CHAPTER 2

\( F(R) \) theory

2.1 The problem and an attempt to cure:

At this end it must have been clear that if present SNeIa observations is explained as the acceleration of the universe due to the presence of dark energy, then there are two distinct approaches to handle the situation. One is the modification of the energy-momentum tensor and the other is the modification of geometry itself i.e. General Theory of Relativity. The second one, in its simplest form is dubbed as \( F(R) \) theory of gravity. Now, there is sufficient observational evidence that the state parameter may be even less than \(-1\) at present which requires K-essence or Phantom models. These models often suffer from the disease of a finite time future singularity called the BIG-RIP. Modification of gravity does not encounter such singularity and therefore is much safer to encounter the problems stated.

It has been shown in the literature that E-H action with \(-\frac{1}{R}\) term can explain the late time acceleration [91] - [94] of the universe without any kind of exotic matter field. However, \(-\frac{1}{R}\) term fails to produce Newtonian gravity in the weak energy limit and so it is not consistent with the solar test [95]. Further, it also shows unavoidable instabilities within matter in the weak gravity limit [96] and fails to explain big bang nucleosynthesis (BBN) [97]. In fact \( R^{1+n}\) theory of gravity suffered initial setback under synthesis of light elements, shift of the horizon size at matter-radiation equality and perihelion-precession observation of Mercury [98]. All these data together puts up severe constraint on \( \delta \), viz., \( 0 < \delta < 7.2 \times 10^{-19} \). Further, solar system also puts up a severe constraints on alternative theories of gravity [99, 100]. Particularly, for an action

\[
A = \int \sqrt{-g} d^4x R^n
\]  

the gravitational potential [119] in the weak field limit is expressed as [98]

\[
\phi(r) = -\frac{Gm}{2r} \left[ 1 + \left( \frac{r}{r_c} \right)^\beta \right]
\]

where, \( r_c \) is an arbitrary parameter varying within the range \((1 - 10^4)\) AU, taking into account the velocity of the earth to be \(30Km s^{-1}\) [99] while \( \beta \) is related to \( n \) as

\[
\beta = \frac{12n^2 - 7n - 1 - \sqrt{36n^4 + 12n^3 - 83n^2 + 50n + 1}}{6n^2 - 4n + 2}.
\]

Clearly, for \( n = 1 \) and \( \beta = 0 \), Newtonian gravitational field is recovered. Any other value of \( n \), which appreciably differs from 1 is ruled out from light bending data in the sun limb and planetary periods [99]. The problem was alleviated [101] by considering an action in the form

\[
A = \int \left[ \beta R^m + \alpha R + \gamma R^{-n} \right] \sqrt{-g} d^4x,
\]

\( m > 0, n > 0 \), which passes solar test and therefore is suitable to explain the cosmological evolution right from the inflationary era through to late time accelerated epoch. At the initial stage, \( R^m \) term dominates and a de-Sitter solution is realizable for \( m = 2 \) in particular, explaining inflationary epoch without invoking phase transition [27, 102]. In the middle, the linear term dominates giving way to the standard BBN and structure formation and finally \( R^{-n} \) term dominates and late stage of accelerated cosmological expansion is realized, without invoking dark energy. However, \( R^{-n} \)
term is not distinguished at all, since neither it is generated by one-loop quantum gravitational corrections nor from any other physical consequence. Rather, it was considered just to invoke late time accelerated expansion.

So, it is required to find some other scalar invariant term, suitable to explain late time cosmic acceleration, having Newtonian analogue in the weak field limit to pass the solar test. Such a theory, dubbed as $F(R)$ theory of gravity [103, 104]. The very advantage of this model is that curvature can’t be detected, so any form of $F(R)$ may be responsible for cosmological evolution. But the problem is that out of indefinitely many curvature invariant terms, one requires an elegant method to choose suitable ones that fit all cosmological data at hand. In this regard the best choice is to invoke Noether symmetry (see review [120] and the references therein).

2.2 Noether symmetry of $F(R)$, treating $R$ as a constraint

Noether symmetry when applied for the first time in scalar-tensor theory of gravity [121], the idea was to find a form of the potential that might give rise to a cyclic co-ordinate and hence a conserved current. That is, out of indefinitely many choice of the potential, Noether symmetry selects one, corresponding to which there exists a cyclic coordinate that can’t be found a-priori through inspection. In the Robertson-Walker background metric the exponential form of the potential thus found was rather encouraging, because it could trigger inflation in the early universe. Since then, it has been applied in nonminimally coupled scalar-tensor theory of gravity both in the background of isotropic and several anisotropic models [122] - [140], together with nonminimally coupled scalar-tensor theory in the presence of $R^2$ term in the action [141] and recently, to classify different dark energy models [142].

