CHAPTER 3

GROWTH ANALYSIS OF FUNCTIONS
ANALYTIC IN THE UNIT POLYDISC
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FUNCTIONS ANALYTIC IN THE
UNIT POLYDISC

3.1 Introduction, Definitions and Notations.

A function $f$ analytic in the unit disc $U = \{z : |z| < 1\}$ is said to be of finite Nevanlinna order [30] if there exists a number $\mu$ such that the Nevanlinna characteristic function

$$T_f(r) = \frac{1}{2\pi} \int_{0}^{2\pi} \log^+ |f(re^{i\theta})| d\theta$$

satisfies $T_f(r) < (1 - r)^{-\mu}$ for all $r$ in $0 < r_0(\mu) < r < 1$. The greatest lower bound of all such numbers $\mu$ is called the Nevanlinna order of $f$. Thus the Nevanlinna order $\rho_f$ of $f$ is given by

$$\rho_f = \limsup_{r \to 1} \frac{\log T_f(r)}{-\log (1 - r)}.$$

Similarly, the Nevanlinna lower order $\lambda_f$ of $f$ are given by

$$\lambda_f = \liminf_{r \to 1} \frac{\log T_f(r)}{-\log (1 - r)}.$$

L. Bernal introduced the relative order between two entire functions of single variables to avoid comparing growth just with the exponential function $\exp z$. In this connection, Banerjee and Dutta [5] gave the following definition in a unit disc:

The results of this chapter have been accepted for publication and to appear in the International Journal of Analysis and Applications, see [25].
Definition 3.1.1 [5] If \( f \) be analytic in \( U \) and \( g \) be entire, then the relative order of \( f \) with respect to \( g \) denoted by \( \rho_g(f) \) is defined by

\[
\rho_g(f) = \inf \left\{ \mu > 0 : T_f(r) < T_g \left( \left( \frac{1}{1-r} \right)^\mu \right) \right\} \text{ for all } 0 < r_0(\mu) < r < 1.
\]

Similarly, one may define \( \lambda_g(f) \), the relative lower order of \( f \) with respect to \( g \). When \( g(z) = \exp z \), the definition coincides with the definition of Nevanlinna order of \( f \).

Analogously,

\[
\lambda_g(f) = \liminf_{r \to 1} \log \frac{T_g^{-1} T_f(r)}{- \log (1 - r)}.
\]

Extending the notion of single variables to several variables, let \( f(z_1, z_2, \ldots, z_n) \) be a non-constant analytic function of \( n \) complex variables \( z_1, z_2, \ldots, z_{n-1} \) and \( z_n \) in the unit polydisc

\[
U = \{(z_1, z_2, \ldots, z_n) : |z_j| \leq 1, \ j = 1, 2, \ldots, n; \ r_1 > 0, r_2 > 0, \ldots, r_n > 0\}.
\]

Now in the line of Nevanlinna order [30], in this chapter we introduce the generalised \( n \) variables based \( p \)-th Nevanlinna order and the generalised \( n \) variables \( p \)-th Nevanlinna lower order for functions of \( n \) complex variables analytic in a unit polydisc as follows:

\[
\rho_f^p(r_1, r_2, \ldots, r_n) = \limsup_{r_1, r_2, \ldots, r_n \to 1} \frac{\log T_f^p(r_1, r_2, \ldots, r_n)}{- \log \left( (1 - r_1)(1 - r_2)\ldots(1 - r_n) \right)}
\]

and

\[
\lambda_f^p(r_1, r_2, \ldots, r_n) = \liminf_{r_1, r_2, \ldots, r_n \to 1} \frac{\log T_f^p(r_1, r_2, \ldots, r_n)}{- \log \left( (1 - r_1)(1 - r_2)\ldots(1 - r_n) \right)}
\]

where \( \log^k x = \log(\log^{k-1} x) \) for \( k = 1, 2, 3, \ldots \) and \( \log^0 x = x \).

When \( n = p = 1 \), the above definition reduces to the definition of Juneja and Kapoor [30].

Likewise, one may introduce the generalised \( n \) variables based \( p \)-th relative Nevanlinna order (generalised \( n \) variables based \( p \)-th relative Nevanlinna lower order) for functions of \( n \) complex variables analytic in a unit polydisc in the following manner:
Definition 3.1.2 Let $T_f(r_1, r_2, \ldots, r_n)$ denote the Nevanlinna’s characteristic function of $f$ of $n$ variables. The generalised $n$ variables based $p$-th relative Nevanlinna order $\rho_g^{[p]}f(r_1, r_2, \ldots, r_n)$ and generalised $n$ variables based $p$-th relative Nevanlinna lower order $\lambda_g^{[p]}f(r_1, r_2, \ldots, r_n)$ of an analytic function $f$ in $U$ with respect to another entire function $g$ in $n$ complex variables are defined in the following way:

$$
\rho_g^{[p]}f(r_1, r_2, \ldots, r_n) = \limsup_{r_1, r_2 \to \infty} \frac{\log^{[p]} T_g^{-1}T_f(r_1, r_2, \ldots, r_n)}{\log [(1 - r_1)(1 - r_2) \ldots (1 - r_n)]}
$$

and

$$
\lambda_g^{[p]}f(r_1, r_2, \ldots, r_n) = \liminf_{r_1, r_2 \to \infty} \frac{\log^{[p]} T_g^{-1}T_f(r_1, r_2, \ldots, r_n)}{\log [(1 - r_1)(1 - r_2) \ldots (1 - r_n)]}
$$

where $n$ and $p$ are any two positive integers.

If we consider $p = n = 1$ in Definition 3.1.2 then it coincides with Definition 3.1.1.

In this chapter, we establish some results relating to the composition of two non-constant analytic functions, of $n$ complex variables in the unit polydisc $U = \{(z_1, z_2, \ldots, z_n) : |z_j| \leq 1, j = 1, 2, \ldots, n; r_1 > 0, r_2 > 0, \ldots, r_n > 0\}$.

Also we prove a few theorems related to generalised $n$ variables based $p$-th relative Nevanlinna order $\rho_g^{[p]}f(r_1, r_2, \ldots, r_n)$ (generalised $n$ variables based $p$-th relative Nevanlinna lower order $\lambda_g^{[p]}f(r_1, r_2, \ldots, r_n)$) of an analytic function $f$ with respect to an entire function $g$ of $n$ complex variables which are in fact some extensions of earlier results as proved in [16] and [18]. For the detailed theory of functions of several complex variables, one can see [26] and [31].

### 3.2 Theorems

In this section we present the main results of the chapter.

