APPENDIX I

UNIQUESS OF SOLUTION OF THE INVERSE PROBLEM

For a two-dimensional harmonic function $H$ with asymptotic behaviour $H = O(r^n)$, $n \geq 1$, $r \to \infty$ defined in the upper half-space domain $B_i$ bounded below by a half-space boundary $S(=S_x+S_L+S_u$, Fig. 3.2.1), given $H$ over $S$, there exists a double layer boundary density $\mu$ over $S$ that reproduces the field $H$ in $B_i$ (Laskar 1984) as

$$H(P) = -\int_S \log|q-P|\mu(q)dq, P \in B_i.$$  \hspace{1cm} (i)

As $P \to p \in S$, following Jaswon and Symm (1977), the formula (i) yields the boundary relation between $H$ and $\mu$ as

$$H(p) = \pi \mu(p) - \int_S \log|q-p|\mu(q)dq, p \in S.$$ \hspace{1cm} (ii)

Given $H$ over $S$, the equation (ii) formulates a Dirichlet problem in $\mu$ for the upper half-space domain $B_i$ in terms of $H$ specified over $S$. That the equation has a unique $\mu$ over $S$, can be shown considering $B_i$ as an interior domain enclosed by $\partial B = S+S_3+S_u$ being a semicircle of radius $R$ with ends over $S$ (Fig. 3.2.1). Since the interior Dirichlet problem in $\mu$ represented by

$$H(p) = \pi \mu(p) - \int_{S_u} \log|q-p|\mu(q)dq, p \in \partial B,$$

has a unique solution (Jaswon and Symm, 1977) and $\mu = O(H)$, $r \to \infty$, shown in Subsection 2.4.3, we find the integrand over $S_u$ vanishes as $R \to \infty$ and the above equation takes the form of equation (ii) with a unique $\mu$ over $S$. This $\mu$ reproduces the field $H$ on and above $S$ as its potential.
Let us now assume that the field $H$ be specified over a horizontal half-space boundary $\overline{S} (= \overline{S}_o + \overline{S}_u + \overline{S}_o, \text{Fig.3.2.1})$ and the curved continuation boundary $S$, having a central concave part $S_L$ with its ends common to those of $\overline{S}_o$, extend to infinity along $\overline{S}_o$ on both sides of $\overline{S}_o$. Now for $P \in \overline{S}_o$, excluding its end points, the half-space formula (i) yields

$$H(P) = - \left[ \int_{S_i} \log |q - P| \mu(q) dq + \int_{\overline{S}_o} \log |q - P| \mu(q) dq \right]$$
$$= - \int_{S_i} \log |q - P| \mu(q) dq, \quad P \in \overline{S}_o, \quad \text{(iii)}$$

the integral over $\overline{S}_o$ having no contribution to $H$ at $P \in \overline{S}_o$. This is evident from the fact that $\log |q - P| = 0$ for $P, q \in \overline{S}$ and $P \neq q$. Once the $\mu$ over $S_L$ is obtained as solution of the equation (iii), $\mu$ over $\overline{S}_o$ is given by (ii) rewritten as

$$\pi \mu(P) = H(P) + \int_{S_i + S_o} \log |q - P| \mu(q) dq, P \in \overline{S}_o,$$
$$= H(P) + \int_{S_o} \log |q - P| \mu(q) dq, P \in \overline{S}_o, \quad \text{(iv)}$$

the integral over $\overline{S}_o$ being zero.

