2.1 Potential due to simple sources

2.1.1 Potential due to a simple source

For a logarithmic simple source $m$ placed at point $q$ in a $xoz$ plane (Fig. 2.1.1) the potential $\phi$ due to it at a point $P$ in the same plane is given by

$$\phi(P) = -m \log r = -m \log |P - q|,$$  

(2.1.1)

where $P$ and $q$ are the position vectors specifying the points $P$ and $q$ respectively with respect to an arbitrary reference point $O$ and $r$ is the distance between $P$ and $q$.

Fig. 2.1.1: The source point $q$ and the field point $P$ in a $xoz$ reference frame with $z$-axis upward
Properties of Simple source potential

- The potential $\phi$ is defined everywhere except at the point $q$ where it has a singularity.
- $\nabla^2 \phi = 0$ everywhere except at the source point $q$, i.e., the potential $\phi$ satisfies Laplace’s equation everywhere except at $q$.
- The $\phi$ at infinity shows the behaviour 
  \[
  \phi(P) = -\left[ m \log|P| - |P|^{-1} (P \cdot q) m + O(|P|^{-2}) \right] 
  \]
  as $|P| \to \infty$.

2.1.2 Potential due to simple sources over a closed contour

For a smooth closed contour $\partial B$ defining the periphery of a vertical section of an infinitely long closed surface of density $\sigma$ (Fig. 2.1.2), the potential $\phi$ at a point $P$ in the plane of the contour is expressed as

\[
\phi(P) = -\int_{\partial B} \log|P - q| \sigma(q) dq,
\]  
(2.1.2)

Fig. 2.1.2: Interior domain $B_i$ is enclosed by the closed contour $\partial B$. Exterior domain $B_e$ lies outside $B_i$. Unit vectors $\hat{i}$ and $\hat{e}$ are internal and external normals respectively to $\partial B$ at $q$. 

\[\text{Diagram showing interior and exterior domains with source point } q\]
where \( |P - q| \) defines the distance between the points \( P \) and \( q \), \( \sigma(q) \) represents the line density at the boundary point \( q \) and \( dq \) represents the elementary arc length at \( q \). For the sake of further mathematical analysis, let us omit the negative sign to the integral and write the logarithmic potentials as

\[
\phi(P) = \int_{\partial B} \log |P - q| \sigma(q) dq, \quad P \in B_i
\]

and

\[
\phi(P) = \int_{\partial B} \log |P - q| \sigma(q) dq, \quad P \in B_e
\]

These define harmonic functions in \( B_i, B_e \) respectively and they remain continuous at \( \partial B \) as

\[
\phi(p) = \int_{\partial B} \log |P - q| \sigma(q) dq, \quad p \in \partial B
\]

It is evident from (2.1.3), (2.1.4) and (2.1.5) that

- The potential \( \phi \) is continuous everywhere including the boundary
- \( \nabla^2 \phi = 0 \) everywhere except at the boundary.
- The \( \phi \) at infinity shows the behaviour

\[
\phi(P) = \log |P| \int_{\partial B} \sigma(q) dq - |P|^{-1} \int_{\partial B} (P \cdot q) \sigma(q) dq + O(|P|^{-2})
\]

as \( |P| \to \infty \).

The tangential derivatives exist and continuous at \( p \in \partial B \) provided \( \sigma \) is Hölder continuous at \( p \), but the normal derivatives are discontinuous.

We write

\[
\frac{\partial}{\partial n_i} \log |p - q| = \log |p - q| = \log |q - p|,
\]
for the interior and exterior derivatives of \( \log|p - q| \) at \( p \) keeping \( q \) fixed. These have equal status and are connected by

\[
\log|p - q| + \log|q - p| = 0, \quad p \in \partial B \tag{2.1.6}
\]

