII.** Let \( p < b, q < a, b - p < a - q \).

Therefore it follows from (3.3) for all sufficiently large values of \( r \) that

\[
\log^{p+a-q+a-q-b+p} \mu(r, f_n) \leq \log^{[a]} M(r, f_{n-4}) + O(1)
\]

\[\text{........... } \text{........... } \text{........... } \text{........... } \]

\[\text{........... } \text{........... } \text{........... } \text{........... } . \]

Therefore,

\[
\log^{p+a-q+a-q-b+q} \mu(r, f_n) \leq \log^{[a]} M(r, g) + O(1) \text{ when } n \text{ is even.}
\]

Similarly,

\[
\log^{p+a-q+a-q-b+a} \mu(r, f_n) \leq (\rho_g(a, b) + \varepsilon) \log^{[b]} M(r, f) + O(1)\]

when \( n \) is odd and \( n \neq 1 \).

This establishes the seventh and the sixteenth part of the lemma respectively.
Case VIII. Let \( p > b, q > a, q - a = p - b \).
We obtain from \([3.1]\) for all sufficiently large values of \( r \) that
\[
\log^{[p]} \mu (r, f_n) \leq (\rho_f (p, q) + \varepsilon) \log^{[q]} M (r, g_{n-1})
\]
i.e.,
\[
\log^{[p]} \mu (r, f_n) \leq (\rho_f (p, q) + \varepsilon) \log^{[q-a]} \log^{[a]} M (r, g_{n-1})
\]
i.e.,
\[
\log^{[p]} \mu (r, f_n) \leq (\rho_f (p, q) + \varepsilon) \log^{[q-a]} \left[ \log^{[a]} M (M (r, g_{n-2}) + g) \right]
\]
i.e.,
\[
\log^{[p]} \mu (r, f_n) \leq (\rho_f (p, q) + \varepsilon) \log^{[q-a]} \left[ (\rho_g (a, b) + \varepsilon) \log^{[b]} M (r, f_{n-2}) \right]
\]
i.e.,
\[
\log^{[p]} \mu (r, f_n) \leq (\rho_f (p, q) + \varepsilon) \log^{[p]} M (M (r, g_{n-3}), f) + O(1)
\]
.........
Equate the two equations to get
\[
\log^{[p]} \mu (r, f_n) \leq (\rho_f (p, q) + \varepsilon) \log^{[q]} M (r, g) + O(1) \quad \text{when } n \text{ is even}.
\]
Analogously ,
\[
\log^{[p]} \mu (r, f_n) \leq (\rho_f (p, q) + \varepsilon) \log^{[q]} M (r, f) + O(1) \quad \text{when } n \text{ is odd and } n \neq 1,
\]
from which the eighth and the seventeenth part of the lemma follows.

Case IX. Let \( p > b, q > a, q - a < p - b \).
We get from \([3.4]\) for all sufficiently large values of \( r \) that
\[
\log^{[p]} \mu (r, f_n) \leq (\rho_f (p, q) + \varepsilon) \log^{[q-a+b]} M (r, f_{n-2}) + O(1)
\]
i.e.,
\[
\log^{[p+a-b-q]} \mu (r, f_n) \leq \log^{[p]} M (M (r, g_{n-3}), f) + O(1)
\]
.........
Therefore,
\[
\log^{[p+\frac{n-3}{2}(p+a-b-q)]} \mu (r, f_n) \leq (\rho_f (p, q) + \varepsilon) \log^{[q]} M (r, g) + O(1) \quad \text{when } n \text{ is even}.
\]
Similarly,
\[
\log^{[p+\frac{n-1}{2}(p+a-b-q)]} \mu (r, f_n) \leq \log^{[p]} M (r, f) + O(1) \quad \text{when } n \text{ is odd and } n \neq 1.
\]
This proves the nineth and the eighteenth part of the lemma respectively.
Thus the lemma follows. 

Lemma 3.2.3 \([13]\) Let \( g \) be an entire function. Then for any \( \delta > 0 \) the function \( r^{\lambda_g + \delta - \lambda_g (r)} \) is an increasing function of \( r \).

Lemma 3.2.4 \([13]\) Let \( g \) be an entire function. Then for any \( \delta > 0 \) the function \( r^{\nu_g + \delta - \nu_g (r)} \) is an increasing function of \( r \).
3.3 Theorems.

In this section we present the main results of the chapter.

Theorem 3.3.1 If \( f \) and \( g \) be any two entire functions such that \( \rho_f(p, q) \) and \( \rho_g(l) \) are both finite where \( p, q, l \) are positive integers with \( p \geq q \) and \( l \geq 2 \), then for any even number \( n \) and \( \beta > 1 \),

\[
\begin{align*}
(i) \quad & \liminf_{r \to \infty} \frac{\log^p(r, f_n)}{\log^{l-1}(r, g)} \leq \rho_f(p, q) \cdot [2\beta]^{\lambda_g[l]} \quad \text{if } p > 1 \text{ and } q = l > 2, \\
(ii) \quad & \liminf_{r \to \infty} \frac{\log^{p}(r, f_n)}{\log^{l}(r, g)} \leq 3\rho_f(p, q) \cdot [2\beta]^{\lambda_g[l]} \quad \text{if } p > 1 \text{ and } q = l = 2, \\
(iii) \quad & \liminf_{r \to \infty} \frac{\log^{p}(r, f_n)}{\log^{l-1}(r, g)} \leq [\rho_f(p, q)]^{\lambda_g[l]} \cdot [2\beta]^{\lambda_g[l]} \quad \text{if } p > 1, \quad q > l > 2 \text{ and } q - l = p - 1, \\
(iv) \quad & \liminf_{r \to \infty} \frac{\log^{p+(n-2)(p+1-q)}(r, f_n)}{\log^{l-1}(r, g)} \leq \rho_f(p, q) \cdot [2\beta]^{\lambda_g[l]} \quad \text{if } p > 1, \quad q > l > 2 \text{ and } q - l < p - 1, \\
(v) \quad & \liminf_{r \to \infty} \frac{\log^{p+(n-2)(p+1-q)}(r, f_n)}{\log^{l-1}(r, g)} \leq 3\rho_f(p, q) \cdot [2\beta]^{\lambda_g[l]} \quad \text{if } p > 1, \quad q > l = 2 \text{ and } q - l < p - 1, \\
(vi) \quad & \liminf_{r \to \infty} \frac{\log^{p+(n-2)(p+1-q)}(r, f_n)}{\log^{l-1}(r, g)} \leq [\rho_f(p, q)]^{\lambda_g[l]} \quad \text{if } q < l - 1 \text{ and } 1 < p, \\
(vii) \quad & \liminf_{r \to \infty} \frac{\log^{p+(n-2)(p+1-q)}(r, f_n)}{\log^{l-1}(r, g)} \leq [2\beta]^{\lambda_g[l]} \quad \text{if } p = q = 1, \quad l - 1 > q \text{ and } n > 2
\end{align*}
\]

and

\[
(viii) \quad \liminf_{r \to \infty} \frac{\log^{n(l-1)}(r, f_n)}{\log^{l-1}(r, g)} \leq [2\beta]^{\lambda_g[l]} \quad \text{if } p = l \text{ and } l - 1 > q = 1.
\]
Proof. Putting $R = \beta r$ in the inequality $M(r, f) \leq \frac{R}{\pi} \mu(R, f)$ \{cf. [52]\} and in view of the inequality $T(r, g) \leq \log^+ M(r, g)$ we get that

$$\log \mu(\beta r, g) + O(1) \geq \log M(r, g) \geq T(r, g)$$

i.e.,

$$\log^{[l-1]} \mu(r, g) + O(1) \geq \log^{[l-2]} T(\frac{r}{\beta}, g).$$

Let $l > 2$. Since $\liminf_{r \to \infty} \frac{\log^{[l-2]} T(r, g)}{\log^{[l-2]} T(\frac{r}{\beta}, g)} = 1$, for given $\varepsilon (0 < \varepsilon < 1)$ we get for a sequence of values of $r$ tending to infinity that

$$\log^{[l-2]} T(r, g) < (1 + \varepsilon) r^{\lambda_{g}^{[l]}(r)}$$

and for all large positive numbers of $r$,

$$\log^{[l-2]} T(r, g) > (1 - \varepsilon) r^{\lambda_{g}^{[l]}(r)}.$$

Since $\log M(r, g) \leq 3T(2r, g)$, for a sequence of values of $r$ tending to infinity we get for any $\delta (> 0)$ that

$$\frac{\log^{[l-1]} M(r, g)}{\log^{[l-2]} T(\frac{r}{\beta}, g)} \leq \frac{\log^{[l-2]} T(2r, g) + O(1)}{\log^{[l-2]} T(\frac{r}{\beta}, g)}$$

$$\leq \frac{(1 + \varepsilon)}{(1 - \varepsilon)} \cdot \frac{(2r)^{\lambda_{g}^{[l]} + \delta}}{(2r)^{\lambda_{g}^{[l]} + \delta - \lambda_{g}^{[l]}(2r)}} \cdot \frac{1}{\left(\frac{r}{\beta}\right)^{\lambda_{g}^{[l]}(\frac{r}{\beta})}} + O(1)$$

$$\leq \frac{(1 + \varepsilon)}{(1 - \varepsilon)} \cdot \frac{(2\beta)^{\lambda_{g}^{[l]} + \delta}}{(2\beta)^{\lambda_{g}^{[l]} + \delta - \lambda_{g}^{[l]}(2\beta)}} \cdot \frac{1}{\left(\frac{r}{\beta}\right)^{\lambda_{g}^{[l]}(\frac{r}{\beta})}} + O(1)$$

$$\leq \frac{(1 + \varepsilon)}{(1 - \varepsilon)} \cdot (2\beta)^{\lambda_{g}^{[l]} + \delta}$$

because $r^{\lambda_{g}^{[l]} + \delta - \lambda_{g}^{[l]}(r)}$ is an increasing function of $r$ by Lemma 3.2.3.

Since $\varepsilon (> 0)$ and $\delta (> 0)$ are both arbitrary, we get from above that

$$\liminf_{r \to \infty} \frac{\log^{[l-1]} M(r, g)}{\log^{[l-2]} T(\frac{r}{\beta}, g)} \leq (2\beta)^{\lambda_{g}^{[l]}}.$$

Again let $l = 2$.