To explore Noether symmetry, it is first required to express the action in canonical form, which is possible by introducing auxiliary variable in higher order theory of gravity. This has been attempted by several authors [143] - [150] for $F(R)$ theory of gravity, treating $R - 6\frac{\dddot{a}}{a} + 6\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = 0$ as a constraint in the Robertson-Walker minisuperspace model. In the process, the Lagrangian is spanned by a set of configuration space variables $(a, \dot{a}, R, \dot{R})$, i.e., $R$ is treated as an auxiliary variable. Here we start with the following action,

$$A_1 = \int BF(R)\sqrt{-g} \, d^4x - 2B \int \left[ \sqrt{h} F, R K \right] \, d^3x$$

(2.2.1)

where, $\Sigma = 2B \int \left[ \sqrt{h} F, R K \right] \, d^3x$, is the surface term for $F(R)$ theory of gravity that emerges under variational principle (metric formalism) [151]. In the above, $h$ is the determinant of the metric on three space, $K$ is the trace of the extrinsic curvature and $F, R$ is the derivative of $F(R)$ with respect to $R$. Now, in the background of isotropic and homogeneous Robertson-Walker line element, the above action in the absence of a matter field is written as

$$A = \int \left[ BF(R) - \lambda \left( R - 6\frac{\dddot{a}}{a} + 6\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) \right] a^3 \, dt - 2B \int \left[ \sqrt{h} F, R K \right] \, d^3x$$

(2.2.2)

where, the expression for $R$ has been introduced as a constraint through Lagrange multiplier $\lambda$. Now varying the above action with respect to $R$, one gets $\lambda = BF, R$, which is substituted in the action. Under integration by parts the action may be expressed as,

$$A = \int B \left[ a^3(F - RF, R) - 6a\dot{a}^2F, R - 6a^2\dot{a} F, RR \dot{R} + 6kaF, R \right] \, dt$$

(2.2.3)
while the Noether equation is,

\[
XL = \mathcal{L}X = B\alpha \left(3a^2(\dot{F} - RF) - 6\dot{a}^2F - 12a\ddot{a}\dot{R}F_{,RR} + 6kF_R\right)
+ B\beta \left[6ka - a^3R - 6\dot{a}^2\right]F_{,RR} - 6a^2\dot{a}F_{,RRR}
+ B \left(\frac{\partial\alpha}{\partial a} + \frac{\partial\alpha}{\partial R}\right)\left(-12a\ddot{a}F_{,R} - 6\dot{a}^2\dot{R}F_{,RR}\right) + B \left(\frac{\partial\beta}{\partial a} - 6a^2\dot{a}F_{,RR} \frac{\partial\beta}{\partial R}\right) = 0,
\]

(2.2.4)

where, \(X\) is the vector field given by

\[
X = \alpha(a, R) \frac{\partial}{\partial a} + \beta(a, R) \frac{\partial}{\partial R} + \dot{\alpha}(a, R) \frac{\partial}{\partial \dot{a}} + \dot{\beta}(a, R) \frac{\partial}{\partial \dot{R}}.
\]

(2.2.5)

Now we equate the coefficients of \(\ddot{a}^2\), \(\dot{R}^2\), \(\dot{a}\dot{R}\) and others respectively to zero as usual to obtain,

\[
6F_{,R}(\alpha + 2a\dot{a}) + 6(\beta + a\dot{\beta})aF_{,RR} = 0.
\]

(2.2.6)

\[
6a^2F_{,RRR} \alpha R = 0.
\]

(2.2.7)

\[
6\beta a^2F_{,RRR} + 6(2\alpha + a\dot{a})aF_{,RR} + 12F_{,R}a\alpha R = 0.
\]

(2.2.8)

\[
a^3(\dot{F} - RF)a^2 + 6kF_{,R} + \beta(6k - a^2R)aF_{,RR} = 0.
\]

(2.2.9)

It is then straightforward to show that Noether symmetry of \(F(R)\) theory of gravity [143] - [150], yields \(F(R) \propto R^2\), admitting an explicit solution in the form

\[
a = \sqrt{a_0 t^4 + a_3 t^3 + a_2 t^2 + a_1 t}.
\]

(2.2.10)

In the absence of matter this solution may be treated as that of early universe depicting that the universe was at radiation dominated era initially, but then it enters power law inflationary era. However, in the presence of dust the same solution is arrived at, depicting that it may be treated as a constraint to explain late time acceleration. Question is in which way it should be treated. Note that \(R^2\) is not realized at one-loop level, rather it is an outcome of Noether symmetry. Naturally, such term should be treated to explain late time cosmic acceleration.