**Theorem 3.2.1** Let $f$ and $g$ be any two non-constant analytic functions of $n$ complex variables in the unit polydisc $U$ such that $0 < \lambda^{[p]}_{fg}(r_1, r_2, \ldots, r_n) \leq$
\[ \rho_{f\circ g}^{[p]}(r_1, r_2, \ldots, r_n) < \infty \text{ and } 0 < \lambda_g^{[q]}(r_1, r_2, \ldots, r_n) \leq \rho_{g}^{[q]}(r_1, r_2, \ldots, r_n) < \infty. \]

Then

\[
\frac{\lambda_{f\circ g}^{[p]}(r_1, r_2, \ldots, r_n)}{\rho_{g}^{[q]}(r_1, r_2, \ldots, r_n)} \leq \liminf_{r_1, r_2, \ldots, r_n \to 1} \frac{\log^{[p]} T_{f\circ g}(r_1, r_2, \ldots, r_n)}{\log^{[q]} T_{g}(r_1, r_2, \ldots, r_n)} \leq \frac{\lambda_{f\circ g}^{[p]}(r_1, r_2, \ldots, r_n)}{\lambda_{g}^{[q]}(r_1, r_2, \ldots, r_n)}
\]

\[
\leq \limsup_{r_1, r_2, \ldots, r_n \to 1} \frac{\log^{[p]} T_{f\circ g}(r_1, r_2, \ldots, r_n)}{\log^{[q]} T_{g}(r_1, r_2, \ldots, r_n)} \leq \frac{\rho_{f\circ g}^{[p]}(r_1, r_2, \ldots, r_n)}{\lambda_{g}^{[q]}(r_1, r_2, \ldots, r_n)}
\]

where \( p \) and \( q \) are any two positive integers.

**Proof.** From the definition of generalised \( n \) variables based \( p \)-th Nevanlinna order and generalised \( n \) variables based \( p \)-th Nevanlinna lower order of analytic functions in the unit polydisc \( U \), we have for arbitrary positive \( \epsilon \) and for all sufficiently large values of \( (\frac{1}{1-r_1}), (\frac{1}{1-r_2}), \ldots \) and \( (\frac{1}{1-r_n}) \) that

\[
\log^{[p]} T_{f\circ g}(r_1, r_2, \ldots, r_n) \\
\geq \left( \lambda_{f\circ g}^{[p]}(r_1, r_2, \ldots, r_n) - \epsilon \right) \left[ - \log [(1 - r_1)(1 - r_2)\ldots(1 - r_n)] \right] \quad (3.1)
\]

and

\[
\log^{[q]} T_{g}(r_1, r_2, \ldots, r_n) \\
\leq \left( \rho_{g}^{[q]}(r_1, r_2, \ldots, r_n) + \epsilon \right) \left[ - \log [(1 - r_1)(1 - r_2)\ldots(1 - r_n)] \right]. \quad (3.2)
\]

Now from (3.1) and (3.2), it follows for all sufficiently large values of \( (\frac{1}{1-r_1}), (\frac{1}{1-r_2}), \ldots \) and \( (\frac{1}{1-r_n}) \) that

\[
\frac{\log^{[p]} T_{f\circ g}(r_1, r_2, \ldots, r_n)}{\log^{[q]} T_{g}(r_1, r_2, \ldots, r_n)} \geq \frac{\lambda_{f\circ g}^{[p]}(r_1, r_2, \ldots, r_n) - \epsilon}{\rho_{g}^{[q]}(r_1, r_2, \ldots, r_n) + \epsilon}. \quad (3.3)
\]

As \( \epsilon (> 0) \) is arbitrary, we obtain that

\[
\liminf_{r_1, r_2, \ldots, r_n \to 1} \frac{\log^{[p]} T_{f\circ g}(r_1, r_2, \ldots, r_n)}{\log^{[q]} T_{g}(r_1, r_2, \ldots, r_n)} \leq \frac{\lambda_{f\circ g}^{[p]}(r_1, r_2, \ldots, r_n)}{\rho_{g}^{[q]}(r_1, r_2, \ldots, r_n)}. \quad (3.4)
\]

Again for a sequence of values of \( (\frac{1}{1-r_1}), (\frac{1}{1-r_2}), \ldots \) and \( (\frac{1}{1-r_n}) \) tending to infinity,

\[
\log^{[p]} T_{f\circ g}(r_1, r_2, \ldots, r_n) \\
\leq \left( \lambda_{f\circ g}^{[p]}(r_1, r_2, \ldots, r_n) + \epsilon \right) \left[ - \log [(1 - r_1)(1 - r_2)\ldots(1 - r_n)] \right] \quad (3.4)
\]
and for all sufficiently large values of \((\frac{1}{1-r_1}), (\frac{1}{1-r_2}), \ldots \text{ and } (\frac{1}{1-r_n})\),

\[
\log^q T_g (r_1, r_2, \ldots, r_n) \\
\geq \left( \lambda^q_g (r_1, r_2, \ldots, r_n) - \epsilon \right) \left[ - \log [(1 - r_1) (1 - r_2) \ldots (1 - r_n)] \right]. \tag{3.5}
\]

So combining (3.4) and (3.5), we get for a sequence of values of \((\frac{1}{1-r_1}), (\frac{1}{1-r_2}), \ldots \text{ and } (\frac{1}{1-r_n})\) tending to infinity that

\[
\frac{\log^q T_{fog} (r_1, r_2, \ldots, r_n)}{\log^q T_g (r_1, r_2, \ldots, r_n)} \leq \frac{\lambda^p_{fog} (r_1, r_2, \ldots, r_n) + \epsilon}{\lambda^q_g (r_1, r_2, \ldots, r_n) - \epsilon}.
\]

Since \(\epsilon (> 0)\) is arbitrary, it follows that

\[
\liminf_{r_1, r_2, \ldots, r_n \to 1} \frac{\log^q T_{fog} (r_1, r_2, \ldots, r_n)}{\log^q T_g (r_1, r_2, \ldots, r_n)} \leq \frac{\lambda^p_{fog} (r_1, r_2, \ldots, r_n)}{\lambda^q_g (r_1, r_2, \ldots, r_n)}.
\tag{3.6}
\]

Also for a sequence of values of \((\frac{1}{1-r_1}), (\frac{1}{1-r_2}), \ldots \text{ and } (\frac{1}{1-r_n})\) tending to infinity, we get that

\[
\log^q T_g (r_1, r_2, \ldots, r_n) \\
\leq \left( \lambda^q_g (r_1, r_2, \ldots, r_n) + \epsilon \right) \left[ - \log [(1 - r_1) (1 - r_2) \ldots (1 - r_n)] \right]. \tag{3.7}
\]

Now from (3.1) and (3.7), we obtain for a sequence of values of \((\frac{1}{1-r_1}), (\frac{1}{1-r_2}), \ldots \text{ and } (\frac{1}{1-r_n})\) tending to infinity that

\[
\frac{\log^p T_{fog} (r_1, r_2, \ldots, r_n)}{\log^q T_g (r_1, r_2, \ldots, r_n)} \geq \frac{\lambda^p_{fog} (r_1, r_2, \ldots, r_n) - \epsilon}{\lambda^q_g (r_1, r_2, \ldots, r_n) + \epsilon}.
\]