Since $\liminf_{r \to \infty} \frac{T(r, g)}{\log^{[2]}(r)} = 1$, in view of condition (v) of Definition 2.3.7 it follows for a sequence of values of $r$ tending to infinity and for a given $\varepsilon (0 < \varepsilon < 1)$ that

$$T(r, g) < (1 + \varepsilon) r^{\lambda_{g}(r)}.$$
and for all sufficiently large values of $r$,
\[ T(r, g) > (1 - \varepsilon)r^{\lambda_y(r)}. \]
As $\log M(r, g) \leq 3T(2r, g)$, for a sequence of values of $r$ tending to infinity we get for any $\delta (> 0)$ that
\[
\frac{\log M(r, g)}{T(\frac{r}{2}, g)} \leq 3(1 + \varepsilon) \frac{(2r)^{\lambda_y + \delta}}{(2r)^{\lambda_y + \delta - \lambda_y(2r)}} \frac{1}{\left(\frac{r}{2}\right)^{\lambda_y(\frac{r}{2})}} + O(1)
\]
i.e.,
\[
\frac{\log M(r, g)}{T(\frac{r}{2}, g)} \leq 3(1 + \varepsilon) \frac{(2\beta)^{\lambda_y + \delta}}{(1 - \varepsilon)\beta^{\lambda_y}}.
\] (3.8)
because $r^{\lambda_y + \delta - \lambda_y(r)}$ is an increasing function of $r$ by Lemma 3.2.3.
Since $\varepsilon (> 0)$ and $\delta (> 0)$ are both arbitrary, we get from (3.8) that
\[
\liminf_{r \to \infty} \frac{\log M(r, g)}{T(\frac{r}{2}, g)} \leq 3. (2)^{\lambda_y}.
\] (3.9)

**Case I.** Let $p > 1$ and $q = l > 2$.
Then from the third part of Lemma 3.2.2 we obtain for all sufficiently large values of $r$ that
\[
\log_m^[(p-1)+1] \mu (r, f_n) \leq (\rho_f(p, q) + \varepsilon) \log^{[l-1]} M (r, g) + O(1).
\]
Since $\varepsilon (> 0)$ we get from (3.6) and above that
\[
\liminf_{r \to \infty} \frac{\log_m^[(p-1)+1] \mu (r, f_n)}{\log^{[l-1]} \mu(r, g)} \leq \rho_f(p, q) \liminf_{r \to \infty} \frac{\log^{[l-1]} M (r, g)}{\log^{[l-2]} T(\frac{r}{2}, g)}.
\] (3.10)

**Case II.** Let $p > 1$, $q > l > 2$ and $q - l = p - 1$.
Then from the eighth part of Lemma 3.2.2 we obtain for all sufficiently large values of $r$ that
\[
\log_m^p \mu (r, f_n) \leq (\rho_f(p, q) + \varepsilon) \frac{\log^{[l-1]} M (r, g)}{\log^{[l-2]} T(\frac{r}{2}, g)} + O(1).
\]
Since $\varepsilon (> 0)$, we get from (3.6) and above that
\[
\liminf_{r \to \infty} \frac{\log_m^p \mu (r, f_n)}{\log^{[l-1]} \mu(r, g)} \leq [\rho_f(p, q)] \frac{\log^{[l-1]} M (r, g)}{\log^{[l-2]} T(\frac{r}{2}, g)}.
\] (3.11)

**Case III.** Let $p > 1$, $q > l$ and $q - l < p - 1$.
Then from the ninth part of Lemma 3.2.2, we obtain for all sufficiently large values of $r$ that
\[
\log_m^{[p+\frac{n-2}{2}(p+l-1-q)]} \mu (r, f_n) \leq (\rho_f(p, q) + \varepsilon) \log^{[l-1]} M (r, g) + O(1).
\]
Since $\varepsilon (> 0)$, we get from (3.6) and above that
\[
\liminf_{r \to \infty} \frac{\log_m^{[p+\frac{n-2}{2}(p+l-1-q)]} \mu (r, f_n)}{\log^{[l-1]} \mu(r, g)} \leq \rho_f(p, q) \liminf_{r \to \infty} \frac{\log^{[l-1]} M (r, g)}{\log^{[l-2]} T(\frac{r}{2}, g)}.
\] (3.12)
Case IV. Let \( q < l - 1 \) and \( 1 < p \).
Then from the first part of Lemma [3.2.2](#) and (3.6), we get for all sufficiently large values of \( r \) that
\[
\log^{[l-q-1]} \log^{\left[ \frac{n-2}{2}(l-q)+\frac{5}{2}(p-1)+1 \right]} \mu (r, f_n) \\
\leq \log^{[l-q-1]} \left\{ (\rho_f (p, q) + \varepsilon) \log^{[q]} M (r, g) + O(1) \right\} \\
i.e., \quad \log^{\left[ \frac{5}{2}(l-q)+\frac{5}{2}(p-1) \right]} \mu (r, f_n) \leq \log^{[l-1]} M (r, g) + O(1) \\
i.e., \quad \frac{\log^{\left[ \frac{5}{2}(l-q)+\frac{5}{2}(p-1) \right]} \mu (r, f_n)}{\log^{[l-1]} \mu (r, g)} \leq \frac{\log^{[l-1]} M (r, g) + O(1)}{\log^{[l-2]} T(\frac{r}{\beta}, g)} .
\]
Therefore we get from above that
\[
\liminf_{r \to \infty} \frac{\log^{\left[ \frac{5}{2}(l-q)+\frac{5}{2}(p-1) \right]} \mu (r, f_n)}{\log^{[l-1]} \mu (r, g)} \leq \liminf_{r \to \infty} \frac{\log^{[l-1]} M (r, g)}{\log^{[l-2]} T(\frac{r}{\beta}, g)} . \tag{3.13}
\]

Case V. Let \( p = q = 1 \), \( l - 1 > q \) and \( n > 2 \).
Then from the second part of Lemma [3.2.2](#) and (3.6), we get for all sufficiently large values of \( r \) that
\[
\log^{[l-2]} \log^{\left[ \frac{n-2}{2}(l-q)+\frac{3}{2}(p-1)+1 \right]} \mu (r, f_n) \\
\leq \log^{[l-2]} \left\{ (\rho_f (1, 1) + \varepsilon) \log M (r, g) \right\} \\
i.e., \quad \log^{\left[ \frac{3}{2}(l-q)+\frac{3}{2}(p-1) \right]} \mu (r, f_n) \leq \log^{[l-1]} M (r, g) + O(1) \\
i.e., \quad \frac{\log^{\left[ \frac{3}{2}(l-q)+\frac{3}{2}(p-1) \right]} \mu (r, f_n)}{\log^{[l-1]} \mu (r, g)} \leq \frac{\log^{[l-1]} M (r, g) + O(1)}{\log^{[l-2]} T(\frac{r}{\beta}, g)} .
\]
Therefore we get from above that
\[
\liminf_{r \to \infty} \frac{\log^{\left[ \frac{3}{2}(l-q)+\frac{3}{2}(p-1) \right]} \mu (r, f_n)}{\log^{[l-1]} \mu (r, g)} \leq \liminf_{r \to \infty} \frac{\log^{[l-1]} M (r, g)}{\log^{[l-2]} T(\frac{r}{\beta}, g)} . \tag{3.14}
\]

Case VI. Let \( p = l \) and \( l - 1 > q = 1 \).
Then from (3.6) and the fifth part of Lemma [3.2.2](#) we get for all sufficiently large values of \( r \) that
\[
\log^{[l-2]} \log^{\left[ \frac{n-2}{2}(l-q)+\frac{3}{2}(p-1)+1 \right]} \mu (r, f_n) \\
\leq \log^{[l-2]} \left\{ (\rho_f (p, q) + \varepsilon) \log M (r, g) + O(1) \right\} \\
i.e., \quad \log^{\left[ \frac{3}{2}(l-q)+\frac{3}{2}(p-1) \right]} \mu (r, f_n) \leq \log^{[l-1]} M (r, g) + O(1) \\
i.e., \quad \frac{\log^{\left[ \frac{3}{2}(l-q)+\frac{3}{2}(p-1) \right]} \mu (r, f_n)}{\log^{[l-1]} \mu (r, g)} \leq \frac{\log^{[l-1]} M (r, g) + O(1)}{\log^{[l-2]} T(\frac{r}{\beta}, g)} .
\]
Therefore we get from above that
\[
\liminf_{r \to \infty} \frac{\log^{\left[ \frac{3}{2}(l-q)+\frac{3}{2}(p-1) \right]} \mu (r, f_n)}{\log^{[l-1]} \mu (r, g)} \leq \liminf_{r \to \infty} \frac{\log^{[l-1]} M (r, g)}{\log^{[l-2]} T(\frac{r}{\beta}, g)} . \tag{3.15}
\]
Now from (3.10) of Case I and (3.7), it follows that
\[
\liminf_{r \to \infty} \frac{\log \left[ \frac{p+1}{2} \right] \mu(r, f_n)}{\log \left[ \frac{p-1}{2} \right] \mu(r, g)} \leq \rho_f(p, q). [2\beta]^{[l]}.
\]
This proves the first part of the theorem.
Again for \( l = 2 \), we obtain in view of (3.9) and (3.10) of Case I that
\[
\liminf_{r \to \infty} \frac{\log \left[ \frac{p+1}{2} \right] \mu(r, f_n)}{\log \left[ \frac{p-1}{2} \right] \mu(r, g)} \leq 3. \rho_f(p, q). [2\beta]^{[l]}.
\]
Thus the second part of the theorem follows.
Again from (3.11) of Case II and (3.7), it follows that
\[
\liminf_{r \to \infty} \frac{\log \mu(r, f_n)}{\log \mu(r, g)} \leq \rho_f(p, q). [2\beta]^{[l]}.
\]
This proves the third part of the theorem.
Similarly, from (3.12) of Case III and (3.7), it follows that
\[
\liminf_{r \to \infty} \frac{\log \left[ \frac{p+1}{2} \right] \mu(r, f_n)}{\log \left[ \frac{p-1}{2} \right] \mu(r, g)} \leq 3. \rho_f(p, q). [2\beta]^{[l]}.
\]
This proves the fourth part of the theorem.
Again for \( l = 2 \), we obtain in view of (3.9) and (3.12) of Case III that
\[
\liminf_{r \to \infty} \frac{\log \left[ \frac{p+1}{2} \right] \mu(r, f_n)}{\log \mu(r, g)} \leq 3. \rho_f(p, q). [2\beta]^{[l]}.
\]
Thus the fifth part of the theorem follows.
Also from (3.13) of Case IV and (3.7), it follows that
\[
\liminf_{r \to \infty} \frac{\log \left[ \frac{p+1}{2} \right] \mu(r, f_n)}{\log \left[ \frac{p-1}{2} \right] \mu(r, g)} \leq 2\beta [2\beta]^{[l]}.
\]
Hence the sixth part of the theorem is established.
Similarly, from (3.14) of Case V and (3.7), it follows that
\[
\liminf_{r \to \infty} \frac{\log \left[ \frac{p+1}{2} \right] \mu(r, f_n)}{\log \left[ \frac{p-1}{2} \right] \mu(r, g)} \leq 2\beta [2\beta]^{[l]}.
\]
This concludes the seventh part of the theorem.
Also from (3.15) of Case VI and (3.7), we obtain that
\[
\liminf_{r \to \infty} \frac{\log \left[ \frac{p+1}{2} \right] \mu(r, f_n)}{\log \left[ \frac{p-1}{2} \right] \mu(r, g)} \leq 2\beta [2\beta]^{[l]}.
\]
Thus the eighth part of the theorem is proved. ■
Corollary 3.3.1 Under the same conditions of Theorem 3.3.1, if \( l = 2 \) then