Now, we want to check that if Noether symmetry along with the solutions presented by [145, 146] exists, when action containing \(R^2\) term is spanned by such a set of configuration space variables, i.e., whether Noether symmetry is an artifact of a particular choice of configuration space variables or not. One may expect that since Noether symmetry of \(F(R)\) theory yields \(R^2\), so \(\mathcal{L}X L = 0\) will be exactly solved if \(F(R) \propto R^2\) is assumed a-priori. In this regard we follow the standard formalism, where a form of curvature invariant term is fixed (here \(R^2\) term), the configuration space is different viz., \((a, \dot{a}, q, \dot{q})\). Here, \(q\) is an auxiliary variable chosen as the derivative of the action with respect to the highest derivative of the variable appearing in the action, that has not been possible to integrate by parts to produce a surface term. The action under consideration in the Robertson-Walker minisuperspace model is,

\[
A = B \int R^2 \sqrt{-g} dt^4 + 2B \int \left[\sqrt{\dot{R}^2} F_{,R} R\right] dt^3
\]

(2.11.11)

\[
= B \int 6^2[a\ddot{a} + \dot{a}^2 + k]^2 dt - 9\sqrt{6} B[a\ddot{a} + \dot{a}^2 + k]^2 a\ddot{a}.
\]

Now let us define the auxiliary variable \(q\) as,

\[
q = \frac{\partial A}{\partial \ddot{a}} = 9\sqrt{6} B[a\ddot{a} + \dot{a}^2 + k]^2 a = 9Ba^2\sqrt{R},
\]

(2.2.12)
which is radically different from $R$. Finally, it has been found that Noether symmetry of $R^{\frac{2}{3}}$ via this standard technique yields the same solution (2.2.10) as obtained by [145, 146].

Understanding that $R^{\frac{2}{3}}$ is an outcome of Noether symmetry of $F(R)$ theory of gravity, it is important to learn its role in explaining the cosmological evolution. Despite the fact that $R^{\frac{2}{3}}$ theory of gravity has some attractive features, particularly in explaining the late time cosmic evolution, nevertheless, it suffers from some severe problems. Firstly, the solution in the radiation era ($a \propto t^{\frac{2}{3}}$), is substantially different from the standard Friedmann solution $a \propto t^{\frac{1}{2}}$, which indicates much faster (1.5 times) expansion rate of the universe. This clearly puts up severe problem in Nucleosynthesis. Secondly, the solution in the early matter dominated era tracks $a \propto t^{\frac{2}{3}}$, like any $R^{-n}$, with $n > 0$ model, which again creates problem in structure formation and fitting WMAP data, as discussed in [110]. These diseases may be cured if such an action containing $F(R)$ would have been supplemented by Einstein-Hilbert term. Hence, we try to explore Noether symmetry of $F(R)$ theory of gravity being supplemented by the Einstein-Hilbert term. This is done to understand if the technique of finding such symmetry is correct. If it is, then the same old result is expected, since $\alpha R + \beta F(R) = \gamma F_1(R)$ and this is what we have obtained.

### 2.3 Viability of Noether Symmetry of $F(R)$

In search of a better form of $F(R)$, we took a theory of gravity consisting of scalar field being nonminimally coupled with $F(R)$ as

$$S = \int d^4x\sqrt{-g} \left[ h(\phi)F(R) - \frac{w(\phi)}{2} \phi_{,\mu}\phi^{,\mu} - V(\phi) \right]. \quad (2.3.1)$$

Following the usual technique of finding Noether symmetry [143] - [150]. But we ended up with the result that Noether symmetry for $F_{,RR} \neq 0$ is obscure. Next we took a nonminimally coupled scalar-tensor theory of gravity along with $F(R)$ term as

$$S = \int \sqrt{-g}d^4x \left[ f(\phi)R + BF(R) - \frac{\omega(\phi)}{2} \phi_{,\mu}\phi^{,\mu} - V(\phi) \right]. \quad (2.3.2)$$

Here also we obtained the same result that Noether symmetry for $F_{,RR} \neq 0$ cannot be realized. This is surprising, since, Noether symmetry for the same action, with $F(R) = f(\phi)R + \beta R^2$ and $\omega(\phi) = 1$, in the conformally flat Robertson-Walker metric already exists in the literature [141].