Choosing \(\epsilon \to 0\), we get that

\[
\limsup_{r_1, r_2, \ldots, r_n \to 1} \frac{\log^p T_{fog} (r_1, r_2, \ldots, r_n)}{\log^q T_g (r_1, r_2, \ldots, r_n)} \geq \frac{\lambda^p_{fog} (r_1, r_2, \ldots, r_n)}{\lambda^q_g (r_1, r_2, \ldots, r_n)}.
\tag{3.8}
\]

Also for all sufficiently large values of \((\frac{1}{1-r_1}), (\frac{1}{1-r_2}), \ldots \text{ and } (\frac{1}{1-r_n})\),

\[
\log^p T_{fog} (r_1, r_2, \ldots, r_n) \\
\leq \left( \rho^p_{fog} (r_1, r_2, \ldots, r_n) + \epsilon \right) \left[ - \log [(1 - r_1) (1 - r_2) \ldots (1 - r_n)] \right]. \tag{3.9}
\]
So from (3.5) and (3.9), it follows for all sufficiently large values of \( \left( \frac{1}{1-r_1} \right) \), 
\( \left( \frac{1}{1-r_2} \right) \), ... and \( \left( \frac{1}{1-r_n} \right) \) that
\[
\frac{\log^{[p]} T_{fog} (r_1, r_2, \ldots, r_n)}{\log^{[q]} T_g (r_1, r_2, \ldots, r_n)} \leq \frac{\rho^{[p]}_{fog} (r_1, r_2, \ldots, r_n) + \varepsilon}{\lambda^{[q]}_g (r_1, r_2, \ldots, r_n) - \varepsilon}.
\]
As \( \varepsilon (\varepsilon > 0) \) is arbitrary, we obtain that
\[
\limsup_{r_1, r_2, \ldots, r_n \to 1} \frac{\log^{[p]} T_{fog} (r_1, r_2, \ldots, r_n)}{\log^{[q]} T_g (r_1, r_2, \ldots, r_n)} \leq \frac{\rho^{[p]}_{fog} (r_1, r_2, \ldots, r_n)}{\lambda^{[q]}_g (r_1, r_2, \ldots, r_n)}.
\]
Thus the theorem follows from (3.3), (3.6), (3.8) and (3.10).

**Remark 3.2.1** If we take \( n = p = 1 \) then Theorem 3.2.1 reduces to Theorem 1 of Datta and Deb \{cf. [16]\} and if we take \( n = 1 \) then Theorem 3.2.1 reduces to a result of Datta and Jerin \{cf. [18]\}.

The following theorem can be proved in the line of Theorem 3.2.1 and so its proof is omitted.

**Theorem 3.2.2** Let \( f \) and \( g \) be any two non-constant analytic functions of \( n \) complex variables in the unit polydisc \( U \) with \( 0 < \lambda^{[p]}_f (r_1, r_2, \ldots, r_n) \leq \rho^{[p]}_{fog} (r_1, r_2, \ldots, r_n) < \infty \) and \( 0 < \lambda^{[l]}_f (r_1, r_2, \ldots, r_n) \leq \rho^{[l]}_{fog} (r_1, r_2, \ldots, r_n) < \infty \) where \( p \) and \( l \) are any two positive integers. Then
\[
\frac{\lambda^{[p]}_{fog} (r_1, r_2, \ldots, r_n)}{\rho^{[l]}_f (r_1, r_2, \ldots, r_n)} \leq \liminf_{r_1, r_2, \ldots, r_n \to 1} \frac{\log^{[p]} T_{fog} (r_1, r_2, \ldots, r_n)}{\log^{[l]} T_f (r_1, r_2, \ldots, r_n)} \leq \limsup_{r_1, r_2, \ldots, r_n \to 1} \frac{\log^{[p]} T_{fog} (r_1, r_2, \ldots, r_n)}{\log^{[l]} T_f (r_1, r_2, \ldots, r_n)} \leq \frac{\lambda^{[p]}_{fog} (r_1, r_2, \ldots, r_n)}{\rho^{[l]}_f (r_1, r_2, \ldots, r_n)}.
\]

**Theorem 3.2.3** Let \( f \) and \( g \) be any two non-constant analytic functions of \( n \) complex variables in the unit polydisc \( U \) such that \( 0 < \rho^{[p]}_{fog} (r_1, r_2, \ldots, r_n) < \infty \) and \( 0 < \rho^{[l]}_g (r_1, r_2, \ldots, r_n) < \infty \). Then
\[
\liminf_{r_1, r_2, \ldots, r_n \to 1} \frac{\log^{[p]} T_{fog} (r_1, r_2, \ldots, r_n)}{\log^{[l]} T_g (r_1, r_2, \ldots, r_n)} \leq \frac{\rho^{[p]}_{fog} (r_1, r_2, \ldots, r_n)}{\rho^{[q]}_g (r_1, r_2, \ldots, r_n)} \leq \limsup_{r_1, r_2, \ldots, r_n \to 1} \frac{\log^{[p]} T_{fog} (r_1, r_2, \ldots, r_n)}{\log^{[l]} T_g (r_1, r_2, \ldots, r_n)}
\]
where \( p \) and \( q \) are any two positive integers.
**Proof.** From the definition of generalised $n$ variables based $p$-th Nevanlinna order, we get for a sequence of values of \( \left( \frac{1}{1-r_1} \right) \), \( \left( \frac{1}{1-r_2} \right) \), ... and \( \left( \frac{1}{1-r_n} \right) \) tending to infinity that
\[
\log^{[q]} T_g (r_1, r_2, ..., r_n) \geq \left( \rho_g^{[q]} (r_1, r_2, ..., r_n) - \varepsilon \right) \left[ - \log [ (1 - r_1) (1 - r_2) ... (1 - r_n)] \right].
\] (3.11)

Now from (3.9) and (3.11), it follows for a sequence of values of \( \left( \frac{1}{1-r_1} \right) \), \( \left( \frac{1}{1-r_2} \right) \), ... and \( \left( \frac{1}{1-r_n} \right) \) tending to infinity that
\[
\log^{[p]} T_{fog} (r_1, r_2, ..., r_n) \leq \frac{\log^{[p]} T_g (r_1, r_2, ..., r_n)}{\log^{[q]} T_g (r_1, r_2, ..., r_n)} \leq \frac{\rho_{fog}^{[p]} (r_1, r_2, ..., r_n)}{\rho_g^{[q]} (r_1, r_2, ..., r_n)} + \varepsilon.
\] (3.12)

As \( \varepsilon (>0) \) is arbitrary, we obtain that
\[
\liminf_{r_1, r_2, ..., r_n \to 1} \log^{[p]} T_{fog} (r_1, r_2, ..., r_n) \leq \frac{\rho_{fog}^{[p]} (r_1, r_2, ..., r_n)}{\rho_g^{[q]} (r_1, r_2, ..., r_n)}.
\] (3.13)