For an interior point \( P \), the derivative of \( \phi \) at \( p \) in the direction \( \hat{n} \) is given by

\[
\frac{\partial}{\partial \hat{n}} \phi(P) = \phi_n(P) = \int \frac{\partial}{\partial \hat{n}} \log|P - q| \sigma(q) dq
\]

\[
= \int \log|q - P| \sigma(q) dq, \quad P \in B, \tag{2.1.7}
\]

exists and continuous in \( B \), for the integrand being regular and uniformly convergent in \( P \). As \( P \rightarrow p \in \partial B \), the integrand in (2.1.7) has a singularity at \( p \). Following Kellogg (1929), it can be established for \( \sigma \) satisfying Hölder continuity at \( p \) and \( n \) representing the interior normal \( i \) at \( \partial B \),

\[
\frac{\partial}{\partial i} \phi(p) = \phi_i(p) = \pi \sigma(p) + \int \log|p - q| \sigma(q) dq, \quad p \in \partial B. \tag{2.1.8}
\]

Following the sign convention of Jaswon (1963), treating both sides of \( \partial B \) as positive,

\[
\frac{\partial}{\partial \hat{n}_e} \phi(p) = \phi_{e}^i(p) = \pi \sigma(p) + \int \log|p - q| \sigma(q) dq, \quad p \in \partial B, \tag{2.1.9}
\]

for \( \phi \) defined in the exterior domain \( B_e \) (Fig. 2.1.2) and \( \sigma \) satisfying Hölder continuity at \( p \in \partial B \).
2.2 Potential due to double sources

2.2.1 Potential due to a double source

For a dipole of strength $\mu$ placed at a point $q$ in the direction $\hat{n}$ (Fig. 2.2.1), the potential $W$ due to it at a point $P$ is given by

$$W(P) = -\mu \log |P-q|_n.$$  
(2.2.1)

Fig 2.2.1: The doublet of strength $\mu$ having direction $\hat{n}$ is placed at the source point $q$ and $P$ defines the field point at a distance $r$ from $q$

For the sake of further mathematical analysis, let us omit the negative sign to the integral and write the logarithmic potentials as

$$W(P) = \mu \log |P-q|.$$ 

Properties of Double source potential

- The potential $W$ is defined everywhere except at the point $q$ where it has a singularity.
- $\nabla^2 W = 0$ everywhere except at the dipole at $q$.
- The $W$ vanishes at infinity with asymptotic behaviour $W = O\left(|P_n^{-1}| |P|\right)$ as $|P| \to \infty$.
2.2.2 Potential due to double sources over a closed contour

A continuous distribution of double sources of strength \( \mu \) over \( \partial B \) generates the double layer logarithmic potentials

\[
W(P) = \int_{\gamma_i} \log|P - q|\,\mu(q)\,dq, \quad P \in B_i
\]

and

\[
W(P) = \int_{\gamma_c} \log|P - q|\,\mu(q)\,dq, \quad P \in B_c
\]

These are harmonic functions in \( B_i, B_c \) respectively and

\[
W(P) = O(|P|^{-1}), \quad \text{as} \quad |P| \to \infty.
\]

The integral (2.2.2) suffers a discontinuity at \( \partial B \) as

\[
\lim_{P_i \to P} W(P_i) = W(p) - \pi\mu(p)
\]

and

\[
\lim_{P_c \to P} W(P_c) = W(p) + \pi\mu(p),
\]

where \( P_i \) and \( P_c \) are points on \( \gamma_i \) and \( \gamma_c \) respectively both emanating from \( p \in \partial B \).

It is evident from (2.2.2) and (2.2.5) that

- The potential \( W \) is continuous everywhere except at the boundary
- The potential \( W \) jumps by an amount \( \pi\mu \) at the boundary
- The \( W \) vanishes at infinity with asymptotic behaviour \( W = O(|P|^{-1})|P| \to \infty \)
2.3 Formulation of Dirichlet and Neumann Problems

2.3.1 Interior Problems

(a) Interior Dirichlet Problems

For a two-dimensional harmonic function \( \phi \) given over a smooth closed contour \( \partial B \), \( \phi \) in the interior domain \( B \), can be reproduced by simple layer logarithmic boundary density \( \sigma \) as

\[
\phi(P) = \int_{\partial B} \log|P - q| \sigma(q) dq, \quad P \in B. \tag{2.3.1}
\]

As \( P \to p \in \partial B \), we obtain the boundary relation

\[
\phi(p) = \int_{\partial B} \log|p - q| \sigma(q) dq, \quad p \in \partial B. \tag{2.3.2}
\]

Given \( \phi \) over \( \partial B \), (2.3.2) formulates a Dirichlet problem in an integral equation of the first kind in \( \sigma \) in terms of \( \phi \) over \( \partial B \).