In doing so, we had used an auxiliary variable $Q$, different from $R$, and the Lagrangian was spanned by the set of configuration space variables $(a, Q, \phi, \dot{a}, \dot{Q}, \dot{\phi})$, instead. Thus, the absence of Noether symmetry in both the cases automatically raises doubt about the technique of treating $R - 6(\frac{\dot{a}^2}{a} + \frac{\dot{b}^2}{b^2} + \frac{\dot{k}^2}{k^2}) = 0$, as a constraint of the theory, which finally treats $R$ as a variable and as we know that variation of the gravitational action with respect to $R$ yields naught. To check if indeed the technique is problematic, we also took spatially symmetric Kantowski-Sachs (K-S), Bianchi-I (B-I) and Bianchi-III (B-III) metric, which can be expressed altogether as,

$$ds^2 = -dt^2 + a^2dr^2 + b^2[d\theta^2 + f_k^2d\phi^2], \quad (2.3.3)$$

where, $f_k = \sin \theta \Rightarrow k = +1$ (K-S), $f_k = \theta \Rightarrow k = 0$ (B - I), $f_k = \sinh \theta \Rightarrow k = -1$ (B - III), and for which

$$4R = 2 \left( \frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + 2 \frac{\dot{a}\dot{b}}{ab} + \frac{\dot{b}^2}{b^2} + \frac{k}{b^2} \right). \quad (2.3.4)$$

The action under consideration is

$$A = \int \left[ BF(R) - \lambda \left( 4R - 2 \left( \frac{\ddot{a}}{a} + 2 \frac{\ddot{b}}{b} + 2 \frac{\dot{a}\dot{b}}{ab} + \frac{\dot{b}^2}{b^2} + \frac{k}{b^2} \right) \right) \right] \sqrt{-g} dt. \quad (2.3.5)$$
This also increases the configuration space variables \((a, b, R, \dot{a}, \dot{b}, \dot{R})\), to explore Noether symmetry of \(F(R)\) theory in vacuum. Here again we find that \(F(R)\) does not admit any nonlinear form. Thus we conclude that symmetry obtained for \(F(R) = R^{3/2}\) in vacuum and matter dominated era, was an accident, but the technique [143]-[150] is somehow problematic, and it is not possible to explore Noether symmetry of \(F(R)\) theory of gravity, in general. We would also like to mention at this stage that attempting Noether gauge symmetry [152, 153] and treating Born-Infeld action being coupled to \(F(R)\) [154], no other symmetry has been found to exists for \(F(R)\) theory of gravity. This means that it is very special out of all the curvature invariants.

2.4 What is so special in \(R^{3/2}\)?

Nevertheless, the more interesting issue is that, out of infinite number of curvature invariant terms, Noether symmetry selects \(R^{3/2}\) term in particular. It has been shown that if we start with the basic variable, \(h_{ij} = a^2 = z\) then the expression of Ricci scalar is given by,

\[
R = 6 \left( \frac{\ddot{z}}{2z} + \frac{k}{z} \right).
\]  

(2.4.1)

The action now reads,

\[
S = \int 3\sqrt{3}(\ddot{z} + 2k)\frac{\dot{z}}{2} dt.
\]  

(2.4.2)

The auxiliary variable is

\[
Q = \frac{\partial S}{\partial \ddot{z}} = \frac{9\sqrt{3}}{2}(\ddot{z} + 2k)\frac{\dot{z}}{2} = \frac{9}{2}a\sqrt{R}.
\]  

(2.4.3)

Then, the canonical form of the action is

\[
S = \int \left[ Q(\ddot{z} + 2k) - \frac{4Q^3}{729} \right] dt = \int \left[ -\dot{Q}\dot{z} + 2kQ - \frac{4Q^3}{729} \right] dt.
\]  

(2.4.4)

The definition of the auxiliary variable is restored immediately from the \(Q\) variation equation, while the variable \(z\) turns out to be cyclic. Thus the field equations are oversimplified to,

\[
\ddot{Q} = 0,
\]  

(2.4.5)

\[
\dot{z}\dot{Q} + 2kQ - \frac{4Q^3}{729B^2} = 0,
\]  

(2.4.6)

In view of the definition of the auxiliary variable \(Q\) given in (2.4.3), equation (2.4.5) immediately gives the following solution,

\[
a = \left[ a_4 t^4 + a_3 t^3 + \left( \frac{3a_3^2}{8a_4^2} - k \right) t^2 + a_2 t + a_1 \right]^\frac{1}{2},
\]  