Again for a sequence of values of \( \left( \frac{1}{1-r_1} \right) \), \( \left( \frac{1}{1-r_2} \right) \), ... and \( \left( \frac{1}{1-r_n} \right) \) tending to infinity,
\[
\log^{[p]} T_{fog} (r_1, r_2, ..., r_n) \geq \left( \rho_{fog}^{[p]} (r_1, r_2, ..., r_n) - \varepsilon \right) \left[ - \log [ (1 - r_1) (1 - r_2) ... (1 - r_n)] \right].
\] (3.14)

So combining (3.2) and (3.13), we get for a sequence of values of \( \left( \frac{1}{1-r_1} \right) \), \( \left( \frac{1}{1-r_2} \right) \), ... and \( \left( \frac{1}{1-r_n} \right) \) tending to infinity that
\[
\log^{[p]} T_{fog} (r_1, r_2, ..., r_n) \geq \frac{\log^{[p]} T_{fog} (r_1, r_2, ..., r_n)}{\log^{[q]} T_g (r_1, r_2, ..., r_n)} \geq \frac{\rho_{fog}^{[p]} (r_1, r_2, ..., r_n)}{\rho_g^{[q]} (r_1, r_2, ..., r_n)} - \varepsilon.
\] (3.14)

Since \( \varepsilon (>0) \) is arbitrary, it follows that
\[
\limsup_{r_1, r_2, ..., r_n \to 1} \log^{[p]} T_{fog} (r_1, r_2, ..., r_n) \geq \frac{\rho_{fog}^{[p]} (r_1, r_2, ..., r_n)}{\rho_g^{[q]} (r_1, r_2, ..., r_n)}.
\] (3.14)

Thus the theorem follows from (3.12) and (3.14).
Remark 3.2.2 Taking \( n = p = 1 \), Theorem 3.2.3 reduces to Theorem 2 of Datta and Deb \( \text{cf. [16]} \) and for \( n = 1 \), Theorem 3.2.3 reduces to a result of Datta and Jerin \( \text{cf. [18]} \).

The following theorem can be carried out in the line of Theorem 3.2.3 and therefore we omit its proof.

**Theorem 3.2.4** Let \( f \) and \( g \) be any two non-constant analytic functions of \( n \) complex variables in the unit polydisc \( U \) with \( 0 < \rho_{(r_1,r_2,...,r_n)}^{[p]} < \infty \) and \( 0 < \rho_{(r_1,r_2,...,r_n)}^{[l]} < \infty \) where \( p \) and \( l \) are any two positive integers. Then

\[
\liminf_{r_1,r_2,...,r_n \to 1} \frac{\log^{[p]} T_{fog} (r_1, r_2, ..., r_n)}{\log^{[l]} T_f (r_1, r_2, ..., r_n)} \leq \frac{\rho_{(r_1,r_2,...,r_n)}^{[p]}}{\rho_{(r_1,r_2,...,r_n)}^{[l]}} \leq \limsup_{r_1,r_2,...,r_n \to 1} \frac{\log^{[p]} T_{fog} (r_1, r_2, ..., r_n)}{\log^{[l]} T_f (r_1, r_2, ..., r_n)}.
\]

The following theorem is a natural consequence of Theorem 3.2.1 and Theorem 3.2.3.

**Theorem 3.2.5** Let \( f \) and \( g \) be any two non-constant analytic functions of \( n \) complex variables in the unit polydisc \( U \) such that \( 0 < \lambda_{(r_1,r_2,...,r_n)}^{[p]} \leq \rho_{(r_1,r_2,...,r_n)}^{[p]} < \infty \) and \( 0 < \lambda_{(r_1,r_2,...,r_n)}^{[q]} \leq \rho_{(r_1,r_2,...,r_n)}^{[q]} < \infty \). Then

\[
\liminf_{r_1,r_2,...,r_n \to 1} \frac{\log^{[p]} T_{fog} (r_1, r_2, ..., r_n)}{\log^{[q]} T_g (r_1, r_2, ..., r_n)} \leq \min \left\{ \frac{\lambda_{(r_1,r_2,...,r_n)}^{[p]}}{\lambda_{(r_1,r_2,...,r_n)}^{[q]}}, \frac{\rho_{(r_1,r_2,...,r_n)}^{[p]}}{\rho_{(r_1,r_2,...,r_n)}^{[q]}}, \frac{\lambda_{(r_1,r_2,...,r_n)}^{[p]}}{\rho_{(r_1,r_2,...,r_n)}^{[q]}}, \frac{\rho_{(r_1,r_2,...,r_n)}^{[p]}}{\lambda_{(r_1,r_2,...,r_n)}^{[q]}} \right\} \leq \max \left\{ \frac{\lambda_{(r_1,r_2,...,r_n)}^{[p]}}{\lambda_{(r_1,r_2,...,r_n)}^{[q]}}, \frac{\rho_{(r_1,r_2,...,r_n)}^{[p]}}{\rho_{(r_1,r_2,...,r_n)}^{[q]}}, \frac{\lambda_{(r_1,r_2,...,r_n)}^{[p]}}{\rho_{(r_1,r_2,...,r_n)}^{[q]}}, \frac{\rho_{(r_1,r_2,...,r_n)}^{[p]}}{\lambda_{(r_1,r_2,...,r_n)}^{[q]}} \right\} \leq \limsup_{r_1,r_2,...,r_n \to 1} \frac{\log^{[p]} T_{fog} (r_1, r_2, ..., r_n)}{\log^{[q]} T_g (r_1, r_2, ..., r_n)}
\]

where \( p \) and \( q \) are any two positive integers.
Remark 3.2.3 Considering \( n = p = 1 \), one can see that Theorem 3.2.5 reduces to Theorem 3 of Datta and Deb \( \text{cf. [16]} \) and also for \( n = 1 \), Theorem 3.2.5 reduces to a result of Datta and Jerin \( \text{cf. [18]} \).

Analogously one may state the following theorem without its proof.

**Theorem 3.2.6** Let \( f \) and \( g \) be any two non-constant analytic functions of \( n \) complex variables in the unit polydisc \( U \) with \( 0 < \lambda_{fog}^{[p]} (r_1, r_2, ..., r_n) \leq \rho_{fog}^{[p]} (r_1, r_2, ..., r_n), \lambda_f^{[l]} (r_1, r_2, ..., r_n) \leq \rho_f^{[l]} (r_1, r_2, ..., r_n) \leq \rho_{fog}^{[l]} (r_1, r_2, ..., r_n) < \infty \) where \( p \) and \( l \) are any two positive integers. Then

\[
\liminf_{r_1, r_2, ..., r_n \to 1} \frac{\log^{[p]} T_{fog} (r_1, r_2, ..., r_n)}{\log^{[l]} T_f (r_1, r_2, ..., r_n)} \leq \min \left\{ \frac{\lambda_{fog}^{[p]} (r_1, r_2, ..., r_n)}{\rho_f^{[l]} (r_1, r_2, ..., r_n)}, \frac{\rho_{fog}^{[p]} (r_1, r_2, ..., r_n)}{\rho_f^{[l]} (r_1, r_2, ..., r_n)} \right\} \leq \max \left\{ \frac{\lambda_f^{[l]} (r_1, r_2, ..., r_n)}{\rho_{fog}^{[l]} (r_1, r_2, ..., r_n)}, \frac{\rho_f^{[l]} (r_1, r_2, ..., r_n)}{\rho_{fog}^{[l]} (r_1, r_2, ..., r_n)} \right\} \leq \limsup_{r_1, r_2, ..., r_n \to 1} \frac{\log^{[p]} T_{fog} (r_1, r_2, ..., r_n)}{\log^{[l]} T_f (r_1, r_2, ..., r_n)}.
\]