Let us now assume that given $H$ over $\overline{S} (= \overline{S}_o + \overline{S}_u + \overline{S}_o)$, the equation (iii) i.e.

$$H(P) = - \int_{S_i} \log |q - P| \mu(q) dq, P \in \overline{S}_o. \quad \text{(v)}$$

has a $\overline{\mu}$ over $S_L$ as its solution. Once the $\overline{\mu}$ over $S_L$ is known, the $\overline{\mu}$ over $\overline{S}_o$ can be obtained from (iv)
\[ \pi \tilde{\mu}(p) = H(p) + \int_{S} \log |q - p| \tilde{\mu}(q) dq, p \in \bar{S} \]  

This \( \tilde{\mu} \) belonging to \( S \) reproduce the \( H \) in \( B \), including the boundary \( \bar{S} \). If \( \bar{H} \) be the potential in \( B \), due to \( \tilde{\mu} \) over \( S \) (=\( \bar{S}_o + S_l + \bar{S}_n \), Fig. 3.2.1) we find, following the above conclusion

\[ \bar{H}(P) = H(P), P \in \bar{S}. \]  

(vii)

Now let us construct a harmonic function \( \delta H \) in \( B \), as

\[ \delta H(P) = \bar{H}(P) = -\int_{\bar{S}} \log |q - P| \delta \mu(q) dq, P \in B, \]  

(viii)

where \( \delta H(P) = H(P) - \bar{H}(P), P \in B \), and \( \delta \mu(q) = \mu(q) - \tilde{\mu}(q), q \in S \). Since \( H = \bar{H} \) over \( \bar{S} \), we obtain

\[ 0 = H(P) - \bar{H}(P) = -\int_{\bar{S}} \log |q - P| \delta \mu(q) dq, P \in \bar{S}. \]  

(ix)

Since \( \delta H(P) = 0 \) over the half-space boundary \( \bar{S} \) it must be zero at infinity. This means, \( \delta H = 0 \) in upper half-space domain \( \bar{B} \) bounded below by \( \bar{S} \). This leads to

\[ \delta H(P) = H(P) - \bar{H}(P) = 0, P \in \bar{S}. \]  

(x)

Since \( H \) and \( \bar{H} \) are potentials due to \( \mu \) and \( \tilde{\mu} \) respectively, both belonging to the boundary \( S \), \( H \) and \( \bar{H} \) must satisfy Laplace's equation in the upper half-space domain \( B \), bounded below by \( S \) and at an interior point \( P(x_0, z_0) \) of \( \bar{S}_o \), both \( H \) and \( \bar{H} \) and their respective normal derivatives \( H' \) and \( \bar{H}' \) are analytic functions. Hence, considering the origin of reference frame at \( P \), by (ix) and (x), we obtain

\[ H(x_0) = \bar{H}(x_0) = H_0(x_0) \text{ say,} \]

and \[ H'(x_0) = \bar{H}'(x_0) = H_1(x_0) \text{ say,} \]
where \( H_0 \) and \( H_1 \) are two different analytic functions on the portion of \( \overline{S_u} \) containing \( P \). Now following Cauchy-Kowalevsky existence theorem (Kellogg, 1929, p.245), we conclude in the present case that there exists a two-dimensional neighbourhood \( N \) of \( P \) and a function \( U(x,z) \) which is harmonic in \( N \) and which assumes on the portion of \( \overline{S_u} \) in \( N \) the same values as the function \( H_0(x) \) and whose normal derivative assumes on the same portion of \( \overline{S_u} \) the values \( H_1(x) \). There is only one such function. Here we would like to mention that unlike other existence theorems Cauchy-Kowalevsky theorem asserts continuation of \( U \) across the portion of \( \overline{S_u} \) containing \( P \). This means, \( H = \overline{H} = U \) in two-dimensional neighbourhood of the portion of \( \overline{S_u} \) containing \( P \). This conclusion on \( H \) and \( \overline{H} \) remains true over all other portions of \( \overline{S_u} \) and as such it leads to

\[
H(P) = \overline{H}(P), \quad P \in S_{int},
\]

where \( S_{int} \) is a half-space boundary with its central part immediately below \( \overline{S_u} \) and arms coinciding with \( \overline{S_o} \) of \( \overline{S} \).

