CHAPTER 6

GENERALIZED RELATIVE LOWER ORDER OF ENTIRE FUNCTIONS
Chapter 6

GENERALIZED RELATIVE LOWER ORDER OF ENTIRE FUNCTIONS

6.1 Introductory Remarks.

Let $f$ and $g$ be any two entire functions defined in the open complex plane $\mathbb{C}$ and $M_f(r) = \max \{|f(z)| : |z| = r\}$, $M_g(r) = \max \{|g(z)| : |z| = r\}$. Sato [52] defined the generalized order $\rho_f^{(l)}$ and generalized lower order $\lambda_f^{(l)}$ of an entire function $f$ for any integer $l \geq 2$. Bernal [1] and [2] introduced the definition of relative order of $g$ with respect to $f$, denoted by $\rho_f^{(l)}(g)$.

During the past decades, several authors {cf. [36],[38],[39] and [47]} made close investigations on the properties of relative order of entire functions. In this chapter we wish to investigate some basic properties of generalized relative lower order of entire functions.

It is well known that the order of products and sums of two entire functions are not greater than the maximal order of two functions. Bernal [2], Lahiri and Banerjee [47] extended these results for relative order and generalized relative order. In this chapter, the basic properties of generalized relative lower order of entire functions are discussed. In fact, we improve here some results of Datta, Biswas and Biswas [16]. The existing literature, relevant definitions and examples of this chapter have already been discussed in Chapter One and Chapter Two respectively. Therefore, we need not mention those here again.

6.2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

**Lemma 6.2.1** [2]Suppose $f$ be a nonconstant entire function. Also let $\alpha > 1, 0 < \beta < \alpha$, $s > 1, 0 < \mu_0 < \lambda$ and $n$ be a positive integer. Then

---

The results of this chapter have been published in *Matematicki Vesnik*, see[27].
(a) $M_f (or) > \beta M_f (r)$.
(b) There exists $K = K(s, f) > 0$ such that $(M_f (r))^s \leq KM_f (r^s)$ for $r > 0$.
(c) $\lim_{r \to \infty} \frac{M_f (r^s)}{M_f (r)} = \infty = \lim_{r \to \infty} \frac{M_f (r^n)}{M_f (r)}$ and
(d) If $f$ is transcendental then
$$\lim_{r \to \infty} \frac{M_f (r^s)}{M_f (r)} = \infty = \lim_{r \to \infty} \frac{M_f (r^n)}{M_f (r)}.$$  

**Lemma 6.2.2** Let $f, g$ and $h$ be any three entire functions. For all sufficiently large values of $r$, if $M_g (r) \leq M_h (r)$ then
$$\lambda_h^{|l|} (f) \leq \lambda_g^{|l|} (f),$$
where $l \geq 1$.

**Proof.** As $M_g (r) \leq M_h (r)$ and $M_f (r)$ is an increasing function of $r$ we get for all sufficiently large values of $r$ that
$$M_h^{-1} (r) \leq M_g^{-1} (r)$$
i.e., $M_h^{-1} M_f (r) \leq M_g^{-1} M_f (r)$
i.e., \( \liminf_{r \to \infty} \frac{\log |l| M_h^{-1} M_f (r)}{\log r} \leq \liminf_{r \to \infty} \frac{\log |l| M_g^{-1} M_f (r)}{\log r} \)
i.e., $\lambda_h^{|l|} (f) \leq \lambda_g^{|l|} (f)$.

This proves the lemma. 

**Lemma 6.2.3** [47]Every entire function $f$ satisfying the Property (A) is transcendental.

**Lemma 6.2.4** [41]Let $f(z)$ be holomorphic in the circle $|z| = 2eR (R > 0)$ with $f(0) = 1$ and $\eta$ be an arbitrary positive number not exceeding $\frac{3e}{2}$. Then inside the circle $|z| = R$, but outside of a family of excluded circles the sum of whose radii is not greater than $4\eta R$, we have
$$\log |f(z)| > -T(\eta) \log M_f (2eR),$$
for $T(\eta) = 2 + \log \frac{3e}{2\eta}$.

### 6.3 Theorems.

In this section we present the main results of the chapter.

**Theorem 6.3.1** If $f_1, f_2, ..., f_n$ ($n \geq 2$) and $g$ are entire functions, then
$$\lambda_f^{|l|} (g) \geq \lambda_{f_i}^{|l|} (g),$$
where $l \geq 1$, $f = f_1 \pm \sum_{k=2}^{n} f_k$ and $\lambda_{f_i}^{|l|} (g) = \min \left\{ \lambda_{f_k}^{|l|} (g) \mid k = 1, 2, ..., n \right\}$. The sign of equality holds when $\lambda_{f_i}^{|l|} (g) = \lambda_{f_k}^{|l|} (g)$ where $k = 1, 2, ..., n$ and $k \neq i$.
Proof. If \(\lambda^{|l|}_f(g) = \infty\) then the result is obvious. So we suppose that \(\lambda^{|l|}_f(g) < \infty\).