**Theorem 3.2.7** Let \( f \) and \( g \) be any two non-constant analytic functions of \( n \) complex variables in the unit polydisc \( U \) such that \( \rho_f^{[l]} (r_1, r_2, ..., r_n) < \infty \) and \( \lambda_{fog}^{[p]} (r_1, r_2, ..., r_n) = \infty \). Then

\[
\lim_{r_1, r_2, ..., r_n \to 1} \frac{\log^{[p]} T_{fog} (r_1, r_2, ..., r_n)}{\log^{[l]} T_f (r_1, r_2, ..., r_n)} = \infty
\]

where \( p \) and \( l \) are any two positive integers.

**Proof.** Let us suppose that the conclusion of the theorem do not hold. Then we can find a constant \( \beta > 0 \) such that for a sequence of values of \( \left( \frac{1}{1-r_1} \right), \left( \frac{1}{1-r_2} \right), \ldots \) and \( \left( \frac{1}{1-r_n} \right) \) tending to infinity,

\[
\log^{[p]} T_{fog} (r_1, r_2, ..., r_n) \leq \beta \log^{[l]} T_f (r_1, r_2, ..., r_n). \tag{3.15}
\]
Again from the definition of \( \rho_f^{[l]} (r_1, r_2, ..., r_n) \), it follows for all sufficiently large values of \( \left( \frac{1}{1-r_1} \right), \left( \frac{1}{1-r_2} \right), \ldots \) and \( \left( \frac{1}{1-r_n} \right) \) that
\[
\log^{[l]} T_f (r_1, r_2, ..., r_n) \\
\leq \left[ \rho_f^{[l]} (r_1, r_2, ..., r_n) + \epsilon \right] \left[ - \log \left( (1 - r_1) (1 - r_2) ... (1 - r_n) \right) \right].
\] (3.16)

Thus from (3.15) and (3.16), we have for a sequence of values of \( \left( \frac{1}{1-r_1} \right), \left( \frac{1}{1-r_2} \right), \ldots \) and \( \left( \frac{1}{1-r_n} \right) \) tending to infinity that
\[
\log^p T_{fog} (r_1, r_2, ..., r_n) \\
\leq \beta \left[ \rho_f^{[l]} (r_1, r_2, ..., r_n) + \epsilon \right] \left[ - \log \left( (1 - r_1) (1 - r_2) ... (1 - r_n) \right) \right]
\]
i.e.,
\[
\frac{\log^p T_{fog} (r_1, r_2, ..., r_n)}{\left[ - \log \left( (1 - r_1) (1 - r_2) ... (1 - r_n) \right) \right]} \\
\leq \frac{\beta \left[ \rho_f^{[l]} (r_1, r_2, ..., r_n) + \epsilon \right] \left[ - \log \left( (1 - r_1) (1 - r_2) ... (1 - r_n) \right) \right]}{\left[ - \log \left( (1 - r_1) (1 - r_2) ... (1 - r_n) \right) \right]}
\]
i.e.,
\[
\liminf_{r_1, r_2, ... r_n \to 1} \frac{\log^p T_{fog} (r_1, r_2, ..., r_n)}{\left[ - \log \left( (1 - r_1) (1 - r_2) ... (1 - r_n) \right) \right]} = \lambda_{fog}^{[p]} (r_1, r_2, ..., r_n) < \infty.
\]

This is a contradiction.

Hence the theorem follows. □

**Remark 3.2.4** Theorem 3.2.7 is also valid with “limit superior” instead of “limit” if \( \lambda_{fog}^{[p]} (r_1, r_2, ..., r_n) = \infty \) is replaced by \( \rho_{fog}^{[p]} (r_1, r_2, ..., r_n) = \infty \) and the other conditions remain the same.

**Corollary 3.2.1** Under the assumptions of Theorem 3.2.7 and Remark 3.2.4,
\[
\lim_{r_1, r_2, ... r_n \to 1} \frac{T_{fog} (r_1, r_2, ..., r_n)}{T_f (r_1, r_2, ..., r_n)} = \infty \hspace{5mm} \text{and} \hspace{5mm} \limsup_{r_1, r_2, ... r_n \to 1} \frac{T_{fog} (r_1, r_2, ..., r_n)}{T_f (r_1, r_2, ..., r_n)} = \infty
\]
respectively hold if \( p = l \).

The proof is omitted.

Analogously one may also state the following theorem and corollaries without their proofs as those may be carried out in the line of Remark 3.2.4, Theorem 3.2.7 and Corollary 3.2.1 respectively.
Theorem 3.2.8 Let \( f \) and \( g \) be any two non-constant analytic functions of \( n \) complex variables in the unit polydisc \( U \) with \( \rho^{[q]}_g(r_1, r_2, \ldots, r_n) < \infty \) and \( \rho^{[p]}_{fog}(r_1, r_2, \ldots, r_n) = \infty \) where \( p \) and \( q \) are any two positive integers. Then

\[
\limsup_{r_1, r_2, \ldots, r_n \to 1} \frac{\log^{[p]} T_{fog}(r_1, r_2, \ldots, r_n)}{\log^{[q]} T_g(r_1, r_2, \ldots, r_n)} = \infty.
\]

Corollary 3.2.2 Theorem 3.2.8 is also valid with “limit” instead of “limit superior” if \( \rho^{[p]}_{fog}(r_1, r_2, \ldots, r_n) = \infty \) is replaced by \( \lambda^{[p]}_{fog}(r_1, r_2, \ldots, r_n) = \infty \) and the other conditions remain the same.

Corollary 3.2.3 Under the assumptions of Theorem 3.2.7 and Corollary 3.2.2,

\[
\limsup_{r_1, r_2, \ldots, r_n \to 1} \frac{T_{fog}(r_1, r_2, \ldots, r_n)}{T_g(r_1, r_2, \ldots, r_n)} = \infty \quad \text{and} \quad \lim_{r_1, r_2, \ldots, r_n \to 1} \frac{T_{fog}(r_1, r_2, \ldots, r_n)}{T_g(r_1, r_2, \ldots, r_n)} = \infty
\]

respectively hold if \( p = q \).

In the next three theorems we establish some comparative growth properties related to the generalised \( n \) variables based \( p \)-th relative Nevanlinna order (generalised \( n \) variables based \( p \)-th relative Nevanlinna lower order) of an analytic function with respect to an entire function in the unit polydisc \( U \).