\[ (i) \liminf_{r \to \infty} \frac{\log^{\frac{n}{2}(p-1)+2} \mu(r, f_n)}{\log^{[q]} \mu(r, g)} \leq 1 \text{ if } p > 1 \text{ and } q = l \]

and

\[ (ii) \liminf_{r \to \infty} \frac{\log^{p+\frac{n-2}{2}(p+1-q)+1} \mu(r, f_n)}{\log^{[q+1]} \mu(r, g)} \leq 1 \text{ if } p > 1, \ q > l \text{ and } q - l < p - 1. \]

Proof. In view of (3.5) and from (3.8), we have for a sequence of values of \( r \) tending to infinity that

\[ \log M(r, g) \leq \left\{ \frac{3(1 + \varepsilon)}{(1 - \varepsilon)} \cdot A^{q+s} \right\} T \left( \frac{r}{\beta}, g \right) \]

i.e., \( \log^{[q+1]} M(r, g) \leq \log^{[q]} T \left( \frac{r}{\beta}, g \right) + O(1) \)

i.e., \( \log^{[q+1]} M(r, g) \leq \log^{[q+1]} \mu(r, g) + O(1). \)  \( (3.16) \)

**Case I.** Let \( p > 1 \) and \( q = l. \)

Then from the third part of Lemma 3.2.2, we obtain for all sufficiently large values of \( r \) that

\[ \log^{\frac{n}{2}(p-1)+1} \mu(r, f_n) \leq (\rho_f(p, q) + \varepsilon) \log^{[q]} M(r, g) + O(1) \]

i.e., \( \log^{\frac{n}{2}(p-1)+2} M(r, f_n) \leq \log^{[3]} M(r, g) + O(1). \)  \( (3.17) \)

Now combining (3.16) and (3.17), it follows for a sequence of values of \( r \) tending to infinity that

\[ \log^{\frac{n}{2}(p-1)+2} \mu(r, f_n) \leq \log^{[3]} \mu(r, g) + O(1) \]

i.e., \( \frac{\log^{\frac{n}{2}(p-1)+2} \mu(r, f_n)}{\log^{[3]} \mu(r, g)} \leq 1 + \frac{O(1)}{\log^{[3]} \mu(r, g)}. \)

So from above we obtain that

\[ \liminf_{r \to \infty} \frac{\log^{\frac{n}{2}(p-1)+2} \mu(r, f_n)}{\log^{[3]} \mu(r, g)} \leq 1. \]

Thus the first part of the corollary follows.

**Case II.** Let \( p > 1, \ q > l \) and \( q - l < p - 1. \)

Then from the ninth part of Lemma 3.2.2, we obtain for all sufficiently large values of \( r \) that

\[ \log^{p+\frac{n-2}{2}(p+1-q)+1} \mu(r, f_n) \leq \log^{[q+1]} M(r, g) + O(1). \]  \( (3.18) \)
Therefore combining (3.16) and (3.18), it follows for a sequence of values of \( r \) tending to infinity that

\[
\log^{[p+\frac{n}{2}(p+1-q)+1]} \mu (r, f_n) \leq \log^{[q+1]} \mu (r, g) + O(1)
\]

i.e.,

\[
\frac{\log^{[p+\frac{n}{2}(p+1-q)+1]} \mu (r, f_n)}{\log^{[q+1]} \mu (r, g)} \leq 1 + \frac{O(1)}{\log^{[q+1]} \mu (r, g)}.
\]

So from above we obtain that

\[
\liminf_{r \to \infty} \frac{\log^{[p+\frac{n}{2}(p+1-q)+1]} \mu (r, f_n)}{\log^{[q+1]} \mu (r, g)} \leq 1.
\]

Thus the second part of the corollary follows.

In the line of Theorem 3.3.1, we may state the following theorem without its proof:

**Theorem 3.3.2** Let \( f \) be entire and \( g \) be entire such that \( \rho_g (a, b) \) and \( \rho_f^{[l]} \) are both finite where \( a, b, l \) are positive integers with \( a \geq b \) and \( l \geq 2 \). Then for any \( \beta > 1 \),

\[
(i) \quad \liminf_{r \to \infty} \frac{\log^{[l+\frac{n}{2}(a-1)+\frac{n}{2}(l-b-2)+1]} \mu (r, f_n)}{\log^{[l-1]} \mu (r, f)} \leq [2\beta]^{\gamma} \text{ if } a > 1 \text{ and } b < l - 1,
\]

\[
(ii) \quad \liminf_{r \to \infty} \frac{\log^{[l+\frac{n}{2}(l-b)+1]} \mu (r, f_n)}{\log^{[l-1]} \mu (r, f)} \leq [2\beta]^{\gamma} \text{ if } a = 1 \text{ and } b < l - 1,
\]

\[
(iii) \quad \liminf_{r \to \infty} \frac{\log^{[n(l-1)-b+1]} \mu (r, f_n)}{\log^{[l-1]} \mu (r, f)} \leq [2\beta]^{\gamma} \text{ if } l = a > 1 \text{ and } l - 1 > b = 1,
\]

\[
(iv) \quad \liminf_{r \to \infty} \frac{\log^{[l+a-1]} \mu (r, f_n)}{\log^{[l-1]} \mu (r, f)} \leq \rho_g (a, b). [2\beta]^{\gamma} \text{ if } l = b > 2 \text{ and } a > 1,
\]

\[
(v) \quad \liminf_{r \to \infty} \frac{\log^{[l+a-1]+2]} \mu (r, f_n)}{\log^{[l-1]} \mu (r, f)} \leq 3. \rho_g (a, b). [2\beta]^{\gamma} \text{ if } a = b = l = 2,
\]

\[
(vi) \quad \liminf_{r \to \infty} \frac{\log^{[l+a-1]} \mu (r, f_n)}{\log^{[l-1]} \mu (r, f)} \leq [\rho_g (a, b)]^{\frac{n-1}{2}}. [2\beta]^{\gamma} \text{ if } a > 1, \quad b > l \text{ and } b - l = \gamma - 1.
\]
\[(vii) \liminf_{r \to \infty} \frac{\log^{[l+a+\frac{a-b}{2}(a-b)-1]} \mu (r, f_n)}{\log^{[l-1]} \mu (r, f)} \leq \rho_g (a, b). [2\beta]_{r}^{[l]}
\]
if \(a > 1, \ b > l > 2\) and \(b - l < a - 1\)

and

\[(viii) \liminf_{r \to \infty} \frac{\log^{[a+\frac{a}{2}(a-b)+1]} \mu (r, f_n)}{\log \mu (r, f)} \leq 3. \rho_g (a, b). [2\beta]^{[r]}
\]
if \(a > 1, \ b > l = 2\) and \(b - l < a - 1\)

when \(n\) is odd and \(n \neq 1\).

**Corollary 3.3.2**  Under the same conditions of Theorem 3.3.2, if \(l = 2\) then

\[(i) \liminf_{r \to \infty} \frac{\log^{[\frac{n}{2}+2]} \mu (r, f_n)}{\log \mu (r, f)} \leq 1\] if \(a = b = l\)

and

\[(ii) \liminf_{r \to \infty} \frac{\log^{[a+\frac{a}{2}(a-b)+2]} \mu (r, f_n)}{\log^{[b+1]} \mu (r, f)} \leq 1\] if \(a > 1, \ b > l\) and \(b - l < a - 1\).

The proof of the Corollary 3.3.2 is omitted as it can be carried out in the line of Theorem 3.3.2 and Corollary 3.3.1.