We can clearly assume that \(\lambda^{|l|}_f(g)\) is finite. By hypothesis, \(\lambda^{|l|}_f(g) \leq \lambda^{|l|}_{f_k}(g)\) for all \(k = 1, 2, \ldots, i, \ldots, n\). We can suppose \(\lambda^{|l|}_f(g) > 0\) as the case \(\lambda^{|l|}_f(g) = 0\) is quite obvious.

Now for any arbitrary \(\varepsilon > 0\), we get for all sufficiently large values of \(r\) that

\[
M_{f_k} \left[ \exp^{[l-1]} \left( \frac{r}{\lambda^{|l|}_{f_k}(g) - \varepsilon} \right) \right] < M_g(r) \quad \text{where } k = 1, 2, \ldots, n
\]

i.e., \(M_{f_k}(r) < M_g \left( \left( \log^{[l-1]} r \right) \left( \frac{1}{\lambda^{|l|}_{f_k}(g) - \varepsilon} \right) \right) \) where \(k = 1, 2, \ldots, n\)

i.e., \(M_{f_k}(r) \leq M_g \left( \left( \log^{[l-1]} r \right) \left( \frac{1}{\lambda^{|l|}_{f_i}(g) - \varepsilon} \right) \right) \) where \(k = 1, 2, \ldots, n\). (6.1)

Now for all sufficiently large values of \(r\),

\[
M_f(r) < \sum_{k=1}^{n} M_{f_k}(r)
\]

i.e., \(M_f(r) < \sum_{k=1}^{n} M_g \left( \left( \log^{[l-1]} r \right) \left( \frac{1}{\lambda^{|l|}_{f_k}(g) - \varepsilon} \right) \right) \)

i.e., \(M_f(r) < nM_g \left( \left( \log^{[l-1]} r \right) \left( \frac{1}{\lambda^{|l|}_{f_i}(g) - \varepsilon} \right) \right) \) . (6.2)

Now in view of the first part of Lemma 6.2.1, we obtain from (6.2) for all sufficiently large values of \(r\) that

\[
M_f(r) < M_g \left( (n + 1) \left( \log^{[l-1]} r \right) \left( \frac{1}{\lambda^{|l|}_{f_i}(g) - \varepsilon} \right) \right)
\]

i.e., \(M_f \left[ \exp^{[l-1]} \left( \frac{r}{n + 1} \right) \left( \frac{1}{\lambda^{|l|}_{f_i}(g) - \varepsilon} \right) \right] < M_g(r) \)

i.e., \(\exp^{[l-1]} \left( \frac{r}{n + 1} \right) \left( \frac{1}{\lambda^{|l|}_{f_i}(g) - \varepsilon} \right) < M_f^{-1}M_g(r) \)

i.e., \(\lambda^{|l|}_{f_i}(g) - \varepsilon \log \left( \frac{r}{n + 1} \right) < \log^{[l]} M_f^{-1}M_g(r) \)

i.e., \(\lambda^{|l|}_{f_i}(g) - \varepsilon < \frac{\log^{[l]} M_f^{-1}M_g(r)}{\log r + O(1)} \)

i.e., \(\frac{\log^{[l]} M_f^{-1}M_g(r)}{\log r + O(1)} > \left( \lambda^{|l|}_{f_i}(g) - \varepsilon \right) \).
So
\[ \lambda_f^{[l]} (g) = \liminf_{r \to \infty} \frac{\log^l M_f^{-1} M_g (r)}{\log r + O(1)} \geq \lambda_{f_i}^{[l]} (g) - \varepsilon. \]

Since \( \varepsilon > 0 \) is arbitrary,
\[ \lambda_f^{[l]} (g) \geq \lambda_{f_i}^{[l]} (g). \quad (6.3) \]

Next let \( \lambda_{f_i}^{[l]} (g) < \lambda_{f_k}^{[l]} (g) \) where \( k = 1, 2, ... n \) and \( k \neq i \).