Theorem 3.2.9 Let \( f, h \) be any two analytic functions of \( n \) complex variables in \( U \) and \( g \) be entire in \( n \) complex variables such that \( 0 < \lambda^{[p]f}_g(r_1, r_2, \ldots, r_n) \leq \rho^{[p]}_g(r_1, r_2, \ldots, r_n) < \infty \) and \( 0 < \lambda^{[p]h}_g(r_1, r_2, \ldots, r_n) \leq \rho^{[p]}_g(r_1, r_2, \ldots, r_n) < \infty \). Then

\[
\frac{\lambda^{[p]f}_g(r_1, r_2, \ldots, r_n)}{\rho^{[p]}_g(r_1, r_2, \ldots, r_n)} \leq \liminf_{r_1, r_2, \ldots, r_n \to 1} \frac{\log^{[p]} T_g^{-1} T_f(r_1, r_2, \ldots, r_n)}{\log^{[p]} T_g^{-1} T_h(r_1, r_2, \ldots, r_n)} \leq \frac{\lambda^{[p]f}_g(r_1, r_2, \ldots, r_n)}{\lambda^{[p]h}_g(r_1, r_2, \ldots, r_n)}
\]

\[
\leq \limsup_{r_1, r_2, \ldots, r_n \to 1} \frac{\log^{[p]} T_g^{-1} T_f(r_1, r_2, \ldots, r_n)}{\log^{[p]} T_g^{-1} T_h(r_1, r_2, \ldots, r_n)} \leq \frac{\rho^{[p]f}_g(r_1, r_2, \ldots, r_n)}{\lambda^{[p]h}_g(r_1, r_2, \ldots, r_n)}
\]

where \( p \) is any positive integer.

Proof. From the definition of generalised \( n \) variables based \( p \)-th relative Nevanlinna order and generalised \( n \) variables based \( p \)-th relative Nevanlinna
lower order of an analytic function with respect to an entire function in an
unit polydisc \( U \), we have for arbitrary positive \( \epsilon \) and for all sufficiently large
values of \( \left( \frac{1}{1-r_1} \right), \left( \frac{1}{1-r_2} \right), \ldots \) and \( \left( \frac{1}{1-r_n} \right) \) that

\[
\log^{[p]} T_f^{-1}(r_1, r_2, \ldots, r_n) \\
\geq \left[ \lambda_g^{[p]} f (r_1, r_2, \ldots, r_n) - \epsilon \right] \left[ - \log [(1 - r_1)(1 - r_2) \ldots (1 - r_n)] \right] \tag{3.17}
\]

and

\[
\log^{[p]} T_h^{-1}(r_1, r_2, \ldots, r_n) \\
\leq \left[ \rho_g^{[p]} h (r_1, r_2, \ldots, r_n) + \epsilon \right] \left[ - \log [(1 - r_1)(1 - r_2) \ldots (1 - r_n)] \right]. \tag{3.18}
\]

Now from (3.17) and (3.18), it follows for all sufficiently large values of
\( \left( \frac{1}{1-r_1} \right), \left( \frac{1}{1-r_2} \right), \ldots \) and \( \left( \frac{1}{1-r_n} \right) \) that

\[
\frac{\log^{[p]} T_f^{-1}(r_1, r_2, \ldots, r_n)}{\log^{[p]} T_h^{-1}(r_1, r_2, \ldots, r_n)} \geq \frac{\lambda_g^{[p]} f (r_1, r_2, \ldots, r_n) - \epsilon}{\rho_g^{[p]} h (r_1, r_2, \ldots, r_n) + \epsilon}.
\]

As \( \epsilon (> 0) \) is arbitrary, we obtain that

\[
\liminf_{r_1, r_2, \ldots, r_n \to \infty} \frac{\log^{[p]} T_f^{-1}(r_1, r_2, \ldots, r_n)}{\log^{[p]} T_h^{-1}(r_1, r_2, \ldots, r_n)} \geq \frac{\lambda_g^{[p]} f (r_1, r_2, \ldots, r_n)}{\rho_g^{[p]} h (r_1, r_2, \ldots, r_n)}. \tag{3.19}
\]

Again we have for a sequence of values of \( \left( \frac{1}{1-r_1} \right), \left( \frac{1}{1-r_2} \right), \ldots \) and \( \left( \frac{1}{1-r_n} \right) \)
tending to infinity that

\[
\log^{[p]} T_f^{-1}(r_1, r_2, \ldots, r_n) \leq \left[ \lambda_g^{[p]} f (r_1, r_2, \ldots, r_n) + \epsilon \right] \left[ - \log (1 - r) \right] \tag{3.20}
\]

and for all sufficiently large values of \( \left( \frac{1}{1-r_1} \right), \left( \frac{1}{1-r_2} \right), \ldots \) and \( \left( \frac{1}{1-r_n} \right) \),

\[
\log^{[p]} T_h^{-1}(r_1, r_2, \ldots, r_n) \\
\geq \left[ \lambda_g^{[p]} h (r_1, r_2, \ldots, r_n) - \epsilon \right] \left[ - \log [(1 - r_1)(1 - r_2) \ldots (1 - r_n)] \right]. \tag{3.21}
\]

So combining (3.20) and (3.21), we get for a sequence of values of \( \left( \frac{1}{1-r_1} \right), \left( \frac{1}{1-r_2} \right), \ldots \) and \( \left( \frac{1}{1-r_n} \right) \), tending to infinity that