This general equation was formulated by Hamel (1949) and Volterra (1959) without any further discussion on it. It has been shown by Jaswon and Symm (1977) that the equation (2.3.2) has a general solution

\[
\sigma = \sigma_0 + k \lambda, \tag{2.3.3}
\]

where \( \sigma_0 \) is a particular solution of (2.3.2), \( k \) is an arbitrary constant and \( \lambda \) satisfies

\[
1 = \phi(p) = \int_{\partial B} \log|p - q| \lambda(q) dq, \quad p \in \partial B \tag{2.3.4}
\]

for \( \partial B \neq \Gamma \)-contour for which equation (2.3.4) does not have a solution (Jaswon, 1963). The solution can be made unique on a particular choice of \( k \).
Given $\phi$ over $\partial B$, the interior Dirichlet problem can also be formulated following (2.2.5) by a double layer logarithmic boundary density $\mu$ as

$$\phi(p) = -\pi \mu(p) + \int_{\partial B} \log|p - q| \mu(q) dq, p \in \partial B. \quad (2.3.5)$$

Following Kellogg (1929), equation (2.3.5) in $\mu$ has a solution if

$$\int_{\partial B} \phi(p) \lambda(p) dp = 0, \quad (2.3.6)$$

where $\lambda$ is the solution of the corresponding adjoint homogeneous equation

$$0 = -\pi \lambda(p) + \int_{\partial B} \log|p - q| \lambda(q) dq, p \in \partial B \quad (2.3.7)$$

which is mathematically equivalent to

$$0 = \pi \lambda(p) + \int_{\partial B} \log|p - q| \lambda(q) dq, p \in \partial B \quad (2.3.8)$$

by virtue of (2.1.6). It can be established that the equation (2.3.8) does not have a non-trivial $\lambda$. This $\lambda$ satisfies (2.3.6) for an arbitrary $\phi$ on $\partial B$. Hence, following Kellogg (1929), equation (2.3.5) has a unique solution $\mu$ for an arbitrary $\phi$ over $\partial B$.

(b) **Interior Neumann Problems**

For $\phi\prime$ prescribed over $\partial B$, the $\sigma$ that reproduces the $\phi$ in $B$, $+\partial B$, can be obtained for $\partial B \neq \Gamma$ as a solution of the normal derivative equation

$$\phi\prime(p) = \pi \sigma(p) + \int_{\partial B} \log|p - q| \sigma(q) dq, p \in \partial B, \quad (2.3.9)$$

formed by (2.1.8). Equation (2.3.9) expresses an interior Neumann problem by a Fredholm integral equation of the second kind in $\sigma$ in terms of $\phi\prime$ given on $\partial B$.

Following Kellogg (1929), this has a solution if

$$\int_{\partial B} \phi\prime(p) \lambda(p) dp = 0 \quad (2.3.10)$$
where $\lambda$ is the solution of the corresponding adjoint homogeneous equation

$$0 = \pi \lambda(p) + \int_{\partial B} \log|p-q| \lambda(q) \, dq, \quad p \in \partial B$$  \hspace{1cm} (2.3.11)

which, by virtue of

$$\int_{\partial B} \log|p-q| \, dq = -\pi, \quad p \in \partial B$$  \hspace{1cm} (2.3.12)

has a non-trivial solution $\lambda = 1$ on $\partial B$. On substitution of this $\lambda$ in (2.3.10), we arrive at the Gauss' condition

$$\int_{\partial B} \phi'(p) \, dp = 0$$  \hspace{1cm} (2.3.13)

for a $\phi$ harmonic in $B$. This ensures the existence of a solution of equation (2.3.9).