As \( \varepsilon (> 0) \) is arbitrary, from the definition of generalized lower order it follows for a sequence of values of \( r \) tending to infinity that
\[ M_g (r) < M_{f_i} \left[ \exp^{[l-1]} \left( r^{\lambda_{f_i}^{[l]} (g) + \varepsilon} \right) \right] \]
i.e.,
\[ M_g \left[ \left( \log^{[l-1]} r \right)^{\lambda_{f_i}^{[l]} (g) + \varepsilon} \right] < M_{f_i} (r). \quad (6.4) \]

Since \( \lambda_{f_i}^{[l]} (g) < \lambda_{f_k}^{[l]} (g) \) where \( k = 1, 2, ... n \) and \( k \neq i \), then in view of the third part of Lemma 6.2.1 we obtain that
\[ \lim_{r \to \infty} \frac{M_g \left[ \left( \log^{[l-1]} r \right)^{\lambda_{f_i}^{[l]} (g) + \varepsilon} \right]}{M_g \left[ \left( \log^{[l-1]} r \right)^{\lambda_{f_k}^{[l]} (g) - \varepsilon} \right]} = \infty \quad \text{where} \ k = 1, 2, ... n \ \text{and} \ k \neq i. \quad (6.5) \]

Therefore from (6.5), we obtain for all sufficiently large values of \( r \) that
\[ M_g \left[ \left( \log^{[l-1]} r \right)^{\lambda_{f_i}^{[l]} (g) + \varepsilon} \right] > n M_g \left[ \left( \log^{[l-1]} r \right)^{\lambda_{f_k}^{[l]} (g) - \varepsilon} \right], \quad (6.6) \]

where \( k = 1, 2, ... n \) and \( k \neq i \).

Thus from (6.1), (6.4) and (6.6), we get for a sequence of values of \( r \) tending to infinity that
\[ M_{f_i} (r) > M_g \left[ \left( \log^{[l-1]} r \right)^{\lambda_{f_i}^{[l]} (g) + \varepsilon} \right] \]
i.e.,
\[ M_{f_i} (r) > n M_g \left[ \left( \log^{[l-1]} r \right)^{\lambda_{f_k}^{[l]} (g) - \varepsilon} \right] \]
i.e.,
\[ M_{f_i} (r) > n M_{f_k} (r) \quad \text{for all} \ k = 1, 2, ... n \ \text{with} \ k \neq i. \quad (6.7) \]
So from (6.4) and (6.7) and in view of the first part of Lemma 6.2.1, it follows for a sequence of values of \( r \) tending to infinity that

\[
M_f (r) \geq M_{f_i} (r) - \sum_{k=1, k \neq i}^{n} M_{f_k} (r)
\]

i.e., \( M_f (r) \geq M_{f_i} (r) - \frac{1}{n} \sum_{k=1, k \neq i}^{n} M_{f_k} (r) \)

i.e., \( M_f (r) \geq M_{f_i} (r) - \left( \frac{n-1}{n} \right) M_{f_i} (r) \)

i.e., \( M_f (r) > \left( \frac{1}{n} \right) M_{f_i} (r) \)

i.e., \( M_f (r) > \left( \frac{1}{n} \right) M_g \left[ \left( \log^{[l-1]} r \right) \left( \frac{\lambda_{f_i}^{[l]} (g) + \varepsilon}{\lambda_{f_i}^{[l]} (g) + \varepsilon} \right) \right] \)

i.e., \( M_f (r) > M_g \left[ \frac{\log^{[l-1]} r \left( \frac{\lambda_{f_i}^{[l]} (g) + \varepsilon}{\lambda_{f_i}^{[l]} (g) + \varepsilon} \right)}{n + 1} \right] \)

This gives for a sequence of values of \( r \) tending to infinity that

\[
M_f \left[ \exp^{[l-1]} \{(n + 1) r \} \left( \frac{\lambda_{f_i}^{[l]} (g) + \varepsilon}{\lambda_{f_i}^{[l]} (g) + \varepsilon} \right) \right] > M_g (r)
\]

i.e., \( \{(n + 1) r \} \left( \frac{\lambda_{f_i}^{[l]} (g) + \varepsilon}{\lambda_{f_i}^{[l]} (g) + \varepsilon} \right) > \log^{[l-1]} M_f^{-1} M_g (r) \)

i.e., \( \left( \frac{\lambda_{f_i}^{[l]} (g) + \varepsilon}{\lambda_{f_i}^{[l]} (g) + \varepsilon} \right) > \frac{\log^{[l]} M_f^{-1} M_g (r)}{\log ((n + 1) r)} \)

i.e., \( \left( \frac{\lambda_{f_i}^{[l]} (g) + \varepsilon}{\lambda_{f_i}^{[l]} (g) + \varepsilon} \right) > \frac{\log^{[l]} M_f^{-1} M_g (r)}{\log r + O(1)} \)

i.e., \( \lambda_{f_i}^{[l]} (g) \geq \liminf_{r \to \infty} \frac{\log^{[l]} M_f^{-1} M_g (r)}{\log r + O(1)} \)

i.e., \( \lambda_{f_i}^{[l]} (g) = \liminf_{r \to \infty} \frac{\log^{[l]} M_f^{-1} M_g (r)}{\log r} \leq \lambda_{f_i}^{[l]} (g) \).