On repeated application of the above procedure to subsequent lower boundaries \( S_{int} \), we arrive at

\[
H(P) = \overline{H}(P), \quad P \in S
\]

Or \( \delta H(P) = H(P) - \overline{H}(P) = 0, \quad P \in S. \) (xii)

This implies,

\[
0 = \delta H(p) = \pi \delta \mu(p) - \int_{S} \log|q-p| \delta \mu(q) dq, \quad P \in S. \tag{xiii}
\]
This equation is identical to the homogeneous component of (ii) with \( \mu \) replaced by \( \delta \mu \). Considering the equation in B, enclosed by \( S + S_u, R \to \infty \), it can be shown following Jaswon and Symm (1977) that the equation (xiii) has no non-trivial solution.

This leads to the conclusion that

\[ \delta \mu (q) = \mu (q) - \bar{\mu} (q) = 0, \quad q \in S \]

or

\[ \bar{\mu} (q) = \mu (q), \quad q \in S. \quad (xiv) \]

Since \( \mu \) is unique over \( S \), being a solution of a half-space Dirichlet problem expressed by equation (iii) for \( H \) specified over \( S \), the solution \( \bar{\mu} \) of equation (v) over \( S_L \) and consequently the \( \bar{\mu} \) over \( S (=S_s + S_L + S_n) \) is unique and it is identical to the Dirichlet \( \mu \) over \( S \).
APPENDIX II

DENSITY INTEGRAL OVER THE HALF-SPACE BOUNDARY

In upward continuation of a two-dimensional potential field $H$ with asymptotic behaviour $H = O(r^{-n})$, $n \geq 1$, $r \to \infty$, from a half-space curved boundary $S(= S_o + S_u + S_\circ$, Fig.4.2.1), given $H$ over $S$, following Laskar (1984), $H$ in the upper half space domain $B_u$, bounded below by $S$, can be reproduced as a double layer potential

$$H(P) = - \int_\mathbb{S} \log |q - P| \mu(q) dq, \quad P \in B_u. \quad (i)$$

It is evident from (i) that as $|P| \to \infty$,

$$H(P) = O(|P|^{-1}) \int_\mathbb{S} \mu(q) dq. \quad (ii)$$

For the gravimetric case, $H$ vanishes asymptotically in $O(|P|^{-1})$ as $|P| \to \infty$ and as such, (iv) yields

$$\int_\mathbb{S} \mu(q) dq = O(1), \quad (iii)$$

a constant, not equal to zero, necessarily. This holds for a horizontal boundary $S$, say $\overline{S} (= \overline{S}_o + \overline{S}_u + \overline{S}_\circ$, Fig.4.2.1), a particular case of $S$,

$$\int_\mathbb{S} \overline{\mu}(q) dq = O(1), \quad (iv)$$

where $\overline{\mu}(q)$ is the density over $\overline{S}$.

For the magnetostatic case, $H$ vanishes asymptotically in $O(|P|^{-2})$ as $|P| \to \infty$ and as such, (ii) yields

$$\int_\mathbb{S} \mu(q) dq = 0 \quad (v)$$

For a horizontal $S$, say $\overline{S}$,
\begin{equation}
\int_{S} \bar{\mu}(q) dq = 0, \quad (vi)
\end{equation}

where, as before, $\bar{\mu}(q)$ is the density over $S$.

Now, considering the integrals (iii), (iv), (v), (vi) we rewrite the integral properties of $\mu$ as

\begin{equation}
\int_{S_{a}} \mu(q) dq + \int_{S} \mu(q) dq = \int_{S_{a}} \bar{\mu}(q) dq + \int_{S} \bar{\mu}(q) dq, \quad (vii)
\end{equation}

valid for both gravimetric as well as magnetostatic case with $S_{a}$ extending to infinity at both ends of $S_{a}$.
APPENDIX III

MAGNETIC RESPONSE OF THIN PLATES FORMING A STEP-FAULT

Let three infinitely long thin plates of width AB, CD and EF, each extending from $-\infty$ to $+\infty$ in the direction of y-axis of a Cartesian reference frame with z-axis upward, be placed at depths $h_1$, $h_2$ and $h_3$ respectively below the (x,y) plane such that the plates form a step-fault, its strike pointing in the direction of y-axis. Let the plates be uniformly polarised by downward doublets of strength $\mu$ per unit area and let the doublets be inclined at an angle $\theta$ with the x-axis, as shown in Fig. 1. The plates so arranged produce two-dimensional magnetic field $T_z$ in (x,z) plane as shown in Fig. 1.