The solution can be written as

$$\sigma = \sigma_0 + k\lambda$$  \hspace{1cm} (2.3.14)

where $\sigma_0$ is a particular solution of (2.3.9), $k$ is an arbitrary constant and $\lambda$ is the solution of (2.3.4). This solution when substituted in (2.3.1) produces a series of $\phi$ in $B$, as

$$\phi = \phi_0 + k$$  \hspace{1cm} (2.3.15)

having the interior normal derivative as prescribed on $\partial B$. The solution can be made unique on proper choice of $k$. 
2.3.2 Exterior Problems

(a) Exterior Dirichlet Problems

The boundary density $\sigma$ obtained as solution of equation (2.3.2) i.e.

$$\phi(p) = \int_{\partial B} \log |p - q| \sigma(q) dq, \quad p \in \partial B,$$

(2.3.16)

for $\partial B \neq \Gamma$, generates a potential $V$ in $B$, that solves the interior Dirichlet problem for $\overline{B}$. The $\sigma$ generates an exterior potential $V_o$ characterized by logarithmic behaviour at infinity, whereas the classical existence-uniqueness theorem (Kellogg 1929) specifies $O(1)$ behaviour, implying boundedness on $V_o$ at infinity.

It has been shown by Jaswon and Symm (1977) that the equation (2.3.16) for the exterior domain $B_e$ has a solution

$$\sigma = \sigma_o + k\lambda,$$

(2.3.17)

$$1 = \int_{\partial B} \log |p - q| \lambda(q) dq, \quad p \in \partial B,$$

(2.3.18)

where $\sigma_o$ satisfies

$$\phi_o(p) = \int_{\partial B} \log |p - q| \sigma_o(q) dq, \quad p \in \partial B,$$

(2.3.19)

with existence condition

$$\int_{\partial B} \phi_o(p) \lambda(p) dp = \int_{\partial B} \sigma_o(q) dq = 0$$

(2.3.20)

implying $\phi_o = O(r^{-1}), r \to \infty$, and $\sigma_o$ is the unique solution of (2.3.19).

The solution (2.3.17) can be made unique on proper choice of $k$.
(a) Exterior Neumann Problems

For \( \phi_e \), defining the exterior normal derivative of an exterior harmonic function \( \phi \), with asymptotic behaviour \( \phi = O(r^{-1}) \), \( r \to \infty \), given \( \phi_e \) over \( \partial B \) (Fig. 2.1.2), \( \phi \) in \( B_e \) can be obtained as a potential due to a simple layer boundary density \( \sigma \) on \( \partial B \) (\( \neq \Gamma \) contour) as

\[
\phi(p) = \int_{\partial B} \log|p - q| \sigma(q) dq, \quad p \in B_e \tag{2.3.21}
\]

Following (2.1.9), the \( \sigma \) of (2.3.21) is related to \( \phi_e \) on \( \partial B \) as

\[
\phi_e(p) = \pi \sigma(p) + \int_{\partial B} \log|p - q| \sigma(q) dq, \quad p \in \partial B \tag{2.3.22}
\]

Given \( \phi_e \) over \( \partial B \), equation (2.3.22) expresses an exterior Neumann problem in a Fredholm boundary integral equation of the second kind for \( \sigma \) in terms of \( \phi_e \) over \( \partial B \). This equation has a solution if

\[
\int_{\partial B} \phi_e(p) \lambda(p) dp = 0 \tag{2.3.23}
\]

where \( \lambda \) satisfies the adjoint homogeneous equation

\[
0 = \pi \lambda(p) + \int_{\partial B} \log|p - q| \lambda(q) dq, p \in \partial B \tag{2.3.24}
\]

That the homogeneous component of (2.3.22)

\[
0 = \pi \sigma(p) - \int_{\partial B} \log|p - q| \sigma(q) dq, p \in \partial B,
\]

does not have a non-trivial solution. Hence, following Kellogg (1929), we conclude that the adjoint homogeneous equation (2.3.24) does not have a non-trivial solution. This implies, the condition (2.3.23) is satisfied for an arbitrary \( \phi_e \) on \( \partial B \). Consequently, the equation (2.3.22) has a solution for an arbitrary \( \phi_e \) on \( \partial B \) and this solution is unique.
2.4 Green's Formulae

2.4.1 Green's Formulae for interior domain

For a harmonic function $\phi$ defined in an interior domain $B$, bounded by a smooth closed contour $\partial B$, Green's formula in two dimensions takes the form

$$\int_{\partial B} \log|P-q|\phi(q) dq - \int_{\partial B} \log|P-q|\phi'(q) dq = -2\pi \phi(P), \; P \in B, \quad (2.4.1)$$