**Theorem 3.3.3**  If \(f\) be entire and \(g\) be entire such that \(\rho_f (p, q)\) and \(\rho_g^{[l]}\) are both finite where \(p, q, l\) are positive integers with \(p \geq q\) and \(l \geq 2\). Then for any even number \(n\),

\[(i) \liminf_{r \to \infty} \frac{\log^{[\frac{p}{2}(p-1)+1]} \mu (r, f_n)}{\log^{[l-1]} \mu (r, g)} \leq \rho_f (p, q). [2\beta]_{p}^{[l]}
\]
if \(p > 1\) and \(q = l > 2\),

\[(ii) \liminf_{r \to \infty} \frac{\log^{[\frac{q}{2}(q-1)+1]} \mu (r, f_n)}{\log \mu (r, g)} \leq 3. \rho_f (p, q). [2\beta]^{[q]}
\]
if \(p > 1\) and \(q = l = 2\),

\[(iii) \liminf_{r \to \infty} \frac{\log^{[p]} \mu (r, f_n)}{\log^{[l-1]} \mu (r, g)} \leq [\rho_f (p, q)]^{\frac{p}{2}} \cdot [2\beta]^{[q]}
\]
if \(p > 1, \ q > l > 2\) and \(q - l = p - 1\),
\[(iv) \liminf_{r \to \infty} \frac{\log \left[p + \frac{n}{2} (p+1-q)\right]}{\log \mu(r, g)} \leq \rho_f(p, q) \cdot [2\beta]^{r^g}
\]

if \(p > 1, q > l > 2\) and \(q - l < p - 1\),

\[(v) \liminf_{r \to \infty} \frac{\log \left[p + \frac{n}{2} (p+1-q)\right]}{\log \mu(r, g)} \leq 3 \rho_f(p, q) \cdot [2\beta]^{r^g}
\]

if \(p > 1, q = l = 2\) and \(q - l < p - 1\),

\[(vi) \liminf_{r \to \infty} \frac{\log \left[p \cdot (l-1) + \frac{n}{2} (p-1)\right]}{\log \mu(r, g)} \leq [2\beta]^{r^g}
\]

if \(q < l - 1\) and \(1 < p\),

\[(vii) \liminf_{r \to \infty} \frac{\log [p \cdot (l-1)]}{\log \mu(r, g)} \leq [2\beta]^{r^g}
\]

if \(p = q = 1, l - 1 > q\) and \(n > 2\)

and

\[(viii) \liminf_{r \to \infty} \frac{\log [n \cdot (l-1)]}{\log \mu(r, g)} \leq [2\beta]^{r^g}
\]

if \(p = l\) and \(l - 1 > q = 1\).

**Proof. Case I.** Let \(l > 2\). As \(\limsup_{r \to \infty} \frac{\log^{[l-2]} T(r, f)}{\rho_f^g(r)} = 1\), for given \(\varepsilon (0 < \varepsilon < 1)\) we obtain for all sufficiently large values of \(r\) that

\[\log^{[l-2]} T(r, g) < (1 + \varepsilon) \rho_f^g(r)\]

and for a sequence of values of \(r\) tending to infinity,

\[\log^{[l-2]} T(r, g) > (1 - \varepsilon) \rho_f^g(r)\].
Since \( \log M(r, g) \leq 3T(2r, g) \), for a sequence of values of \( r \) tending to infinity we get for any \( \delta > 0 \) that

\[
\frac{\log^{[l-1]} M(r, g)}{\log^{[l-2]} T(\frac{r}{\beta}, g)} \leq \frac{\log^{[l-2]} T(2r, g) + O(1)}{\log^{[l-2]} T(\frac{r}{\beta}, g)}
\]

\[
\leq \frac{(1 + \varepsilon)}{(1 - \varepsilon)} \cdot \frac{(2r)^{\rho^g + \delta}}{(2r)^{\rho^g + \delta - \rho^g(2r)}} \cdot \frac{1}{\left(\frac{r}{\beta}\right)^{\rho^g(\frac{r}{\beta})}} + O(1)
\]

\[
\leq \frac{(1 + \varepsilon)}{(1 - \varepsilon)} \cdot \left(\frac{2\beta}{\beta}\right)^{\rho^g + \delta} \frac{1}{\left(\frac{r}{\beta}\right)^{\rho^g(\frac{r}{\beta})}} + O(1)
\]

\[
\leq \frac{(1 + \varepsilon)}{(1 - \varepsilon)} \cdot (2\beta)^{\rho^g + \delta}
\]

because \( r^{\rho^g + \delta - \rho^g(r)} \) is an increasing function of \( r \) by Lemma 3.2.4.

Since \( \varepsilon > 0 \) and \( \delta > 0 \) are both arbitrary, we get from above that

\[
\liminf_{r \to \infty} \frac{\log^{[l-1]} M(r, g)}{\log^{[l-2]} T(\frac{r}{\beta}, g)} \leq (2\beta)^{\rho^g}.
\]

(3.19)

**Case II.** Let \( l = 2 \).

Since \( \limsup_{r \to \infty} \frac{T(r, g)}{r^{\rho^g(r)}} = 1 \), in view of condition (v) of Definition 2.3.6 it follows for all sufficiently large values of \( r \) and for a given \( \varepsilon > 0 \) that

\[
T(r, g) < (1 + \varepsilon) r^{\rho^g(r)}
\]

and for a sequence values of \( r \) tending to infinity

\[
T(r, g) > (1 - \varepsilon) r^{\rho^g(r)}.
\]

As \( \log M(r, g) \leq 3T(2r, g) \), for a sequence of values of \( r \) tending to infinity we get for any \( \delta > 0 \) that

\[
\frac{\log M(r, g)}{T(\frac{r}{\beta}, g)} \leq \frac{3(1 + \varepsilon)}{(1 - \varepsilon)} \cdot \frac{(2r)^{\rho^g + \delta}}{(2r)^{\rho^g + \delta - \rho^g(2r)}} \cdot \frac{1}{\left(\frac{r}{\beta}\right)^{\rho^g(\frac{r}{\beta})}} + O(1)
\]

\[
\text{i.e.,} \quad \frac{\log M(r, g)}{T(\frac{r}{\beta}, g)} \leq \frac{3(1 + \varepsilon)}{(1 - \varepsilon)} \cdot (2\beta)^{\rho^g + \delta}
\]

because \( r^{\rho^g + \delta - \rho^g(r)} \) is an increasing function of \( r \) by Lemma 3.2.4.

Since \( \varepsilon > 0 \) and \( \delta > 0 \) are both arbitrary, we get from above that

\[
\liminf_{r \to \infty} \frac{\log M(r, g)}{T(\frac{r}{\beta}, g)} \leq 3 \cdot (2\beta)^{\rho^g}.
\]

(3.20)
Therefore from (3.10) and (3.19), it follows that
\[
\liminf_{r \to \infty} \frac{\log \left[ \frac{2^{(p-1)+1}}{n^2} \right] \mu(r, f_n)}{\log \mu(r, g)} \leq \rho_f(p, q) \cdot [2\beta]^{\rho_f[\beta]}
\]
This proves the first part of the theorem.
Again for \( l = 2 \), we obtain in view of (3.10) and (3.20) that
\[
\liminf_{r \to \infty} \frac{\log \left[ \frac{2^{(p-1)+1}}{n^2} \right] \mu(r, f_n)}{\log \mu(r, g)} \leq 3, \rho_f(p, q) \cdot [2\beta]^{\rho_f[\beta]}
\]
Thus the second part of the theorem is established.
Similarly, from (3.13) and (3.19), we get that
\[
\liminf_{r \to \infty} \frac{\log \left[ \frac{2^{(l-q)+2}}{n^2} \right] \mu(r, f_n)}{\log \mu(r, g)} \leq [2\beta]^{\rho_f[\beta]}
\]
Thus the seventh part of the theorem follows.

**Theorem 3.3.4** If \( f \) be entire and \( g \) be entire such that \( \rho_g(a, b) \) and \( \rho_f^n \) are both finite where \( a, b, l \) are positive integers with \( a \geq b \) and \( l \geq 2 \). Then

(i) \[
\liminf_{r \to \infty} \frac{\log \left[ \frac{n^{(a-1)+1}}{n^2} + \frac{n^{(l-b)-1}}{n^2} \right] \mu(r, f_n)}{\log \mu(r, f)} \leq [2\beta]^{\rho_f[\beta]} \quad \text{if } a > 1 \text{ and } b < l - 1,
\]

(ii) \[
\liminf_{r \to \infty} \frac{\log \left[ \frac{n^{(l-b)-1}}{n^2} \right] \mu(r, f_n)}{\log \mu(r, f)} \leq [2\beta]^{\rho_f[\beta]} \quad \text{if } a = 1 \text{ and } b < l - 1,
\]

(iii) \[
\liminf_{r \to \infty} \frac{\log \left[ \frac{n^{(l-1)+1}}{n^2} \right] \mu(r, f_n)}{\log \mu(r, f)} \leq [2\beta]^{\rho_f[\beta]} \quad \text{if } l = a > 1 \text{ and } l - 1 > b = 1,
\]

(iv) \[
\liminf_{r \to \infty} \frac{\log \left[ \frac{n^{(a-1)}}{n^2} \right] \mu(r, f_n)}{\log \mu(r, f)} \leq \rho_g(a, b) \cdot [2\beta]^{\rho_f[\beta]} \quad \text{if } l = b > 2 \text{ and } a > 1,
\]

(v) \[
\liminf_{r \to \infty} \frac{\log \left[ \frac{n^{a+2}}{n^2} \right] \mu(r, f_n)}{\log \mu(r, f)} \leq 3, \rho_g(2, 2) \cdot [2\beta]^{\rho_f} \quad \text{if } a = b = l = 2,
\]
\( \liminf_{r \to \infty} \frac{\log^{[a+1]} \mu(r, f_n)}{\log^{[b]} \mu(r, f)} \leq \frac{[\rho_{\gamma}(a, b)]_{[\frac{b}{a}]}^{[b]}}{[\rho_{\gamma}(a, b)]_{[\frac{b}{a}]}^{[b]}} \) if \( a > 1, \ b > 1 \) and \( b - l = a - 1 \),

\( \liminf_{r \to \infty} \frac{\log^{[a+3]} \mu(r, f_n)}{\log^{[b]} \mu(r, f)} \leq \rho_{\gamma}(a, b) \) if \( a > 1, \ b > 1 \) and \( b - l < a - 1 \)

and

\( \liminf_{r \to \infty} \frac{\log^{[a+3]} \mu(r, f_n)}{\log^{[b]} \mu(r, f)} \leq 3 \rho_{\gamma}(a, b) \) if \( a = b = p = q \) or \( p > b; q > a \) and \( q - a = p - b \),

when \( n \) is odd and \( n \neq 1 \).

The proof of Theorem 3.3.4 is omitted as it can be carried out in the line of Theorem 3.3.3.

