Chapter 6

Discrete SMC Algorithm using UDE

The chapter details design and validation of a delta-operator based discrete SMC algorithm for uncertain systems. The control law is synthesized by estimating states and uncertainties using UDE. The design is tested for motion control problem.

The motivation and idea is introduced in section 6.1. Section 6.2 describes the problem formulation with necessary assumptions. The design of unifying sliding condition is illustrated in 6.3. Section 6.4 includes the model-following control along with uncertainty estimation. The design of observer is explained next in section 6.5. Section 6.6 elaborates the Lyapunov stability analysis. The performance is illustrated by an application to motion control in section 6.7. The chapter concludes with a summary in Section 6.8.

6.1 Introduction

The pervasive use of digital controllers has necessitated a need to generalize the concept of sliding mode to discrete-time control systems (Misawa, 1997a,b). The digital implementation requires a certain sampling interval, which causes not only chattering along the sliding surface but also possible instability with a large gain. The discrete systems may become unstable with a large sampling period. The performance of discrete controllers at moderate sampling periods is of importance.
In discrete-time, the control input is computed at discrete instants of time and avoid instantaneous switching. The system can thus undergo only quasi-sliding motion i.e. the state of the system can approach the switching surface but cannot generally stay on it (Utkin, 1977; Sarpturk et al., 1987). The magnitude of quasi-sliding depends on the extent of uncertainty and sampling period (Furuta, 1990). A large number of control laws for discrete time sliding mode (Bartoszewicz, 1998; Milosavljevic, 1985; Chan, 1999; Gao et al., 1995; Yu and Yu, 2000; Ramirez, 1991) have been proposed in the literature.

The delta-operator (Middleton and Goodwin, 1986) can be used for synthesizing a discrete-time-controller (Jabbari, 1991; Collins, 1999; Veselic et al., 2008). The delta-operator approach offers attractive features like superior finite world length coefficient representation and convergence to its continuous counterpart as the sampling period decreases to zero. The issue of matching conditions in discretized systems is addressed in (Tesfaye and Tomizuka, 1995) and unification of sliding condition in (Ginoya et al., 2015b).

The traditional SMC also requires all system states to be available. The increased number of sensors for state measurement makes overall system complex and expensive. Observer design for systems with uncertainties, disturbances and noise, is a major problem. A sliding mode observer is designed in (Slotine et al., 1987) that uses additional switching terms to counter the effects of uncertainties. The problem of state observation in presence of uncertainties of known bounds is dealt in (Walcott and Zak, 1988). An improvement is proposed in (Chen and Saif, 2006). A combined state and perturbation observer for discrete-time systems is available in (Kwon and Chung, 2003).

In this work, continuous-time SMC combined with UDE is extended to a discrete-time case. The UDE used in combination with SMC makes it possible to use a smooth control without having to employ a smoothing approximation. A notable feature of the proposed control is that it affords control over the magnitude of the quasi-sliding for a given sampling period. A unifying sliding condition is used and control is designed for model-following. The design is robustified by designing a observer that estimates states, uncertainty and disturbance. The control design is validated on a benchmark motion control problem.
6.2 Problem Formulation

Consider a continuous-time linear plant described by,

\[ \dot{x}(t) = A_c x(t) + B_c u(t) + F_c d(t) \]  

(6.1)

where \( A_c = A_{nc} + \Delta A_c \), \( B_c = B_{nc} + \Delta B_c \), with ‘nc’ as the nominal part of uncertain continuous time system, \( x(t) \) is state vector, \( u(t) \) is control input, \( d(t) \) is unknown disturbance and \( \Delta A_c \) and \( \Delta B_c \) are uncertainties.

**Assumption 6.1**  \( A_c \) and \( B_c \) is stabilizable and the uncertainties \( \Delta A_c \) and \( \Delta B_c \) satisfy matching conditions given by,

\[ \Delta A_c = B_{nc} \Delta a \quad \Delta B_c = B_{nc} \Delta b \quad F_c = B_{nc} \Delta f \]  

(6.2)

The plant in (6.1) is discretized using \( \delta \) operator ([Middleton and Goodwin, 1990]) and the modified form given by [Tesfaye and Tomizuka (1995)].

The plant in (6.1) can now be written as,

\[ \delta x_k = A x_k + B u_k + B (b u_k + E_k) \]  

(6.3)

where

\[ E_k = \Delta A x_k + w_k \]

\[ w_k = (1 + \Delta_a B_{nc} T/2) \Delta_f d_k \]

and the system parameters are given by:

\[ A = \frac{e^{A_{nc} T} - I}{T}, \quad B = \frac{1}{T} \int_0^T \frac{e^{A_{nc} \tau} - I}{T} B_{nc} d\tau \]  

(6.4)

\[ b = \Delta_a B_{nc} (1 + \Delta_b) \frac{T}{2} + \Delta_b \]  

(6.5)

\[ \Delta A = \Delta_a [I + (A_{nc} + B_{nc} \Delta_a) \frac{T}{2}] \]  

(6.6)