For the field point $P$ located on $\partial B$, the boundary formula in two-dimensions is written as,

$$\int_{\partial B} \log|P-q|\phi(q) dq - \int_{\partial B} \log|P-q|\phi'(q) dq = -\pi \phi(P), \; P \in \partial B \quad (2.4.2)$$

Given $\phi$ on $\partial B$, equation (2.4.2) expresses an interior Dirichlet problem for $\phi$, in terms of $\phi$, by a Fredholm boundary integral equation of the first kind in $\phi$, as

$$\int_{\partial B} \log|P-q|\phi'(q) dq = \pi \phi(P) + \int_{\partial B} \log|P-q|\phi(q) dq, \; P \in \partial B. \quad (2.4.3)$$

This equation is of the type (2.3.2) which has been proved to have a unique solution.

2.4.2 Green's Formulae for exterior domain

To discuss Green's formulae for the exterior domain $B_e$, let us assume $\phi = O(r^{-1}), r \to \infty$. We know changing of $i$ into $e$ yields the analogous exterior formulae under the new sign convention of Jaswon (1963). For example, under this rule, the formula (2.4.1) yields the exterior formula

$$\int_{\partial B} \log|P-q|\phi(q) dq - \int_{\partial B} \log|P-q|\phi'(q) dq = -2\pi \phi(P), \; P \in B_e \quad (2.4.4)$$

and (2.4.2) yields the boundary formula for the exterior $\phi$ as
where $\pi$ signifies the external angle at $p$.

Given $\phi$ on $\partial B$, the relation (2.4.5) yields the Boundary equation

$$\int_{\partial B} \log|p-q| \phi(q) dq - \int_{\partial B} \log|p-q| \phi(q) dq = -\pi \phi(p), \quad p \in \partial B$$  \hspace{1cm} (2.4.5)

which expresses the exterior Dirichlet Problem for $\phi_e$ in a Fredholm integral equation of the first kind in $\phi_e$ in terms of $\phi$ on $\partial B$.

This equation is of type (2.3.16) for the exterior domain $B_e$ which has a solution for $O(1)$ behaviour of $\phi$ and that can be made unique. Since the $\phi$ under discussion vanishes at infinity, the equation (2.4.6) has a unique solution.

Given $\phi_e$ on $\partial B$, equation (2.4.6) expresses an exterior Neumann Problem in a Fredholm integral equation of the second kind in $\phi$ as

$$\int_{\partial B} \log|p-q| \phi(q) dq + \pi \phi(p) = \int_{\partial B} \log|p-q| \phi(q) dq$$  \hspace{1cm} (2.4.6)

Following Kellogg (1929), this has a solution if and only if

$$\int_{\partial B} \left\{ \int_{\partial B} \log|p-q| \phi(q) dq \right\} \lambda(p) dq = 0,$$

where $\lambda$ satisfies the adjoint homogenous equation

$$0 = \pi \lambda(p) + \int_{\partial B} |p-q| \lambda(q) dq, \quad p \in \partial B$$

This equation does not have a non-trivial solution $\lambda$ as discussed in the equation (2.3.22). Hence, the exterior Neumann problem, expressed by equation (2.4.7) has a unique solution for $\phi \sim O(r^{-1}), r \to \infty$ and $\partial B \neq \Gamma$ contour.
2.4.3 Reproduction of a Harmonic function as Simple and double layer potential

Let a two-dimensional harmonic function $\phi$ with asymptotic behaviour $\phi = O(r^n), n \geq 1$, $r \to \infty$, be defined in the upper half-space domain bounded below by a general half-space boundary $S$. Let us consider the $\phi$ above $S$ in a closed domain $B_i$ as shown in Fig. 2.4.1, bounded above by a semicircle $S_u$ of radius $R$. Given $\phi$ and its interior normal derivative $\phi_i$ over $\partial B(=S+S_u)$, $\phi$ in the interior is given by Green's formula as

$$-2\pi\phi(P) = \int_{\partial B} \log|q-P|\phi(q)\,dq - \int_{\partial B} \log|q-P|\phi'_i(q)\,dq, \quad P \in B_i \quad (2.4.8)$$

Fig. 2.4.1: The closed domain $B_i$ bounded below by $S$ and above by a semicircle $S_u$ of Radius $R$, $R \to \infty$

Let us now consider an exterior harmonic function $f$ with asymptotic behaviour $f = O(r^n), n \geq 1, r \to \infty$ defined in the exterior domain $B_e$ bounded at interior by $\partial B$. 