(2.4.7)

which resembles exactly with the solution (2.2.10) obtained earlier in view of Noether symmetry. Note that, equation (2.4.6) also yields the same above solution. Thus, it is clear that Noether symmetry is in built in \(R^{3/2}\) theory of gravity, since \(z\) turns out to be cyclic automatically. \(R^{3/2}\) although does not arise from one loop quantum gravitational correction, it is clear that it has a very special status amongst all the curvature invariant terms, since it is the only one found in view of Noether symmetry. Further, in contrast to other powers of \(R\), no decay of earth radius has been observed for \(R^{3/2}\) term [155]. So it is important to study the consequence of this term in further detail. Already we learn that \(R^{3/2}\) has problems in explaining cosmic evolution. However, an additional linear term might resolve the problems. But it is extremely difficult to explore
solutions for such an action. Nevertheless, there is an ingenious technique explored by Sanyal [156] to extract solutions in such extremely difficult situation through a general conserved current that exists for scalar-tensor theory of gravity. We follow the procedure to find solution of the action containing Einstein-Hilbert term in addition to $R^2$ term both in the radiation dominated and matter dominated era.

### 2.5 A general conserved current for $F(R)$

From previous discussion, the next question that automatically arises, “is it possible to find an integral of motion in general for $F(R)$ theory of gravity being coupled with a scalar field, including a linear term in addition?”. The answer is yes and already appears in the literature [156], where a general conserved current in connection with a general higher order theory of gravity coupled with a dilatonic scalar has been explored in a metric independent way. We briefly discuss the technique of finding the same and as an example, show how to use such conserved current to find solutions in the present context. For this purpose, we take up action (2.3.2) in addition to a matter Lagrangian $L_m$, to briefly illustrate the existence of the said integral of motion, as

\[
A = \int [f(\phi) R + BF(R) - \omega(\phi) \phi_{,\mu} \phi^{,\mu} - V(\phi) - \kappa L_m] \sqrt{-g} \, d^4x. \tag{2.5.1}
\]

Field equations corresponding to the above action under metric variation method are,

\[
f \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) + f^{,\alpha} \phi_{,\alpha \mu} f_{,\nu} - f_{,\mu \nu} - \omega \phi_{,\mu} \phi_{,\nu} + \frac{1}{2} g_{\mu\nu} \left( \omega \phi_{,\alpha} \phi^{,\alpha} + V(\phi) \right) + B \left[ (F/R) R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \left( F/R \right)_{,\alpha} g_{\alpha\nu} - (F/R)_{,\mu \nu} \right] = \frac{\kappa}{2} T_{\mu\nu}.
\]

\[
R f' + 2 \omega \phi^{,\mu}_{,\mu} + \omega' \phi^{,\mu}_{,\mu} - V'(\phi) = 0. \tag{2.5.3}
\]

The trace of the field equation (2.5.2) is,

\[
R f - 3 f^{,\mu}_{,\mu} - \frac{\omega}{\phi} \phi^{,\mu} \phi_{,\mu} - 2 V - B \left[ R(F/R) + 3(F/R)_{,\alpha}^{,\alpha} - 2 F \right] = - \frac{\kappa}{2} T^{\mu}_{\mu}. \tag{2.5.4}
\]

Combining equations (2.5.3) and (2.5.4) it has been shown [141] that under the following condition,

\[
B \left[ R(F/R) + 3(F/R)_{,\alpha}^{,\alpha} - 2 F \right] = \frac{\kappa}{2} T^{\mu}_{\mu} + \frac{f^3}{f'} \left( \frac{V}{f^2} \right)', \tag{2.5.5}
\]

the above action (2.5.1) carries a conserved current

\[
J^\mu_{\mu} = \left[ (3 f^{,\mu}_{,\mu} + 2 f \dot{\omega}) \dot{\phi}^{,\mu} \right]_{,\mu} = 0, \tag{2.5.6}
\]

which in the case of homogeneous cosmology, takes the following simplified form,

\[
\sqrt{-g} \left( 3 f^{,\mu}_{,\mu} + 2 f \dot{\omega} \right) \dot{\phi} = c, \tag{2.5.7}
\]

where, $c$ is a non-vanishing constant and $g$ is the determinant of the metric. Thus, a general conserved current exists for higher order theory of gravity, without even fixing the forms of the parameters involved, which lies beyond the scope of Noether symmetry, since it is true for the theory of gravitation in general.