So from (6.3) and (6.8), we finally obtain that

\[
\lambda_f^{[l]} (g) = \lambda_{f_i}^{[l]} (g),
\]

whenever \( \lambda_{f_i}^{[l]} (g) \neq \lambda_{f_k}^{[l]} (g) \) for all \( k = 1, 2, \ldots n \) and \( k \neq i \). ■
Theorem 6.3.2 Let \( n, l \) be a positive integer with \( n, l \geq 2 \). Then

\[
\frac{1}{n} \lambda_f^{[n]}(g) \leq \lambda_{f^n}^{[n]}(g) \leq \lambda_f^{[n]}(g).
\]

Proof. From the first and second parts of Lemma 6.2.1 we obtain that

\[
\{ M_f(r) \}^n \leq KM_f(r^n) < M_f((K + 1)r^n), \quad n > 1 \text{ and } r > 0
\]

(6.9)

where \( K = K(n, f) > 0 \). Therefore from (6.9), we obtain that

\[
M_f^{-1}(r^n) < (K + 1) \{ M_f^{-1}(r) \}^n
\]

i.e.,

\[
\frac{1}{(K + 1)} M_f^{-1}(r^n) < \{ M_f^{-1}(r) \}^n.
\]

So

\[
\lambda^{[n]}_{f^n}(g) \geq \frac{\log^{[n]} \frac{1}{(K + 1)} M_f^{-1} M_g(r^n)}{\log r^n}
\]

i.e.,

\[
\lambda^{[n]}_{f^n}(g) \geq \frac{1}{n} \lambda_f^{[n]}(g).
\]

(6.10)

On the other hand since \( \{ M_f(r) \}^n > M_f(r) \) for all sufficiently large values of \( r \), we have by Lemma 6.2.2

\[
\lambda^{[n]}_{f^n}(g) \leq \lambda_f^{[n]}(g).
\]

(6.11)

Thus the theorem follows from (6.10) and (6.11). □

Remark 6.3.1 The following example ensures the validity of the conclusion as obtained in Theorem 6.3.2.

Example 6.3.1 Let \( f = \exp z \), \( g = \exp^{[2]} z \) and \( n = l = 2 \). Then

\[
\lambda_f^{[2]}(g) = \liminf_{r \to \infty} \frac{\log^{[2]} M_f^{-1} M_g(r)}{\log r}
\]

\[
= \liminf_{r \to \infty} \frac{\log^{[2]} \log(\exp^{[2]} r)}{\log r}
\]

\[
= \liminf_{r \to \infty} \frac{\log r}{\log r}
\]

\[
= 1
\]
\[ \lambda_{f}^{[\ell]}(g) = \liminf_{r \to \infty} \frac{\log^2 M_{f}^{-1}(\exp M_{g}(r))}{\log r} \]

\[ = \liminf_{r \to \infty} \frac{\log^2 (\exp r)^2}{\log r} \]

\[ = \liminf_{r \to \infty} \frac{\log(2r)}{\log r} \]

\[ = 1. \]

Also, \( \frac{1}{n} \lambda_{f}^{[\ell]}(g) = \frac{1}{2} \).

**Corollary 6.3.1** Let \( n, \ell \) be a positive integer with \( n, \ell \geq 2 \). Then

\[ \frac{1}{n} \rho_{f}^{[\ell]}(g) \leq \rho_{f}^{[\ell]}(g) \leq \rho_{f}^{[\ell]}(g). \]

The proof is omitted as it can be carried out in the line of Theorem 6.3.2.

**Theorem 6.3.3** Let \( P \) be a polynomial. If \( f \) is transcendental then \( \lambda_{Pf}^{[\ell]}(g) = \lambda_{f}^{[\ell]}(g) \) and if \( g \) is transcendental, then \( \lambda_{g}^{[\ell]}(Pg) = \lambda_{f}^{[\ell]}(g) \). If \( f \) and \( g \) both are transcendental then \( \lambda_{Pf}^{[\ell]}(g) = \lambda_{f}^{[\ell]}(Pg) = \lambda_{g}^{[\ell]}(g) = \lambda_{Pf}^{[\ell]}(Pg) \) where \( Pf \) and \( Pg \) denote the ordinary product of \( P \) with \( f \) and \( g \) respectively and \( \ell \geq 1 \).

**Proof.** Let \( m \) be the degree of \( P(z) \).

Then there exists \( \alpha \) such that \( 0 < \alpha < 1 \) and a positive integer \( n > m \) for which

\[ 2\alpha \leq |P(z)| \leq r^n \]

holds on \( |z| = r \) for all sufficiently large values of \( r \).