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Following Green's identity II, its boundary data satisfy

\[ \int_{\partial B} \log |q - P| f(q) dq - \int_{\partial B} \log |q - P| f'_e(q) dq = 0, \quad P \in B, \]  

(2.4.9)

Superposition of (2.4.9) on formula (2.4.8) yields, by virtue of

\[ \log |q - P| = \log |q - P|, \quad q \in \partial B, \]

\[ \int_{\partial B} \log |q - P| \left( \phi(q) - f(q) \right) dq - \int_{\partial B} \log |q - P| \left( \phi'_e(q) + f'_e(q) \right) dq = -2\pi \phi(P), \quad P \in B, \]

(2.4.10)

already shown by Jaswon and Symm (1977).

Now we consider two distinct possibilities for \( f \)

(a) For \( f = \phi \) over \( \partial B \), we find

\[ \int_{\partial B} \log |q - P| \left( \phi'_e(q) + f'_e(q) \right) dq = 2\pi \phi(P), \quad P \in B, \]

(2.4.11)

This provides a simple layer representation of \( \phi \) in \( B \), with source density

\[ \sigma(q) = \frac{1}{2\pi} \left\{ \phi'_e(q) + f'_e(q) \right\} \]

(2.4.12)

Existence of a unique exterior \( f \) with asymptotic behaviour as assumed above, satisfying \( f = \phi \) over \( \partial B \), is ensured by the exterior Dirichlet existence theorem.

(b) The second possibility \( f'_e = -\phi'_e \) over \( \partial B \), provides the representation

\[ \int_{\partial B} \log |q - P| \left( \phi(q) - f(q) \right) dq = -2\pi \phi(P), \quad P \in B, \]
This is a double layer potential generated by source density

\[ \mu(q) = -\frac{1}{2\pi} \{ \phi(q) - f(q) \}. \]  

(2.4.13)

Existence of a unique \( f \) in \( B_\varepsilon \) satisfying \( f = -\phi \) over \( \partial B \) is ensured by exterior Neumann existence theorem.

For \( f \) and \( \phi \) vanishing at infinity in same order over \( S_u \) (Fig. 2.4.1) as \( r \to \infty \), we find for \( q \in S_u \)

\[ \sigma(q) = O(\phi), |q| \to \infty, \]  

(2.4.14)

and

\[ \mu(q) = O(\phi), |q| \to \infty, \]  

(2.4.15)

from (2.4.12) and (2.4.13) respectively.

A quick verification of (2.4.14) and (2.4.15) comes from the boundary relations of \( \phi \) and \( \sigma \) and that of \( \phi \) and \( \mu \) over a half-space horizontal boundary \( \overline{S} \). For \( \phi \) in \( B_\varepsilon \) (Fig. 2.4.1) given by

\[ \phi(P) = \int_{\overline{S}} \log|P - q| \sigma(q) dq, P \in B_\varepsilon. \]

the normal derivative relation \( \phi \) and \( \sigma \) over \( \overline{S} \) is

\[ \phi'(p) = \pi \sigma(p) + \int_{\overline{S}} \log|p - q| \sigma(q) dq = \pi \sigma(p), p \in \overline{S} \]  

(2.4.16)

the integral over \( \overline{S} \) being zero for both \( p, q \in \overline{S} \). For \( \phi \) in \( B_\varepsilon \), given by
\[ \phi(P) = \int_{\Sigma} \log |P - q| \mu(q) dq, \quad P \in B, \]

and the boundary relation of \( \phi \) and \( \mu \) over \( \overline{S} \) is

\[ \phi(p) = -\pi \mu(p) + \int_{\Sigma} \log |p - q| \mu(q) dq = -\pi \mu(p), \quad p \in \overline{S}, \quad (2.4.17) \]

the integral on the right hand side of (2.4.17) vanishing for \( p, q \in \overline{S} \).