**Theorem 3.3.5** Let \( f \) and \( g \) be any two entire functions such that \( \rho_{\gamma}(a, b) < \lambda_{\gamma}(p, q) \leq \rho_{\gamma}(p, q) < \infty \) where \( a, b, p, q \) are positive integers with \( a \geq b \) and \( p \geq q \). Then

\( (i) \lim_{r \to \infty} \frac{\log^{[p-1]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} = 0 \) if \( p > b \) and \( q = a \),

\( (ii) \lim_{r \to \infty} \frac{\log^{[p]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p]} \mu(\exp^{[q-1]} r, f)} = 0 \) if \( a = b = p = q \)

or \( p > b, q > a \) and \( q - a = p - b \),

\( (iii) \lim_{r \to \infty} \frac{\log^{[p]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p]} \mu(\exp^{[q-1]} r, f)} = 0 \) if \( p > b, q > a \) and \( q - a < p - b \),
Now for all sufficiently large values of $r$,

$$(iv) \lim_{r \to \infty} \frac{\log^{[p+\frac{n-2}{2}(a-q)+\frac{n-2}{2}(p-b)+a-q-1]} \mu \left( \exp^{[b-1]} r, f_n \right)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} = 0$$

if $q < a$ and $b < p$,

$$(v) \lim_{r \to \infty} \frac{\log^{[p+(n-2)(a-q)+a-q-1]} \mu \left( \exp^{[b-1]} r, f_n \right)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} = 0$$

if $p = b \geq q$, $a > q$ and $n > 2$,

$$(vi) \lim_{r \to \infty} \frac{\log^{[p+2a-2q-1]} \mu \left( \exp^{[b-1]} r, f_n \right)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} = 0$$

if $p < b$, $q < a$ and $b - p = a - q$

and

$$(vii) \lim_{r \to \infty} \frac{\log^{[p-q+(a+b)+\frac{3}{2}(a-b)-(q+2)+1]} \mu \left( \exp^{[b-1]} r, f_n \right)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} = 0$$

if $p < b$, $q < a$ and $b - p < a - q$

where $n$ is any even number.

**Proof.** Since $\rho_g(a, b) < \lambda_f(p, q)$, we can choose $\varepsilon (> 0)$ is such a way that

$$\rho_g(a, b) + \varepsilon < \lambda_f(p, q) - \varepsilon. \quad (3.21)$$

Now for all sufficiently large values of $r$,

$$\log^{[a]} M \left( \exp^{[b-1]} r, g \right) \leq (\rho_g(a, b) + \varepsilon) \log^{[b]} \exp^{[b-1]} r$$

i.e.,

$$\log^{[a]} M \left( \exp^{[b-1]} r, g \right) \leq (\rho_g(a, b) + \varepsilon) \log r \quad (3.22)$$

i.e.,

$$\log^{[a]} M \left( \exp^{[b-1]} r, g \right) \leq \log r^{\rho_g(a, b) + \varepsilon}. \quad (3.23)$$

Again for all sufficiently large values of $r$, we obtain that

$$\log^{[p]} \mu(\exp^{[q-1]} r, f) \geq (\lambda_f(p, q) - \varepsilon) \log^{[q]} \exp^{[q-1]} r$$

i.e.,

$$\log^{[p]} \mu(\exp^{[q-1]} r, f) \geq (\lambda_f(p, q) - \varepsilon) \log r \quad (3.24)$$

i.e.,

$$\log^{[p]} \mu(\exp^{[q-1]} r, f) \geq \log r^{\lambda_f(p, q) - \varepsilon} \quad (3.25)$$
Now the following cases may arise:

**Case I.** Let \( p > b \) and \( q = a \).

Then we have from the third part of Lemma 3.2.2 for all sufficiently large values of \( r \) that

\[
\log^{(p-1)+\frac{n-2}{2}(p-b)+1} \mu \left( \exp^{[b-1]} r, f_n \right) \leq (\rho_f (p, q) + \varepsilon) \log^{[a-1]} M \left( \exp^{[b-1]} r, g \right).
\] (3.26)

Now from \((3.23)\) and \((3.26)\), we have for all sufficiently large values of \( r \) that

\[
\log^{(p-1)+\frac{n-2}{2}(p-b)+1} \mu \left( \exp^{[b-1]} r, f_n \right) \leq (\rho_f (p, q) + \varepsilon) r^{(\rho_g(a,b)+\varepsilon)}.
\] (3.27)

**Case II.** Let \( a = b = p = q \).

Now we obtain from the fourth part of Lemma 3.2.2 for all sufficiently large values of \( r \) that

\[
\log^p \mu \left( \exp^{[b-1]} r, f_n \right) \leq (\rho_f (p, q) + \varepsilon)^\frac{n}{2} (\rho_g(a,b) + \varepsilon)^\frac{n-2}{2} \log^{[a-1]} M \left( \exp^{[b-1]} r, g \right).
\] (3.28)

Then from \((3.23)\) and \((3.28)\), we obtain for all sufficiently large values of \( r \) that

\[
\log^p \mu \left( \exp^{[b-1]} r, f_n \right) \leq (\rho_f (p, q) + \varepsilon)^\frac{n}{2} (\rho_g(a,b) + \varepsilon)^\frac{n-2}{2} r^{(\rho_g(a,b)+\varepsilon)}.
\] (3.29)

**Case III.** Let \( p > b, q > a \) and \( q - a = p - b \).

Then we get from Lemma 3.2.2 \((viii)\) for all sufficiently large values of \( r \) that

\[
\log^p \mu \left( \exp^{[b-1]} r, f_n \right) \leq (\rho_f (p, q) + \varepsilon)^\frac{n}{2} \log^{[a-1]} M \left( \exp^{[b-1]} r, g \right).
\] (3.30)

Now from \((3.23)\) and \((3.30)\), we get for all sufficiently large values of \( r \) that

\[
\log^p \mu \left( \exp^{[b-1]} r, f_n \right) \leq (\rho_f (p, q) + \varepsilon)^\frac{n}{2} r^{(\rho_g(a,b)+\varepsilon)}.
\] (3.31)

**Case IV.** Let \( p > b, q > a \) and \( q - a < p - b \).

Now we obtain from Lemma 3.2.2 \((ix)\) for all sufficiently large values of \( r \) that

\[
\log^{p+\frac{n-2}{2}(p+a-b-q)} \mu \left( \exp^{[b-1]} r, f_n \right) \leq (\rho_f (p, q) + \varepsilon) \log^{[a-1]} M \left( \exp^{[b-1]} r, g \right).
\] (3.32)

Then from \((3.23)\) and \((3.32)\), we have for all sufficiently large values of \( r \) that

\[
\log^{p+\frac{n-2}{2}(p+a-b-q)} \mu \left( \exp^{[b-1]} r, f_n \right) \leq (\rho_f (p, q) + \varepsilon) r^{(\rho_g(a,b)+\varepsilon)}.
\] (3.33)
Further from (3.22), it follows for all sufficiently large values of $r$ that
\[
\exp^{[a-q]} \log^{[a]} \mu \left( \exp^{[b-1]} r, g \right) \leq \exp^{[a-q]} \log r_{\rho_2(a,b)+\varepsilon}.
\]
\[i.e., \quad \exp^{[a-q]} \log^{[a]} \mu \left( \exp^{[b-1]} r, g \right) \leq \exp^{[a-q-1]} r_{\rho_2(a,b)+\varepsilon}.
\] (3.34)

**Case V.** Let $q < a$ and $b < p$.
Then for all sufficiently large values of $r$, we get in view of the first part of Lemma 3.2.2 that
\[
\log^{[(p-1)+\frac{a-q}{2}(a-q)+\frac{a-q}{2}(p-b)+1]} \mu \left( \exp^{[b-1]} r, f_n \right) \leq (\rho_f (p, q) + \varepsilon) \exp^{[a-q-1]} r_{\rho_2(a,b)+\varepsilon}.
\] (3.35)

Now from (3.34) and (3.35), we have for all sufficiently large values of $r$ that
\[
\log^{[(p+\frac{a-q}{2}(a-q)+\frac{a-q}{2}(p-b)]} \mu \left( \exp^{[b-1]} r, f_n \right) \leq (\rho_f (p, q) + \varepsilon) \exp^{[a-q-2]} r_{\rho_2(a,b)+\varepsilon}
\]
\[i.e., \quad \log^{[(p+\frac{a-q}{2}(a-q)+\frac{a-q}{2}(p-b)+a-q-1)]} \mu \left( \exp^{[b-1]} r, f_n \right) \leq \exp^{[a-q-2]} r_{\rho_2(a,b)+\varepsilon}.
\] (3.36)

**Case VI.** Let $p = b \geq q$, $a > q$ and $n > 2$.
Now for all sufficiently large values of $r$, we get in view of the second part of Lemma 3.2.2 that
\[
\log^{[p+\frac{a-q}{2}(a-q)]} \mu \left( \exp^{[b-1]} r, f_n \right) \leq (\rho_f (p, q) + \varepsilon) (\rho_g (a, b) + \varepsilon) \exp^{[a-q]} \log^{[a]} M \left( \exp^{[b-1]} r, g \right).
\] (3.37)

Then from (3.34) and (3.37), we have for all sufficiently large values of $r$ that
\[
\log^{[p+\frac{a-q}{2}(a-q)]} \mu \left( \exp^{[b-1]} r, f_n \right) \leq (\rho_f (p, q) + \varepsilon) (\rho_g (a, b) + \varepsilon) \exp^{[a-q-1]} r_{\rho_2(a,b)+\varepsilon}
\]
\[i.e., \quad \log^{[p+\frac{a-q}{2}(a-q)+1]} \mu \left( \exp^{[b-1]} r, f_n \right) \leq \exp^{[a-q-2]} r_{\rho_2(a,b)+\varepsilon}
\]
\[i.e., \quad \log^{[p+\frac{a-q}{2}(a-q)+a-q-1]} \mu \left( \exp^{[b-1]} r, f_n \right) \leq \log^{[a-q-2]} \exp^{[a-q-2]} r_{\rho_2(a,b)+\varepsilon}
\]
\[i.e., \quad \log^{[p+\frac{a-q}{2}(a-q)+a-q-1]} \mu \left( \exp^{[b-1]} r, f_n \right) \leq r_{\rho_2(a,b)+\varepsilon}.
\] (3.38)
Case VII. Let \( p = a > q = b \).
Then for all sufficiently large values of \( r \), we get in view of the fifth part of Lemma 3.2.2 that
\[
\log^{[p+(n-2)(p-q)]} \mu \left( \exp^{[b-1]} r, f_n \right) \leq (\rho_f (p, q) + \varepsilon) \exp^{[a-q]} \log^{[a]} M \left( \exp^{[b-1]} r, g \right).
\] (3.39)