Now by the first part of Lemma 6.2.1 we obtain that

\[ M_{g} \left( \frac{1}{\alpha} \right) > \frac{1}{2\alpha} M_{g}(\alpha r) \]

i.e., \( M_{g}(\alpha r) < 2\alpha M_{g}(r) \).

(6.12)

Now let us consider \( h(z) = P(z) f(z) \). Then from (6.2) and in view of the fourth part of Lemma 6.2.1 we get for any \( s > 1 \) and for all sufficiently large values of \( r \) that

\[ M_{g}(\alpha r) < 2\alpha M_{g}(r) < M_{h}(r) \leq r^n M_{g}(r) < M_{g}(r^s). \]
So
\[
\liminf_{r \to \infty} \frac{\log^{|r|} M_f^{-1} M_g (\alpha r)}{\log r} \leq \liminf_{r \to \infty} \frac{\log^{|r|} M_f^{-1} M_h (r)}{\log r} \leq \liminf_{r \to \infty} \frac{\log^{|r|} M_f^{-1} M_g (r^*)}{\log r}
\]
i.e.,
\[
\liminf_{r \to \infty} \frac{\log^{|r|} M_f^{-1} M_g (\alpha r)}{\log (\alpha r) + O(1)} \leq \liminf_{r \to \infty} \frac{\log^{|r|} M_f^{-1} M_h (r)}{\log r} \leq \liminf_{r \to \infty} \frac{\log^{|r|} M_f^{-1} M_g (r^*)}{\log r^*}.
\]
and letting \( s \to 1^+ \) we get that
\[
\lambda_f^{|r|} (g) \leq \lambda_h^{|r|} (g) \leq s \lambda_f^{|r|} (g),
\]
and letting \( s \to 1^+ \) we get that
\[
\lambda_f^{|r|} (Pg) = \lambda_f^{|r|} (g).
\]
Similarly, when \( g \) is transcendental one can easily prove that
\[
\lambda_f^{|r|} (Pg) = \lambda_f^{|r|} (g).
\]
If \( f \) and \( g \) both are transcendental then the conclusion of the theorem can easily be obtained by combining (6.13) and (6.14).
Thus the theorem follows. \( \blacksquare \)

**Theorem 6.3.4** If \( f_1, f_2, \ldots, f_n \ (n \geq 2) \), \( g \) are entire functions and \( g \) has the Property (A), then
\[
\lambda_f^{|r|} (g) \geq \lambda_{f_k}^{|r|} (g)
\]
where \( f = \prod_{k=1}^{n} f_k \) and \( \lambda_{f_k}^{|r|} (g) = \min \left\{ \lambda_{f_k}^{|r|} (g) \mid k = 1, 2, \ldots, n \right\} \). The equality holds when \( \lambda_{f_k}^{|r|} (g) \neq \lambda_{f_k}^{|r|} (g) \) where \( k = 1, 2, \ldots, n \) and \( k \neq i \). Finally, assume that \( F_1 \) and \( F_2 \) are entire functions such that \( f = \frac{F_1}{F_2} \) is also an entire function. Then \( \lambda_f^{|r|} (g) = \min \left\{ \lambda_{F_1}^{|r|} (g), \lambda_{F_2}^{|r|} (g) \right\} \).

**Proof.** By Lemma \( \boxed{6.2.3} \), \( g \) is transcendental.
Suppose that \( \lambda_f^{|r|} (g) < \infty \). Otherwise if \( \lambda_f^{|r|} (g) = \infty \) then the result is obvious.
We can clearly assume that \( \lambda_f^{|r|} (g) \) is finite. Also suppose that \( \lambda_{f_k}^{|r|} (g) \leq \lambda_{f_k}^{|r|} (g) \) where \( k = 1, 2, \ldots, n \). We can suppose \( \lambda_{f_k}^{|r|} (g) > 0 \) as the case \( \lambda_{f_k}^{|r|} (g) = 0 \) is quite obvious.

Now for any arbitrary \( \varepsilon > 0 \), with \( \varepsilon < \lambda_{f_k}^{|r|} (g) \), we have for all sufficiently large values of \( r \) that
\[
M_{f_k} \left[ \exp^{|r-\varepsilon|} (\lambda_{f_k}^{|r|} (g) - \frac{r}{2}) \right] < M_g (r) \quad \text{where } k = 1, 2, \ldots, n
\]
\[ M_f(r) < M_g \left( \log^{[l-1]} r \right)^{\frac{1}{\lambda_{f_i}^{(l)}(g) - \frac{\epsilon}{2}}} \] where \( k = 1, 2, \ldots, n \)

So, \( M_f(r) \leq M_g \left( \log^{[l-1]} r \right)^{\frac{1}{\lambda_{f_i}^{(l)}(g) - \frac{\epsilon}{2}}} \) where \( k = 1, 2, \ldots, n \). \hspace{1cm} (6.15)