\[
\frac{\log^{[p]} T_f^{-1}(r_1, r_2, \ldots, r_n)}{\log^{[p]} T_h^{-1}(r_1, r_2, \ldots, r_n)} \leq \frac{\lambda_g^{[p]} f (r_1, r_2, \ldots, r_n) + \epsilon}{\lambda_g^{[p]} h (r_1, r_2, \ldots, r_n) - \epsilon}.
\]
Since \( \epsilon (>0) \) is arbitrary, it follows that
\[
\liminf_{r_1, r_2, \ldots, r_n \to 1} \frac{\log^{[p]} T_g^{-1} T_f(r_1, r_2, \ldots, r_n)}{\log^{[p]} T_g^{-1} T_h(r_1, r_2, \ldots, r_n)} \leq \frac{\lambda_g^{[p]f}(r_1, r_2, \ldots, r_n)}{\lambda_g^{[p]h}(r_1, r_2, \ldots, r_n)}. \tag{3.22}
\]
Also for a sequence of values of \( \left( \frac{1}{1-r_1} \right), \left( \frac{1}{1-r_2} \right), \ldots \) and \( \left( \frac{1}{1-r_n} \right) \) tending to infinity,
\[
\log^{[p]} T_g^{-1} T_h(r_1, r_2, \ldots, r_n)
\leq \left[ \lambda_g^{[p]h}(r_1, r_2, \ldots, r_n) + \epsilon \right] \left[ - \log [(1 - r_1)(1 - r_2) \ldots (1 - r_n)] \right]. \tag{3.23}
\]
Now from (3.17) and (3.23), we obtain for a sequence of values of \( \left( \frac{1}{1-r_1} \right), \left( \frac{1}{1-r_2} \right), \ldots \) and \( \left( \frac{1}{1-r_n} \right) \), tending to infinity that
\[
\frac{\log^{[p]} T_g^{-1} T_f(r_1, r_2, \ldots, r_n)}{\log^{[p]} T_g^{-1} T_h(r_1, r_2, \ldots, r_n)} \geq \frac{\lambda_g^{[p]f}(r_1, r_2, \ldots, r_n) - \epsilon}{\lambda_g^{[p]h}(r_1, r_2, \ldots, r_n) + \epsilon}.
\]
Choosing \( \epsilon (>0) \), we get that
\[
\limsup_{r_1, r_2, \ldots, r_n \to 1} \frac{\log^{[p]} T_g^{-1} T_f(r_1, r_2, \ldots, r_n)}{\log^{[p]} T_g^{-1} T_h(r_1, r_2, \ldots, r_n)} \geq \frac{\lambda_g^{[p]f}(r_1, r_2, \ldots, r_n)}{\lambda_g^{[p]h}(r_1, r_2, \ldots, r_n)}. \tag{3.24}
\]
Also for all sufficiently large values of \( \left( \frac{1}{1-r_1} \right), \left( \frac{1}{1-r_2} \right), \ldots \) and \( \left( \frac{1}{1-r_n} \right) \),
\[
\log^{[p]} T_g^{-1} T_f(r_1, r_2, \ldots, r_n)
\leq \left[ \rho_g^{[p]f}(r_1, r_2, \ldots, r_n) + \epsilon \right] \left[ - \log [(1 - r_1)(1 - r_2) \ldots (1 - r_n)] \right]. \tag{3.25}
\]
So from (3.21) and (3.25), it follows for all sufficiently large values of \( \left( \frac{1}{1-r_1} \right), \left( \frac{1}{1-r_2} \right), \ldots \) and \( \left( \frac{1}{1-r_n} \right) \) that
\[
\frac{\log^{[p]} T_g^{-1} T_f(r_1, r_2, \ldots, r_n)}{\log^{[p]} T_g^{-1} T_h(r_1, r_2, \ldots, r_n)} \leq \frac{\rho_g^{[p]f}(r_1, r_2, \ldots, r_n) + \epsilon}{\lambda_g^{[p]h}(r_1, r_2, \ldots, r_n) - \epsilon}.
\]
As \( \epsilon (>0) \) is arbitrary, we obtain from above that
\[
\limsup_{r_1, r_2, \ldots, r_n \to 1} \frac{\log^{[p]} T_g^{-1} T_f(r_1, r_2, \ldots, r_n)}{\log^{[p]} T_g^{-1} T_h(r_1, r_2, \ldots, r_n)} \leq \frac{\rho_g^{[p]f}(r_1, r_2, \ldots, r_n)}{\lambda_g^{[p]h}(r_1, r_2, \ldots, r_n)}. \tag{3.26}
\]
Thus the theorem follows from (3.19), (3.22), (3.24) and (3.26).
Remark 3.2.5 In fact, Theorem 3.2.9 is a generalization of Theorem 4 of Datta and Deb {cf. [16]} for \( n = p = 1 \) and also the same of a result of Datta and Jerin {cf. [18]} for \( n = 1 \).

Theorem 3.2.10 Let \( f, h \) be any two analytic functions of \( n \) complex variables in \( U \) and \( g \) be entire in \( n \) complex variables with \( 0 < \rho_g^{[p]}(r_1, r_2, ..., r_n) < \infty \) and \( 0 < \rho_g^{[p]}(r_1, r_2, ..., r_n) < \infty \) where \( p \) is any positive integer. Then

\[
\liminf_{r_1, r_2, ..., r_n \to 1} \frac{\log^{[p]} T^{-1}_g T_f(r_1, r_2, ..., r_n)}{\log^{[p]} T^{-1}_g T_h(r_1, r_2, ..., r_n)} \leq \frac{\rho_g^{[p]}(r_1, r_2, ..., r_n)}{\rho_g^{[p]}(r_1, r_2, ..., r_n)} \leq \limsup_{r_1, r_2, ..., r_n \to 1} \frac{\log^{[p]} T^{-1}_g T_f(r_1, r_2, ..., r_n)}{\log^{[p]} T^{-1}_g T_h(r_1, r_2, ..., r_n)}.
\]

Proof. From the definition of generalised \( n \) variables based \( p \)-th relative Nevanlinna order, we get for a sequence of values of \( \left( \frac{1}{1-r_1} \right), \left( \frac{1}{1-r_2} \right), \ldots \) and \( \left( \frac{1}{1-r_n} \right) \) tending to infinity that

\[
\log^{[p]} T^{-1}_g T_h(r_1, r_2, ..., r_n) \\
\geq \left[ \rho_g^{[p]}(r_1, r_2, ..., r_n) - \varepsilon \right] \left[ - \log [(1 - r_1)(1 - r_2) \ldots (1 - r_n)] \right]. \tag{3.27}
\]

Now from (3.25) and (3.27), it follows for a sequence of values of \( \left( \frac{1}{1-r_1} \right), \left( \frac{1}{1-r_2} \right), \ldots \) and \( \left( \frac{1}{1-r_n} \right) \) tending to infinity that

\[
\frac{\log^{[p]} T^{-1}_g T_f(r_1, r_2, ..., r_n)}{\log^{[p]} T^{-1}_g T_h(r_1, r_2, ..., r_n)} \leq \frac{\rho_g^{[p]}(r_1, r_2, ..., r_n) + \varepsilon}{\rho_g^{[p]}(r_1, r_2, ..., r_n) - \varepsilon}.
\]

As \( \varepsilon (>) 0 \) is arbitrary, we obtain that

\[
\liminf_{r_1, r_2, ..., r_n \to 1} \frac{\log^{[p]} T^{-1}_g T_f(r_1, r_2, ..., r_n)}{\log^{[p]} T^{-1}_g T_h(r_1, r_2, ..., r_n)} \leq \frac{\rho_g^{[p]}(r_1, r_2, ..., r_n)}{\rho_g^{[p]}(r_1, r_2, ..., r_n)} \tag{3.28}
\]

Again for a sequence of values of \( \left( \frac{1}{1-r_1} \right), \left( \frac{1}{1-r_2} \right), \ldots \) and \( \left( \frac{1}{1-r_n} \right) \) tending to infinity,

\[
\log^{[p]} T^{-1}_g T_f(r_1, r_2, ..., r_n) \\
\geq \left[ \rho_g^{[p]}(r_1, r_2, ..., r_n) - \varepsilon \right] \left[ - \log [(1 - r_1)(1 - r_2) \ldots (1 - r_n)] \right]. \tag{3.29}
\]
So combining (3.18) and (3.29), we get for a sequence of values of \( \frac{1}{1-r_1} \), \( \frac{1}{1-r_2} \), ... and \( \frac{1}{1-r_n} \) tending to infinity that

\[
\frac{\log^{[p]} T_g^{-1}T_f(r_1, r_2, ..., r_n)}{\log^{[p]} T_g^{-1}T_h(r_1, r_2, ..., r_n)} \geq \frac{\rho_g^{[p]f}(r_1, r_2, ..., r_n) - \epsilon}{\rho_g^{[p]h}(r_1, r_2, ..., r_n) + \epsilon}.
\]