Now from (3.34) and (3.39), we have for all sufficiently large values of \( r \) that
\[
\log^{[p+(n-2)(p-q)]} \mu \left( \exp^{[b-1]} r, f_n \right) \leq (\rho_f (p, q) + \varepsilon) \exp^{[a-q]} \rho_q (a, b) + \varepsilon
\]

i.e.,
\[
\log^{[p+(n-2)(p-q)+1]} \mu \left( \exp^{[b-1]} r, f_n \right) \leq \exp^{[a-q-2]} \rho_q (a, b) + \varepsilon
\]
i.e.,
\[
\log^{[p+(n-2)(p-q)+q-1]} \mu \left( \exp^{[b-1]} r, f_n \right) \leq \log^{[a-q-2]} \exp^{[a-q-2]} \rho_q (a, b) + \varepsilon
\]
i.e.,
\[
\log^{[p+(n-2)(p-q)+q-1]} \mu \left( \exp^{[b-1]} r, f_n \right) \leq \rho_q (a, b) + \varepsilon.
\] (3.40)

Case VIII. Let \( p < b, q < a \) and \( b - p = a - q \).
Now for all sufficiently large values of \( r \), we get in view of the sixth part of Lemma 3.2.2 that
\[
\log^{[p+a-q]} \mu \left( \exp^{[b-1]} r, f_n \right) \leq (\rho_g (a, b) + \varepsilon) \frac{n-2}{n} \exp^{[a-q]} \log^{[a]} M \left( \exp^{[b-1]} r, g \right).
\] (3.41)

Then from (3.34) and (3.30), we have for all sufficiently large values of \( r \) that
\[
\log^{[p+a-q]} \mu \left( \exp^{[b-1]} r, f_n \right) \leq (\rho_g (a, b) + \varepsilon) \frac{n-2}{n} \exp^{[a-q-2]} \rho_q (a, b) + \varepsilon
\]
i.e.,
\[
\log^{[p+a-q+1]} \mu \left( \exp^{[b-1]} r, f_n \right) \leq \exp^{[a-q-2]} \rho_q (a, b) + \varepsilon
\]
i.e.,
\[
\log^{[p+a-q+a-q-1]} \mu \left( \exp^{[b-1]} r, f_n \right) \leq \log^{[a-q-2]} \exp^{[a-q-2]} \rho_q (a, b) + \varepsilon
\]
i.e.,
\[
\log^{[p+2a-2q-1]} \mu \left( \exp^{[b-1]} r, f_n \right) \leq \rho_q (a, b) + \varepsilon.
\] (3.42)

Case IX. Let \( p < b, q < a \) and \( b - p < a - q \).
Then for all sufficiently large values of \( r \), we get in view of the seventh part of Lemma 3.2.2 that
\[
\log^{[p+a+\frac{a-2}{a}(a-b)-q]} \mu \left( \exp^{[b-1]} r, f_n \right) \leq \exp^{[a-q]} \log^{[a]} M \left( \exp^{[b-1]} r, g \right).
\] (3.43)

Now from (3.34) and (3.43), we have for all sufficiently large values of \( r \) that
\[
\log^{[p+a+\frac{a-2}{a}(a-b)-q]} \mu \left( \exp^{[b-1]} r, f_n \right) \leq \exp^{[a-q-1]} \rho_q (a, b) + \varepsilon
\]
Similarly combining \( \log \left[ \frac{a + \frac{a}{2} - q + 1}{a} \right] \), we get for all sufficiently large values of \( r \) that

\[
\log \left[ \frac{a + \frac{a}{2} - q + 1}{a} \right] \mu \left( \exp^{\lceil b \rceil} r, f_n \right) \leq \frac{(\rho_f (p, q) + \varepsilon) \left( \rho_g (a, b) + \varepsilon \right) \left( \frac{a}{2} \right) r^{\rho_g (a, b) + \varepsilon}}{r^{(\lambda_f (p, q) - \varepsilon)}}.
\]  

Now in view of \((3.21)\) it follows from \((3.45)\) that

\[
\lim_{r \to \infty} \sup_n \left( \frac{\log \left[ \frac{a + \frac{a}{2} - q + 1}{a} \right] \mu \left( \exp^{\lceil b \rceil} r, f_n \right)}{\log \left[ \frac{a + \frac{a}{2} - q + 1}{a} \right] \mu \left( \exp^{\lceil q \rceil} r, f \right)} \right) = 0.
\]

Similarly combining \((3.21)\), \((3.29)\) of Case II and \((3.25)\), we obtain for all sufficiently large values of \( r \) that

\[
\lim_{r \to \infty} \sup_n \left( \frac{\log \left[ \frac{a + \frac{a}{2} - q + 1}{a} \right] \mu \left( \exp^{\lceil b \rceil} r, f_n \right)}{\log \left[ \frac{a + \frac{a}{2} - q + 1}{a} \right] \mu \left( \exp^{\lceil q \rceil} r, f \right)} \right) = 0.
\]  

Thus the second part of the theorem follows from \((3.46)\) and above. Again combining \((3.33)\) of Case IV and \((3.25)\), we get for all sufficiently large values of \( r \) that

\[
\frac{\log \left[ \frac{a + \frac{a}{2} - q + 1}{a} \right] \mu \left( \exp^{\lceil b \rceil} r, f_n \right)}{\log \left[ \frac{a + \frac{a}{2} - q + 1}{a} \right] \mu \left( \exp^{\lceil q \rceil} r, f \right)} \leq \frac{(\rho_f (p, q) + \varepsilon) \left( \rho_g (a, b) + \varepsilon \right) \left( \frac{a}{2} \right) r^{\rho_g (a, b) + \varepsilon}}{r^{(\lambda_f (p, q) - \varepsilon)}}.
\]  

Now combining \((3.27)\) of Case I and \((3.25)\), we get for all sufficiently large values of \( r \) that

\[
\frac{\log \left[ \frac{a + \frac{a}{2} - q + 1}{a} \right] \mu \left( \exp^{\lceil b \rceil} r, f_n \right)}{\log \left[ \frac{a + \frac{a}{2} - q + 1}{a} \right] \mu \left( \exp^{\lceil q \rceil} r, f \right)} \leq \frac{(\rho_f (p, q) + \varepsilon) \left( \rho_g (a, b) + \varepsilon \right) \left( \frac{a}{2} \right) r^{\rho_g (a, b) + \varepsilon}}{r^{(\lambda_f (p, q) - \varepsilon)}}.
\]  

Now in view of \((3.21)\), it follows from \((3.45)\) that

\[
\lim_{r \to \infty} \sup_n \left( \frac{\log \left[ \frac{a + \frac{a}{2} - q + 1}{a} \right] \mu \left( \exp^{\lceil b \rceil} r, f_n \right)}{\log \left[ \frac{a + \frac{a}{2} - q + 1}{a} \right] \mu \left( \exp^{\lceil q \rceil} r, f \right)} \right) = 0.
\]  

Thus the second part of the theorem follows from \((3.46)\) and above.
Therefore in view of (3.21), we get from (3.47) that

\[ \lim_{r \to \infty} \frac{\log \left[ p + \frac{a}{2} (a-q) + \frac{b}{2} (p-b) + a - q - 1 \right] \mu \left( \exp^{[b-1]} r, f_n \right)}{\log \left[ \exp^{[q-1]} r, f \right]} = 0. \]

This proves the third part of the theorem.

Similarly combining (3.36) of Case V and (3.25), we obtain in view of (3.21) for all sufficiently large values of \( r \) that

\[ \lim_{r \to \infty} \frac{\log \left[ p + \frac{a}{2} (a-q) + \frac{b}{2} (p-b) + a - q - 1 \right] \mu \left( \exp^{[b-1]} r, f_n \right)}{\log \left[ \exp^{[q-1]} r, f \right]} \leq \frac{\rho_p(a,b)+\varepsilon}{\rho_f(p,q)-\varepsilon}. \]

i.e.,

\[ \lim_{r \to \infty} \frac{\log \left[ p + \frac{a}{2} (a-q) + \frac{b}{2} (p-b) + a - q - 1 \right] \mu \left( \exp^{[b-1]} r, f_n \right)}{\log \left[ \exp^{[q-1]} r, f \right]} = 0. \]

This establishes the fourth part of the theorem.

Analogously, in view of (3.21), (3.38) of Case VI and (3.25) it follows for all sufficiently large values of \( r \) that

\[ \lim_{r \to \infty} \frac{\log \left[ p + \frac{a}{2} (a-q) + a - q - 1 \right] \mu \left( \exp^{[b-1]} r, f_n \right)}{\log \left[ \exp^{[q-1]} r, f \right]} \leq \frac{\rho_p(a,b)+\varepsilon}{\rho_f(p,q)-\varepsilon}. \]

i.e.,

\[ \lim_{r \to \infty} \frac{\log \left[ p + \frac{a}{2} (a-q) + a - q - 1 \right] \mu \left( \exp^{[b-1]} r, f_n \right)}{\log \left[ \exp^{[q-1]} r, f \right]} = 0. \]

Thus the fifth part of the theorem follows from above.

Again combining (3.21), (3.40) of Case VII and (3.25) we obtain for all sufficiently large values of \( r \) that

\[ \limsup_{r \to \infty} \frac{\log \left[ p + (n-2)(p-q) + a - q - 1 \right] \mu \left( \exp^{[b-1]} r, f_n \right)}{\log \left[ \exp^{[q-1]} r, f \right]} \leq \frac{\rho_p(a,b)+\varepsilon}{\rho_f(p,q)-\varepsilon}. \]

i.e.,

\[ \lim_{r \to \infty} \frac{\log \left[ p + (n-2)(p-q) + a - q - 1 \right] \mu \left( \exp^{[b-1]} r, f_n \right)}{\log \left[ \exp^{[q-1]} r, f \right]} = 0. \]

whic is the sixth part of the theorem.