Now from (6.15), we have for all sufficiently large values of \( r \) that

\[ M_f(r) < \prod_{k=1}^{n} M_{f_k}(r) \]

\[ i.e., \ M_f(r) < \prod_{k=1}^{n} M_g \left( \log^{[l-1]} r \right)^{\frac{1}{\lambda_{f_i}^{(l)}(g) - \frac{\epsilon}{2}}} \]

\[ i.e., \ M_f(r) < \left[ M_g \left( \log^{[l-1]} r \right)^{\frac{1}{\lambda_{f_i}^{(l)}(g) - \frac{\epsilon}{2}}} \right]^n \]. \hspace{1cm} (6.16)

Observe that

\[ \delta = \frac{\lambda_{f_i}^{(l)}(g) - \frac{\epsilon}{2}}{\lambda_{f_i}^{(l)}(g) - \epsilon} > 1. \hspace{1cm} (6.17) \]

Since \( g \) has the Property (A), in view of Lemma 5.2.2 and (6.17), we obtain from (6.16) for all sufficiently large values of \( r \) that

\[ M_f(r) < M_g \left( \log^{[l-1]} r \right)^{\frac{1}{\lambda_{f_i}^{(l)}(g) - \frac{\epsilon}{2}}} \]

\[ = M_g \left[ \left( \log^{[l-1]} r \right)^{\frac{1}{\lambda_{f_i}^{(l)}(g) - \epsilon}} \right] \]

\[ i.e., \ M_f \left[ \exp^{[l-1]} r^{\lambda_{f_i}^{(l)}(g) - \epsilon} \right] < M_g(r) \]

\[ i.e., \ r^{\lambda_{f_i}^{(l)}(g) - \epsilon} < \log^{[l]} M_f^{-1} M_g(r) \]

\[ i.e., \ \lambda_{f_i}^{[l]}(g) - \epsilon \log r < \log^{[l]} M_f^{-1} M_g(r) \]

\[ i.e., \ \lambda_{f_i}^{[l]}(g) - \epsilon < \frac{\log^{[l]} M_f^{-1} M_g(r)}{\log r} \].

So

\[ \lambda_{f_i}^{[l]}(g) = \liminf_{r \to \infty} \frac{\log^{[l]} M_f^{-1} M_g(r)}{\log r} \geq \lambda_{f_i}^{[l]}(g) - \epsilon. \]
Since $\varepsilon > 0$ is arbitrary,
\[ \lambda_f^{|g|} (g) \geq \lambda_{f_i}^{|g|} (g). \] (6.18)

Next let $\lambda_{f_i}^{|g|} (g) < \lambda_{f_k}^{|g|} (g)$ where $k = 1, 2, \ldots n$ and $k \neq i$.

Fix $\varepsilon > 0$ with $\varepsilon < \frac{1}{2} \min \{ \lambda_{f_k}^{|g|} (g) : k = 1, 2, \ldots n \text{ and } k \neq i \}$. Without loss of any generality, we may assume that $f_k (0) = 1$ where $k = 1, 2, \ldots n$ and $k \neq i$.

Now from the definition of relative lower order, we obtain for a sequence of values of $R$ tending to infinity that
\[ M_g (R) < M_{f_i} \left[ \exp^{1-1} \left( R^\left( \frac{1}{2} \lambda_{f_i}^{|g|} (g) + \varepsilon \right) \right) \right] \]
\[ \text{i.e., } M_{f_i} (R) > M_g \left[ \log^{1-1} R \left( \frac{1}{2} \lambda_{f_i}^{|g|} (g) - \varepsilon \right) \right]. \] (6.19)

Also for all sufficiently large values of $r$, we get that
\[ M_{f_k} \left[ \exp^{1-1} \left( r^\left( \frac{1}{2} \lambda_{f_k}^{|g|} (g) - \varepsilon \right) \right) \right] < M_g (r) \text{ where } k = 1, 2, \ldots n \text{ and } k \neq i
\[ \text{i.e., } M_{f_k} (r) < M_g \left[ \log^{1-1} r \left( \frac{1}{2} \lambda_{f_k}^{|g|} (g) - \varepsilon \right) \right] \text{ where } k = 1, 2, \ldots n \text{ and } k \neq i. \] (6.20)

Since $\lambda_{f_i}^{|g|} (g) < \lambda_{f_k}^{|g|} (g)$, we get from above that
\[ M_{f_k} (r) < M_g \left[ \log^{1-1} r \left( \frac{1}{2} \lambda_{f_k}^{|g|} (g) - \varepsilon \right) \right] \]
\[ \text{where } k = 1, 2, \ldots n \text{ and } k \neq i. \]