Lastly, an attempt had been made via this general prescription to obtain solution for an action containing $f(\phi) R + BR^2$. To handle the equations comfortably, we split the condition (2.5.5) into two, assuming $V \propto f(\phi)^\alpha$, as

\[
B \left[ RF_{,R} + 3(F/R)_{,\alpha}^{,\alpha} - 2 F \right] = \frac{\kappa}{2} T^{\mu}_{\mu}, \tag{2.5.8}
\]
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\[ V = \lambda f^2, \] (2.5.9)

where, \( \lambda \) is a constant. So finally, in the case of homogeneous cosmology, we need to solve following set of five equations,

\[
\begin{align*}
B[R_{F,R} + 3 \Box F_{,R} - 2F] &= \frac{2}{3} T_{\mu}^\mu, \quad (2.5.10) \\
V &= \lambda f^2, \quad (2.5.11) \\
Rf - 3 \Box f - \omega \phi^{,\mu} \phi_{,\mu} - 2V &= 0, \quad (2.5.12) \\
Rf' + 2 \omega \Box \phi + \omega' \phi^{,\mu} \phi_{,\mu} - V'(\phi) &= 0, \quad (2.5.13) \\
\sqrt{-g} (3f'^2 + 2f \omega) \dot{\phi} &= c. \quad (2.5.14)
\end{align*}
\]

Now for \( F(R) = R^2 \),

\[
\Box F_{,R} = -[(\ddot{R} + \Theta \dot{R})F_{,RR} + \dot{R}^2 F_{,RRR}] = - \left[ \frac{3}{4} \frac{\ddot{R}}{\sqrt{R}} - \frac{3}{8} \frac{\dot{R}^2}{R^2} + \frac{27}{4} H \frac{R}{\sqrt{R}} \right], \quad (2.5.15)
\]

and so equation (2.5.10) reduces to,

\[
\left[ \frac{9}{\sqrt{R}} \right. \frac{\ddot{R}}{\sqrt{R}} - \frac{9}{2} \frac{\dot{R}^2}{R^2} + 9H \frac{R}{\sqrt{R}} + 2\dot{R}^2 \left. \right] = \frac{2}{B} T_{\mu}^\mu. \quad (2.5.16)
\]

In the radiation dominated era, taking \( p = \frac{1}{3} \rho \), the trace of the energy momentum tensor \( T_{\mu}^\mu \) vanishes and equation (2.5.16), gets solved to yield \( a \propto \sqrt{t} \). This is a brilliant feature of higher order theory of gravity, that despite being tightly coupled to the scalar field it still produces Friedmann type solution in the radiation era keeping standard big-bang-nucleosynthesis (BBN) unchanged.

In the matter dominated era, on the other end, \( p = 0 \) and the trace of the energy momentum tensor is simply \( T_{\mu}^\mu = \rho \), with \( a^3 \rho = \rho_{m0} \), \( \rho_{m0} \) being the present matter density of the universe. The only solution that we could find for equation (2.5.16) is \( a = a_0 t \), provided the condition \( B = -\frac{\sqrt{6} \rho_{m0}}{90 a_0} \) holds. But, such a coasting solution is ruled out, since it does not admit structure formation. However, this is only a particular solution and one may explore even general solution to ensure viability of the model and also extend the work taking other forms of \( F(R) \).

These issues have been explored in paper-3 & paper-4. Recently it has come to notice that attempt to expatiate numerical solution has been presented by Modak, Sarkar and Sanyal [157] taking into account an action containing both \( R \) and \( R^2 \) terms in the presence of radiation and matter. The result is a continuous transition in the context of the evolution of the universe from \( z > 3200 \) till date. The graphs presented show that at \( z > 3200 \), the universe was in the Friedmann type radiation dominated era with deceleration parameter \( q = 1 \). It then falls slowly to \( q = 0.8 \) at \( z = 1100 \) i.e., at decoupling era. After that it falls sharply to Friedmann type matter dominated era \( (q = .5) \) at around \( z = 250 \). It stays there quite a long time until it again increases to a recent radiation dominated era and then falls of sharply to present accelerating phase. The late time radiation era had been interpreted as reionization of intermediate galactic matter viz., hydrogen and helium. This result imply that an action in the form \( A = \int \left[ \alpha R + \beta R^2 + \gamma R^2 \right] \sqrt{-gd^4x} + A_m \), \( A_m \) being the matter lagrangian, is compatible to explain the history of cosmic evolution.