Further in view of (3.21), (3.42) of Case VIII and (3.25) we get for all sufficiently large values of \( r \) that

\[ \lim_{r \to \infty} \frac{\log \left[ p + 2a - 2q - 1 \right] \mu \left( \exp^{[b-1]} r, f_n \right)}{\log \left[ \exp^{[q-1]} r, f \right]} \leq \frac{\rho_p(a,b)+\varepsilon}{\rho_f(p,q)-\varepsilon}. \]

i.e.,

\[ \lim_{r \to \infty} \frac{\log \left[ p + 2a - 2q - 1 \right] \mu \left( \exp^{[b-1]} r, f_n \right)}{\log \left[ \exp^{[q-1]} r, f \right]} = 0. \]
This proves the seventh part of the theorem.

Similarly, combining (3.21), (3.44) of Case IX and (3.25) it follows for all sufficiently large values of \( r \) that

\[
\frac{\log^{[p+a+\frac{n-2}{2}(a-b)-q+a-q-1]} \mu \left( \exp^{[b-1]} r, f_n \right)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} \leq \frac{r^{\rho_b(a,b)+\varepsilon}}{r^{(\lambda_f(p,q)-\varepsilon)}}
\]

\( i.e., \),

\[
\lim_{r \to \infty} \frac{\log^{[p-q+(a+b)+\frac{2}{q}(a-b)-(q+1)]} \mu \left( \exp^{[b-1]} r, f_n \right)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} = 0.
\]

This establishes the eighth part of the theorem. \( \blacksquare \)

**Remark 3.3.1** The condition \( \rho_g(a,b) < \lambda_f(p,q) \) in Theorem 3.3.5 is essential as we see in the following example.

**Example 3.3.1** Let \( f = g = \exp z \) and \( p = a = n = 2, q = b = 1 \).

Then \( \rho_g(a,b) = \lambda_f(p,q) = \rho_f(p,q) = 1 \).

Now

\[
\log \mu(r, f \circ g) \geq \log \mu \left( \frac{r}{2}, f \circ g \right) + O(1) \geq T \left( \frac{r}{2}, f \circ g \right) + O(1)
\]

\[
= T \left( \frac{r}{2}, \exp^{[2]} z \right) + O(1) \sim \frac{\exp \left( \frac{r}{2} \right)}{(2\pi^3 r)^{\frac{1}{2}}} + O(1) \quad (r \to \infty)
\]

\( i.e., \)

\[
\log^{[2]} \mu(r, f \circ g) \geq \frac{r}{2} - \frac{1}{2} \log r + O(1)
\]

and \( \log \mu(r, f) \leq \log M(r, f) = \log M(r, \exp z) = r \).

Then

\[
\lim_{r \to \infty} \frac{\log^{[p+m-q-1]} \mu \left( \exp^{[n-1]} r, f \circ g \right)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)}
\]

\[
= \lim_{r \to \infty} \frac{\log^{[2]} \mu(r, f \circ g)}{\log \mu(r, f)}
\]

\[
\geq \lim_{r \to \infty} \frac{\frac{r}{2} - \frac{1}{2} \log r + O(1)}{r}
\]

\[
= 1 \neq 0,
\]

which is contrary to Theorem 3.3.5.

**Remark 3.3.2** Theorem 3.3.5 is still valid with "limit inferior" instead of "limit" if we replace the condition "\( \rho_g(a,b) < \lambda_f(p,q) \leq \rho_f(p,q) < \infty \)" by "\( \lambda_g(a,b) < \lambda_f(p,q) \leq \rho_f(p,q) < \infty \)."
Remark 3.3.3 Considering \( f = g = \exp z \) and \( p = a = n = 2, \ q = b = 1 \), one can easily verify that the condition \( \lambda_g(a, b) < \lambda_f(p, q) \) in Remark 3.3.2 is essential.

In the line of Theorem 3.3.5, we may state the following theorem without its proof:

**Theorem 3.3.6** Let \( f \) and \( g \) be any two entire functions such that \( \rho_f(p, q) < \lambda_g(a, b) \leq \rho_g(a, b) < \infty \) where \( a, b, p, q \) are positive integers with \( a \geq b \) and \( p \geq q \). Then

\[
\begin{align*}
(i) & \lim_{r \to \infty} \frac{\log^{(p-1)+\frac{a-2}{2}(p-b)+1} \mu(\exp^{[q-1]} r, f_n)}{\log^{[a-1]} \mu(\exp^{[b-1]} r, g)} = 0 \quad \text{if } p > b \text{ and } q = a, \\
(ii) & \lim_{r \to \infty} \frac{\log[p] \mu(\exp^{[q-1]} r, f_n)}{\log^{[a-1]} \mu(\exp^{[b-1]} r, g)} = 0 \quad \text{if } a = b = p = q \\
& \quad \text{or } p > b, q > a \text{ and } q - a = p - b, \\
(iii) & \lim_{r \to \infty} \frac{\log^{[p+\frac{a-2}{2}(p+a-b)-q]} \mu(\exp^{[q-1]} r, f_n)}{\log^{[a-1]} \mu(\exp^{[b-1]} r, g)} = 0 \\
& \quad \text{if } p > b, q > a \text{ and } q - a < p - b, \\
(iv) & \lim_{r \to \infty} \frac{\log^{[2p+\frac{2}{2}(a-q)+\frac{a-2}{2}(p-b)-b-1]} \mu(\exp^{[q-1]} r, f_n)}{\log^{[a-1]} \mu(\exp^{[b-1]} r, g)} = 0 \\
& \quad \text{if } q < a \text{ and } b < p, \\
(v) & \lim_{r \to \infty} \frac{\log^{[2p+\frac{n-1(a-q)-b-1]} M(\exp^{[q-1]} r, f_n)}}{\log^{[a-1]} M(\exp^{[b-1]} r, g)} = 0 \\
& \quad \text{if } p = b \text{ and } a > q, \\
(vi) & \lim_{r \to \infty} \frac{\log^{[2p+(n-2)(p-q)-b-1]} \mu(\exp^{[q-1]} r, f_n)}{\log^{[a-1]} \mu(\exp^{[b-1]} r, g)} = 0 \\
& \quad \text{if } p = a > q = b,
\end{align*}
\]
\[ (vii) \lim_{r \to -\infty} \frac{\log^{[2p+a-q-b-1]} \mu \left( \exp^{[q-1]} r, f_n \right)}{\log^{[a-1]} \mu \left( \exp^{[b-1]} r, g \right)} = 0 \] 
if \( p < b, q < a \) and \( b - p = a - q \)

and

\[ (viii) \lim_{r \to -\infty} \frac{\log^{[2p+a+\frac{n-2(a-b)-q-b-1]} \mu \left( \exp^{[q-1]} r, f_n \right)}{\log^{[a-1]} \mu \left( \exp^{[b-1]} r, g \right)} = 0 \] 
if \( p < b, q < a \) and \( b - p < a - q \)

when \( n \) is odd and \( n \neq 1 \).

The proof is omitted.

Remark 3.3.4 In Theorem 3.3.6 if we take the condition \( \lambda_f(p, q) < \lambda_g(a, b) \leq \rho_g(a, b) < \infty \) instead of \( \rho_f(p, q) < \lambda_g(a, b) \leq \rho_g(a, b) < \infty \), then also Theorem 3.3.6 remains true with “limit inferior” in place of “limit”.

Theorem 3.3.7 Let \( f \) and \( g \) be any two entire functions such that \( \lambda_f(p, q) \leq \rho_f(p, q) < \infty \) and \( \rho_g(a, b) < \infty \) where \( a, b, p, q \) are positive integers with \( a \geq b \) and \( p \geq q \). Then

\( (i) \limsup_{r \to -\infty} \frac{\log^{[p+\frac{n-2(a-b)-p+q}{2}]} \mu \left( \exp^{[b-1]} r, f_n \right)}{\log^{[p]} \mu \left( \exp^{[q-1]} r, f \right)} \leq \frac{\rho_g(a, b)}{\lambda_f(p, q)} = 0 \) if \( p > b \) and \( q = a \),

\( (ii) \limsup_{r \to -\infty} \frac{\log^{[p+1]} \mu \left( \exp^{[b-1]} r, f_n \right)}{\log^{[p]} \mu \left( \exp^{[q-1]} r, f \right)} \leq \frac{\rho_g(a, b)}{\lambda_f(p, q)} \) if \( a = b = p = q \)

or \( p > b, q > a \) and \( q - a = p - b \),

\( (iii) \limsup_{r \to -\infty} \frac{\log^{[p+\frac{n-2(a-b)-q}{2}]} \mu \left( \exp^{[b-1]} r, f_n \right)}{\log^{[p]} \mu \left( \exp^{[q-1]} r, f \right)} \leq \frac{\rho_g(a, b)}{\lambda_f(p, q)} \) if \( p > b, q > a \) and \( q - a < p - b \).
(iv) \[ \limsup_{r \to \infty} \frac{\log \left[ p + \frac{n-2}{2} (a-q) + \frac{n-2}{2} (p-b) + a - q \right] \mu \left( \exp^{[b-1]} r, f_n \right)}{\log [p] \mu (\exp^{[q-1]} r, f)} \leq \frac{\rho_g(a, b)}{\lambda_f(p, q)} \]

if \( q < a \) and \( b < p \),

(v) \[ \limsup_{r \to \infty} \frac{\log \left[ p + \frac{n-2}{2} (a-q) + a - q \right] \mu \left( \exp^{[b-1]} r, f_n \right)}{\log [p] \mu (\exp^{[q-1]} r, f)} \leq \frac{\rho_g(a, b)}{\lambda_f(p, q)} \]

if \( p = b \geq q, a > q \) and \( n > 2 \),

(vi) \[ \limsup_{r \to \infty} \frac{\log \left[ p + (n-2)(p-q) + a - q \right] \mu \left( \exp^{[b-1]} r, f_n \right)}{\log [p] \mu (\exp^{[q-1]} r, f)} \leq \frac{\rho_g(a, b)}{\lambda_f(p, q)} \]

if \( p = a > q = b \),

(vii) \[ \limsup_{r \to \infty} \frac{\log \left[ p + 2a - 2q \right] \mu \left( \exp^{[b-1]} r, f_n \right)}{\log [p] \mu (\exp^{[q-1]} r, f)} \leq \frac{\rho_g(a, b)}{\lambda_f(p, q)} \]

if \( p < b, q < a \) and \( b - p = a - q \)

and

(viii) \[ \limsup_{r \to \infty} \frac{\log \left[ p + a + \frac{n-2}{2} (a-b) + a - q \right] \mu \left( \exp^{[b-1]} r, f_n \right)}{\log [p] \mu (\exp^{[q-1]} r, f)} \leq \frac{\rho_g(a, b)}{\lambda_f(p, q)} \]

if \( p < b, q < a \) and \( b - p < a - q \)

where \( n \) is any even number.