Now in view of Lemma 6.2.4, if we take $f_k (z)$ for $f (z)$, $\eta = \frac{1}{16}$ and $2R$ for $R$, it follows for the values of $z$ specified in the lemma that
\[ \log |f_k (z)| > -T (\eta) \log M_{f_k} (2e.2R), \]
\[ \text{where} \]
\[ T (\eta) = 2 + \log \left( \frac{3e}{2.16} \right) = 2 + \log (24e). \]

Therefore
\[ \log |f_k (z)| > -(2 + \log (24e)) \log M_{f_k} (4e.R) \]
holds within and on $|z| = 2R$ but outside a family of excluded circles the sum of whose radii is not greater than
\[ 4 \cdot \frac{1}{16} 2R = \frac{R}{2}. \]

If $r \in (R, 2R)$ then on $|z| = r$
\[ \log |f_k (z)| > -7 \log M_{f_k} (4e.R). \] (6.21)
Since \( r > R \), we have from above and (6.19) for a sequence of values of \( r \) tending to infinity that

\[
M_{f_i}(r) > M_{f_i}(R) > M_g \left[ \left( \log^{[l-1]} R \right) \left( \lambda^{[l]}_{R,t}(g) + \epsilon \right) \right]^{\frac{1}{2}}
\]

\[
> M_g \left[ \left( \log^{[l-1]} \left( \frac{r}{2} \right) \right) \left( \lambda^{[l]}_{R,t}(g) + \epsilon \right) \right]^{\frac{1}{2}}
\]

(6.22)

Let \( z_r \) be a point on \( |z| = r \) such that \( M_{f_i}(r) = |f_i(z_r)| \).

Therefore as \( r > R \), from (6.20), (6.21) and (6.22) it follows for a sequence of values of \( r \) tending to infinity that

\[
M_f(r) = \max \{|f(z)| : |z| = r\} = \max \left\{ \prod_{k=1}^{n} |f_k(z)| : |z| = r \right\}.
\]

So

\[
M_f(r) \geq \prod_{k=1}^{n} |f_k(z_r)| \cdot |f_i(z_r)|
\]

i.e., \( M_f(r) \geq \prod_{k=1}^{n} |f_k(z_r)| \cdot M_{f_i}(r) \)

i.e., \( M_f(r) \geq \prod_{k=1}^{n} [M_{f_k}(4eR)]^{-7} M_g \left[ \left( \log^{[l-1]} (4eR) \right) \left( \lambda^{[l]}_{R,t}(g) + \epsilon \right) \right]^{\frac{1}{2}}
\]

\[
= \prod_{k=1}^{n} \left[ M_g \left[ \left( \log^{[l-1]} (4eR) \right) \left( \lambda^{[l]}_{R,t}(g) + \epsilon \right) \right]^{\frac{1}{2}} \right]^{-7} M_g \left[ \left( \log^{[l-1]} \left( \frac{4er}{8e} \right) \right) \left( \lambda^{[l]}_{R,t}(g) + \epsilon \right) \right]^{\frac{1}{2}},
\]

hence

\[
M_f(r) \geq \prod_{k=1}^{n} \left[ M_g \left[ \left( \log^{[l-1]} (4er) \right) \left( \lambda^{[l]}_{R,t}(g) + \epsilon \right) \right]^{\frac{1}{2}} \right]^{-7} M_g \left[ \left( \log^{[l-1]} \left( \frac{4er}{8e} \right) \right) \left( \lambda^{[l]}_{R,t}(g) + \epsilon \right) \right]^{\frac{1}{2}}.
\]

(6.23)

On the other hand, we have

\[
\left( \log^{[l-1]} \left( \frac{4er}{8e} \right) \right) \left( \lambda^{[l]}_{R,t}(g) + \epsilon \right) \geq \left( \log^{[l-1]} (4er) \right) \left( \lambda^{[l]}_{R,t}(g) + \epsilon \right)
\]

asymptotically. By using this fact together with Lemma 5.2.2 (with \( n = 2 \) and \( \delta = \frac{\lambda^{[l]}_{R,t}(g) + 3\epsilon}{\lambda^{[l]}_{R,t}(g) + 2\epsilon} > 1 \)
we get for \( r \) large enough that
\[
M_g \left[ \left( \log^{l-1} \left( \frac{4er}{8\varepsilon} \right) \right) \left( \frac{1}{\lambda_{f_i}^{[j]}(g)+3\varepsilon} \right) \right]^2 \]
\[
\geq M_g \left[ \left( \log^{l-1} \left( 4er \right) \right) \left( \frac{1}{\lambda_{f_i}^{[j]}(g)+3\varepsilon} \right) \right]^2 \] (6.24)