Fig. 1: Vertical component magnetic response and its gradients of step-faults approximation to basement in a geological basin

(Three infinitely long plates of widths extending from $x=-10$ to 5, $x=5$ to 15 and $x=15$ to 25 each polarised at $35^\circ$, lie at depths 3, 1.5 & 1 units respectively below the datum line $z=0$ in a XOZ reference frame with z axis upward. Maximum of $T_\alpha$, point of inflexion of $T_\alpha$ and point of minimum of $T_\alpha$ form a cluster in the vicinity of the fault-trace point.)
(i) Magnetic Response of a Single Plate

For \(q(x,z)\) defining a point on the line \(AB\) with \(A\) and \(B\) defined by \((x_1,z)\) and \((x_2,z)\) respectively in the vertical plane \(y=0\), the magnetostatic potential due to the plate at a point \(P(X,Z)\) is

\[
W(P) = -\int_{AB} \log |q - P| \mu(q) dq
\]

where \(P,q\) define the position vectors of the field point \(P\) and the source point \(q\) respectively, \(dq\) is the arc element at \(q\), \(|q - P|\) is the distance \(r\) between \(P\) and \(q\), \(\log |q - P|\) defines the derivative of \(\log |q - P|\) at the point \(q\) keeping \(P\) fixed in the direction \(\hat{n}\), \(\hat{n}\) defining the direction of the doublet of strength \(\mu\) at \(q\). On further simplification, the equation (i) becomes

\[
W(X,Z) = -\int_{\hat{n}} |q - P|^{-2} (q - P) \cdot \hat{n} \mu(q) dq
\]

\[
= - \int_{x=A}^{B} \frac{(x - X)l + (z - Z)m}{(x - X)^2 + (z - Z)^2} \mu(x) dx
\]

where \(l,m\) are the direction cosines of \(\hat{n}\).

Now, the downward vertical component field \(T_z\) at \(P\) is,

\[
T_z(X,Z) = \frac{\partial W(X,Z)}{\partial Z} = \left[ \frac{(z - Z)l - (x - X)m}{(x - X)^2 + (z - Z)^2} \right]_{x=x_1}^{x_3}.
\]

(ii) Vertical Component Magnetic field due to the Plates forming Step-Faults

Three infinitely long thin plates \(AB\), \(CD\) and \(EF\) of widths extending from \(x_1\) to \(x_2\), \(x_2\) to \(x_3\) and \(x_3\) to \(x_4\) respectively lying at depths \(h_1\), \(h_2\) and \(h_3\) respectively below the \((x,y)\) plane. They provide the simplest possible configuration of a step-fault below the ground plane \(z=0\). Superposition of \(T_z\) response of each plate computed by (iii) provides the \(T_z\) response of the faults at \((X,0)\) on the datum line. Using the \(T_z\) values
calculated for an angle of inclination $\theta = 35^\circ \text{N}$ on the datum line, its horizontal gradient $T_{zx}$ and vertical gradient $T_{zz}$ are computed at level $z=2h$, by the numerical formulae given by Laskar (1999), $z=0$ defining the datum line and $h(=0.125)$ defining the spacing of data over $z=0$. The $T_z$, $T_{zx}$ and $T_{zz}$ so obtained are shown in Fig. 1. It is evident from Fig. 1 that the point of maximum of $T_z$, point of inflexion of $T_{zx}$ and the point of minimum of $T_{zz}$ form a cluster near the fault trace point.