**Proof.** Let \( p > b \) and \( q = a \).

Now from (3.27), we get for all sufficiently large values of \( r \) that

\[ \log \left[ p + \frac{n-2}{2} (p-b) + 1 \right] \mu \left( \exp^{[b-1]} r, f_n \right) \leq (\rho_g(a, b) + \varepsilon) \log r + O(1). \]  

(3.48)

Therefore combining (3.24) and (3.48), we get for all sufficiently large values of \( r \) that

\[ \frac{\log \left[ p + \frac{n-2}{2} (p-b) + 1 \right] \mu \left( \exp^{[b-1]} r, f_n \right)}{\log [p] \mu (\exp^{[q-1]} r, f)} \leq \frac{(\rho_g(a, b) + \varepsilon) \log r + O(1)}{(\lambda_f(p, q) - \varepsilon) \log r} \]

i.e., \[ \limsup_{r \to \infty} \frac{\log \left[ p + \frac{n-2}{2} (p-b) + 1 \right] \mu \left( \exp^{[b-1]} r, f_n \right)}{\log [p] \mu (\exp^{[q-1]} r, f)} \leq \frac{\rho_g(a, b)}{\lambda_f(p, q)}. \]
This proves the first part of the theorem. Further suppose that \(a = b = p = q\). Then from (3.24) and (3.29), we obtain for all sufficiently large values of \(r\) that
\[
\frac{\log^{[p+1]} \mu \left( \exp^{[b-1]} r, f_n \right)}{\log^{[p]} \mu \left( \exp^{[q-1]} r, f \right)} \leq \frac{\left( \rho_g(a, b) + \varepsilon \right) \log r + O(1)}{(\lambda_f(p, q) - \varepsilon) \log r}
\]
i.e.,
\[
\limsup_{r \to \infty} \frac{\log^{[p+1]} \mu \left( \exp^{[b-1]} r, f_n \right)}{\log^{[p]} \mu \left( \exp^{[q-1]} r, f \right)} \leq \frac{\rho_g(a, b)}{\lambda_f(p, q)}.
\] (3.49)

Again let \(p > b, q > a\) and \(q - a = p - b\). Then also combining (3.24) and (3.31), it follows for all sufficiently large values of \(r\) that
\[
\frac{\log^{[p+1]} \mu \left( \exp^{[b-1]} r, f_n \right)}{\log^{[p]} \mu \left( \exp^{[q-1]} r, f \right)} \leq \frac{\left( \rho_g(a, b) + \varepsilon \right) \log r + O(1)}{(\lambda_f(p, q) - \varepsilon) \log r}
\]
i.e.,
\[
\limsup_{r \to \infty} \frac{\log^{[p+1]} \mu \left( \exp^{[b-1]} r, f_n \right)}{\log^{[p]} \mu \left( \exp^{[q-1]} r, f \right)} \leq \frac{\rho_g(a, b)}{\lambda_f(p, q)}.
\]

Thus the second part of the theorem follows from (3.49) and above. Similarly, using the same technique of above one can easily prove the remaining parts of Theorem 3.3.7 from (3.33), (3.36), (3.38), (3.40), (3.42), (3.44) respectively and with the help of the inequality (3.24). Hence their proofs are omitted.

**Remark 3.3.5** The condition \(\rho_g(a, b) < \infty\) in Theorem 3.3.7 is necessary which is evident from the following example.

**Example 3.3.2** Let \(f = \exp z\), \(g = \exp^{[2]} z\) and \(p = a = n = 2, q = b = 1\).

Then \(\lambda_f(p, q) = \rho_f(p, q) = 1\) and \(\rho_g(a, b) = \infty\).

Now
\[
\log^{[3]} \mu(r, f \circ g) \geq \log^{[3]} M \left( \frac{r}{2}, f \circ g \right) + O(1)
\]
i.e.,
\[
\log^{[3]} \mu(r, f \circ g) \geq \log^{[3]} \exp^{[3]} \left( \frac{r}{2} \right) + O(1)
\]
i.e.,
\[
\log^{[3]} \mu(r, f \circ g) = \left( \frac{r}{2} \right) + O(1)
\]
and
\[
\log^{[2]} \mu(r, f) \leq \log^{[2]} M(r, f) = \log r.
\]
Therefore

\[
\limsup_{r \to \infty} \frac{\log[p + p^{n-1}(p-a) + a] \mu \left( \exp^{[b-1]} r, f_n \right]}{\log[p] \mu \left( \exp^{[a-1]} r, f \right)} = \limsup_{r \to \infty} \frac{\log[p] \mu \left( \exp^{[a-1]} r, f \right)}{\log[p] \mu \left( \exp^{[a-1]} r, f \right)}
\]

i.e.,

\[
\limsup_{r \to \infty} \frac{\log[p + p^{n-1}(p-a) + a] \mu \left( \exp^{[b-1]} r, f_n \right]}{\log[p] \mu \left( \exp^{[a-1]} r, f \right)} = \limsup_{r \to \infty} \frac{r^2 + O(1)}{\log r}
\]

which is contrary to Theorem 3.3.7.

**Remark 3.3.6** Theorem 3.3.7 is still valid with "limit inferior" instead of “limit superior” if we replace \( \rho_g(a, b) \) by \( \rho_g(a, b) \).

**Remark 3.3.7** Considering \( f = \exp z \), \( g = \exp^2 z \) and \( p = a = n = 2, q = b = 1 \), one can easily verify that the condition \( \lambda_g(a, b) < \infty \) in Remark 3.3.6 is essential.

**Theorem 3.3.8** Let \( f \) and \( g \) be any two entire functions such that \( \lambda_g(a, b) \leq \rho_g(a, b) < \infty \) and \( \rho_f(p, q) < \infty \) where \( a, b, p, q \) are positive integers with \( a \geq b \) and \( p \geq q \). Then

(i) \[ \limsup_{r \to \infty} \frac{\log[p + p^{n-1}(p-b)+1] \mu \left( \exp^{[q-1]} r, f_n \right]}{\log[p] \mu \left( \exp^{[b-1]} r, g \right)} \leq \rho_f(p, q) \frac{\lambda_g(a, b)}{\lambda_y(a, b)} \]

if \( p > b \) and \( q = a \),

(ii) \[ \limsup_{r \to \infty} \frac{\log[p+1] \mu \left( \exp^{[q-1]} r, f_n \right]}{\log[p] \mu \left( \exp^{[b-1]} r, g \right)} \leq \rho_f(p, q) \frac{\lambda_g(a, b)}{\lambda_y(a, b)} \]

if \( a = b = p = q \)

or \( p > b, q > a \) and \( q - a = p - b \),

(iii) \[ \limsup_{r \to \infty} \frac{\log[p^{n-1}(p-a-b)+1] \mu \left( \exp^{[q-1]} r, f_n \right]}{\log[p] \mu \left( \exp^{[b-1]} r, g \right)} \leq \rho_f(p, q) \frac{\lambda_g(a, b)}{\lambda_y(a, b)} \]

if \( p > b, q > a \) and \( q - a < p - b \),

(iv) \[ \limsup_{r \to \infty} \frac{\log[2p^{n-1}(a-b)+2p^{n-2}(p-b)-b] \mu \left( \exp^{[q-1]} r, f_n \right]}{\log[p] \mu \left( \exp^{[b-1]} r, g \right)} \leq \rho_f(p, q) \frac{\lambda_g(a, b)}{\lambda_y(a, b)} \]

if \( q < a \) and \( b < p \),
\[(v) \limsup_{r \to \infty} \frac{\log \left[ 2p + \frac{n-1}{2}(a-q)-b \right] \mu \left( \exp^{[q-1]} r, f_n \right)}{\log[a] \mu(\exp^{[b-1]} r, g)} \leq \frac{\rho_f(p, q)}{\lambda_g(a, b)} \]

if \( p = b \geq q \) and \( a > q \),

\[(vi) \limsup_{r \to \infty} \frac{\log \left[ 2p+(n-2)(p-q)-b \right] \mu \left( \exp^{[q-1]} r, f_n \right)}{\log[a] \mu(\exp^{[b-1]} r, g)} = \frac{\rho_f(p, q)}{\lambda_g(a, b)} \]

if \( p = a > q = b \),

\[(vii) \limsup_{r \to \infty} \frac{\log \left[ 2p+a-q-b \right] \mu \left( \exp^{[q-1]} r, f_n \right)}{\log[a] \mu(\exp^{[b-1]} r, g)} \leq \frac{\rho_f(p, q)}{\lambda_g(a, b)} \]

if \( p < b, q < a \) and \( b - p = a - q \)

and

\[(viii) \limsup_{r \to \infty} \frac{\log \left[ 2p+a+\frac{n-2}{2}(a-b)-q-b \right] \mu \left( \exp^{[q-1]} r, f_n \right)}{\log[a] \mu(\exp^{[b-1]} r, g)} \leq \frac{\rho_f(p, q)}{\lambda_g(a, b)} \]

if \( p < b, q < a \) and \( b - p < a - q \)

when \( n \) is odd and \( n \neq 1 \).

The proof is omitted.

**Remark 3.3.8** In Theorem 3.3.8 if we replace \( \rho_f(p, q) \) by \( \lambda_f(p, q) \) and the other conditions remaining the same, then also Theorem 3.3.6 remains true with "limit inferior" in place of "limit superior".