Let \( L = \min \{ \lambda_{f_i}^{[j]}(g) : k \neq i \} \). Now, by choosing this time \( \delta = \frac{L-\varepsilon}{\lambda_{f_i}^{[j]}(g)+3\varepsilon} \) (which is \( > 1 \) due to our selection of \( \varepsilon \)), a further application of Lemma 5.2.2 yields
\[
M_g \left[ \left( \log^{l-1} \left( 4er \right) \right) \left( \frac{1}{\lambda_{f_i}^{[j]}(g)+3\varepsilon} \right) \right]^7 \]
\[
\geq \prod_{k=1}^{n} M_g \left[ \left( \log^{l-1} \left( 4er \right) \right) \left( \frac{1}{\lambda_{f_i}^{[j]}(g)+3\varepsilon} \right) \right]^7 \] (6.25)
for \( r \) large enough.

Now from (6.23), (6.24) and (6.25), it follows for a sequence of values of \( r \) tending to infinity that
\[
M_f(r) \geq M_g \left[ \left( \log^{l-1} \left( 4er \right) \right) \left( \frac{1}{\lambda_{f_i}^{[j]}(g)+3\varepsilon} \right) \right]
\]
i.e.,
\[
M_f \left[ \exp^{l-1} \left( \frac{\lambda_{f_i}^{[j]}(g)+3\varepsilon}{r} \right) \right] \geq M_g \left( 4er \right)
\]
i.e.,
\[
\lambda_{f_i}^{[j]}(g) + 3\varepsilon \geq \log^{l-1} \left( 4er \right) M_f^{-1} M_g \left( 4er \right)
\]
i.e.,
\[
\lambda_{f_i}^{[j]}(g) + 3\varepsilon \geq \log^{l-1} \left( 4er \right) M_f^{-1} M_g \left( 4er \right) \frac{\log \left( 4er \right) + O(1)}{\log \left( 4er \right) + O(1)}.
\]

If we let \( \varepsilon \to 0^+ \) then we get
\[
\lambda_{f_i}^{[j]}(g) \geq \lim \inf_{r \to \infty} \frac{\log^{l-1} \left( 4er \right) M_f^{-1} M_g \left( 4er \right)}{\log \left( 4er \right) + O(1)}.
\]
Therefore
\[
\lambda_f^{[j]}(g) = \lim \inf_{r \to \infty} \frac{\log^{l-1} \left( 4er \right) M_f^{-1} M_g \left( r \right)}{\log r} \leq \lambda_{f_i}^{[j]}(g).
\]
So from (6.18) and above, we finally obtain that

\[ \lambda^{[n]}_{f_i}(g) = \lambda^{[n]}_{f_i}(g), \]

if one assume that \( \lambda^{[n]}_{f_i}(g) \neq \lambda^{[n]}_{f_k}(g) \) for all \( k = 1, 2, \ldots, n \) and \( k \neq i \).

Let now \( f = \frac{F_1}{F_2} \) with \( F_1, F_2, f \) as entire and suppose \( \lambda^{[n]}_{F_1}(g) \geq \lambda^{[n]}_{F_2}(g) \). We have \( F_1 = f \cdot F_2 \). Thus \( \lambda^{[n]}_{F_1}(g) = \lambda^{[n]}_{f}(g) \) if \( \lambda^{[n]}_{f}(g) < \lambda^{[n]}_{F_2}(g) \). So it follows that \( \lambda^{[n]}_{F_1}(g) < \lambda^{[n]}_{F_2}(g) \), which contradicts the hypothesis "\( \lambda^{[n]}_{F_1}(g) \geq \lambda^{[n]}_{F_2}(g) \)". Hence \( \lambda^{[n]}_{F_1}(g) = \lambda^{[n]}_{F_2}(g) \geq \lambda^{[n]}_{F_2}(g) = \min \left\{ \lambda^{[n]}_{F_1}(g), \lambda^{[n]}_{F_2}(g) \right\} \). Also suppose that \( \lambda^{[n]}_{F_1}(g) > \lambda^{[n]}_{F_2}(g) \). Then \( \lambda^{[n]}_{F_1}(g) = \min \left\{ \lambda^{[n]}_{f}(g), \lambda^{[n]}_{F_2}(g) \right\} = \lambda^{[n]}_{F_2}(g) \), if \( \lambda^{[n]}_{f}(g) > \lambda^{[n]}_{F_2}(g) \), which is also a contradiction. Thus \( \lambda^{[n]}_{f}(g) = \lambda^{[n]}_{F_1}(g) = \min \left\{ \lambda^{[n]}_{F_1}(g), \lambda^{[n]}_{F_2}(g) \right\} \). This proves the theorem. \( \blacksquare \)

\[ \cdots \Leftrightarrow \otimes \Rightarrow \cdots \]