Since \( \epsilon > 0 \) is arbitrary, it follows that

\[
\limsup_{r_1, r_2, ..., r_n \to 1} \frac{\log^{[p]} T_g^{-1}T_f(r_1, r_2, ..., r_n)}{\log^{[p]} T_g^{-1}T_h(r_1, r_2, ..., r_n)} \geq \frac{\rho_g^{[p]f}(r_1, r_2, ..., r_n)}{\rho_g^{[p]h}(r_1, r_2, ..., r_n)}.
\]

Thus the theorem follows from (3.28) and (3.30).

\[\text{Remark 3.2.6} \] Theorem 3.2.10 is a generalization of Theorem 5 of Datta and Deb \{cf. [16]\} and also the same of a result of Datta and Jerin \{cf. [18]\} respectively for \( n = p = 1 \) and \( n = 1 \).

In view of Theorem 3.2.9 and Theorem 3.2.10 we may state the following theorem without its proof.

**Theorem 3.2.11** Let \( f, h \) be any two analytic functions of \( n \) complex variables in \( U \) and \( g \) be entire in \( n \) complex variables such that \( 0 < \lambda_g^{[p]f}(r_1, r_2, ..., r_n) \leq \rho_g^{[p]f}(r_1, r_2, ..., r_n) < \infty \) and \( 0 < \lambda_g^{[p]h}(r_1, r_2, ..., r_n) \leq \rho_g^{[p]h}(r_1, r_2, ..., r_n) < \infty \). Then

\[
\liminf_{r_1, r_2, ..., r_n \to 1} \frac{\log^{[p]} T_g^{-1}T_f(r_1, r_2, ..., r_n)}{\log^{[p]} T_g^{-1}T_h(r_1, r_2, ..., r_n)} \leq \min \left\{ \frac{\lambda_g^{[p]f}(r_1, r_2, ..., r_n)}{\lambda_g^{[p]h}(r_1, r_2, ..., r_n)}, \frac{\rho_g^{[p]f}(r_1, r_2, ..., r_n)}{\rho_g^{[p]h}(r_1, r_2, ..., r_n)} \right\}
\]

\[
\leq \max \left\{ \frac{\lambda_g^{[p]f}(r_1, r_2, ..., r_n)}{\lambda_g^{[p]h}(r_1, r_2, ..., r_n)}, \frac{\rho_g^{[p]f}(r_1, r_2, ..., r_n)}{\rho_g^{[p]h}(r_1, r_2, ..., r_n)} \right\}
\]

\[
\leq \limsup_{r_1, r_2, ..., r_n \to 1} \frac{\log^{[p]} T_g^{-1}T_f(r_1, r_2, ..., r_n)}{\log^{[p]} T_g^{-1}T_h(r_1, r_2, ..., r_n)}
\]

where \( p \) is any positive integer.
Remark 3.2.7 Theorem 6 of Datta and Deb \textit{cf.} [16]\textsuperscript{1} for \( n = p = 1 \). Also Theorem 3.2.11 impovres a result of Datta and Jerin \textit{cf.} [18]\textsuperscript{2} for \( n = 1 \).

Theorem 3.2.12 Let \( f, h \) be any two analytic functions of \( n \) complex variables in \( U \) and \( g \) be entire in \( n \) complex variables such that \( \rho_h^{[p]}(r_1, r_2, ..., r_n) < \infty \) and \( \lambda_h^{[p]}(r_1, r_2, ..., r_n) = \infty \) where \( p \) is any positive integer. Then

\[
\lim_{r_1, r_2, ..., r_n \to 1} \frac{\log[p] T_h^{-1} T_{fog}(r_1, r_2, ..., r_n)}{\log[p] T_h^{-1} T_f(r_1, r_2, ..., r_n)} = \infty.
\]

The proof is omitted because it can be carried out using the same technique as involved in Theorem 3.2.7.

Remark 3.2.8 Theorem 3.2.12 is also valid with “limit superior” instead of “limit” if \( \lambda_h^{[p]}(r_1, r_2, ..., r_n) = \infty \) is replaced by \( \rho_h^{[p]}(r_1, r_2, ..., r_n) = \infty \) and the other conditions remain the same.

Corollary 3.2.4 Under the assumptions of Theorem 3.2.12 and Remark 3.2.8

\[
\lim_{r_1, r_2, ..., r_n \to 1} \frac{T_h^{-1} T_{fog}(r_1, r_2, ..., r_n)}{T_h^{-1} T_f(r_1, r_2, ..., r_n)} = \infty \quad \text{and} \quad \limsup_{r_1, r_2, ..., r_n \to 1} \frac{T_h^{-1} T_{fog}(r_1, r_2, ..., r_n)}{T_h^{-1} T_g(r_1, r_2, ..., r_n)} = \infty
\]

respectively hold.

The proof is omitted.

Similarly, one may also state the following theorem and corollaries without their proofs as they may be carried out in the line of Remark 3.2.8 Theorem 3.2.12 and Corollary 3.2.4 respectively.

Theorem 3.2.13 Let \( f, h \) be any two analytic functions of \( n \) complex variables in \( U \) and \( g \) be entire in \( n \) complex variables such that \( \rho_h^{[p]}(r_1, r_2, ..., r_n) < \infty \) and \( \rho_h^{[p]}(r_1, r_2, ..., r_n) = \infty \). Then

\[
\limsup_{r_1, r_2, ..., r_n \to 1} \frac{\log[p] T_h^{-1} T_{fog}(r_1, r_2, ..., r_n)}{\log[p] T_h^{-1} T_g(r_1, r_2, ..., r_n)} = \infty
\]

where \( p \) is any positive integer.
Corollary 3.2.5 Theorem 3.2.13 is also valid with "limit" instead of "limit superior" if \( \rho_h^{[p]f} (r_1, r_2, \ldots, r_n) = \infty \) is replaced by \( \lambda_h^{[p]f} (r_1, r_2, \ldots, r_n) = \infty \) and the other conditions remain the same.

Corollary 3.2.6 Under the assumptions of Theorem 3.2.12 and Corollary 3.2.5,

\[
\limsup_{r_1, r_2, \ldots, r_n \to 1} \frac{T_h^{-1} T_{fog} (r_1, r_2, \ldots, r_n)}{T_h^{-1} T_g (r_1, r_2, \ldots, r_n)} = \infty \quad \text{and} \quad \lim_{r_1, r_2, \ldots, r_n \to 1} \frac{T_h^{-1} T_{fog} (r_1, r_2, \ldots, r_n)}{T_h^{-1} T_g (r_1, r_2, \ldots, r_n)} = \infty
\]

respectively hold.