This system can be written as

\[ \delta x_k = A x_k + B u_k + B e_k \]  

(6.7)

where \( e_k \) is the lumped uncertainty.
Remark 6.1  Shift operator and z-transform which forms the basis of most discrete time analysis are inappropriate with fast sampling, have no continuous counterpart (Middleton and Goodwin, 1990). Better correspondence is obtained between continuous and discrete time, if the shift operator is replaced with a different operator, more like derivative.

\[
\delta = \frac{q - 1}{T}
\]

where \( T \) is a sampling period

\[
\delta x_k = \frac{x_{k+1} - x_k}{T}
\] (6.8)

Remark 6.2  Generally \( q \) leads to simpler expressions and emphasizes the sequential nature of sampled signals. On the other hand \( \delta \) leads to models that are more alike models in \( d/dt \). Using the operator \( \delta \), any polynomial in \( q \) of degree \( n \) will be exactly equivalent to some polynomial in \( \delta \) of degree \( n \).

Assumption 6.2  The disturbance vector \( d_k \) is such that its derivatives up to \( r \)-th order are bounded. More specifically

\[
\|\delta^{(r)} e_k\| \leq \mu \quad \text{for} \quad r \geq 0
\] (6.9)

where \( \mu \) is a positive constant and \( \delta^{(r)} e_k \) stands for the \( r \)-th derivative of \( e_k \).

Remark 6.3  This assumption admits a fairly large class of uncertainties. It may be noted that the constant \( \mu \) is not required to be known.

The objective is to design a discrete control \((u_k)\) such that, the uncertain plant follows the desired model inspite of uncertainties and disturbances \((e_k)\). The control is expected to ensure robust performance for varying cases.

The control is to be designed initially, by assuming that the plant states \((x)\) are available. However, the issue of non-availability of states is also expected to be addressed. An observer to estimate states in presence of parametric uncertainties and disturbances is required; and UDE is to be utilized for estimation of states and disturbances. The estimated states and disturbances are to be employed to synthesize a robust discrete observer-controller structure.
6.3 Sliding Condition

Let $s_k$ be the value of the continuous sliding variable $\sigma$ at the $k^{th}$ sampling instant and $\Delta s_k = s_{k+1} - s_k$. For sliding to occur,

$$s_{k+1}^2 < s_k^2$$  \hfill (6.10)
$$s_{k+1}^2 - s_k^2 < 0$$  \hfill (6.11)

Rearranging (6.11),

$$(s_{k+1} - s_k)(s_{k+1} - s_k + 2s_k) < 0$$  \hfill (6.12)

The equation (6.12) can be written as,

$$\Delta s_k(\Delta s_k + 2s_k) < 0$$  \hfill (6.13)
$$\Delta s_k^2 < -2s_k\Delta s_k$$  \hfill (6.14)

The necessary and sufficient condition for (6.14) to hold is derived.

For sliding to occur, if $s_k > 0$, then $\Delta s_k < 0$. Using this in (6.14)

$$\begin{align*}
-2s_k &< \Delta s_k \\
-2s_k &< \Delta s_k < 0
\end{align*}$$  \hfill (6.15)

Similarly if $s_k < 0$, then $\Delta s_k > 0$. Using this in (6.14)

$$\begin{align*}
\Delta s_k &< -2s_k \\
0 &< \Delta s_k < -2s_k \\
2s_k &< -\Delta s_k < 0
\end{align*}$$  \hfill (6.16)

Multiplying (6.15) by $s_k$ and (6.16) by $-s_k$,

$$-2s_k^2 < s_k\Delta s_k < 0$$  \hfill (6.17)

As $\delta s_k = \frac{\Delta s_k}{T}$ the sliding condition (6.17) can be written as,

$$\frac{-2s_k^2}{T} < s_k\delta s_k < 0$$  \hfill (6.18)

Remark 6.4 The condition (6.18) clearly shows that two conditions must be satisfied for sliding to occur in discrete time system. The condition (6.18) further collapses into the familiar $\sigma\dot{\sigma} < 0$, as the sampling time $T \to 0$. 

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6.4 Model Following Control

Consider the discretized continuous system (6.7)

\[ \delta x_k = Ax_k + Bu_k + Be_k \]  

and the sliding surface (Ackermann and Utkin, 1998)

\[ s_k = B^T x_k + z_k \]  

where

\[ \delta z_k = -B^T A_m x_k - B^T B_m u_m k, \quad z_0 = -B^T x_0 \]  

The choice of \( A_m \) and \( B_m \) is made in such a way that

\[ \delta x_m = A_m x_m + B_m u_m \]  

will have desired response, like the model in a model following system.

Assumption 6.3 The choice of model is such that it satisfies the matching condition \( A - A_m = BL \) and \( B_m = BM \), where \( L \) and \( M \) are known matrices of appropriate dimensions.

From (6.20)

\[ \delta s_k = B^T \delta x_k + \delta z_k \]

\[ = B^T A x_k + B^T B u_k + B^T B e_k - B^T A_m x_k - B^T B_m u_m k \]

\[ = B^T B L x_k - B^T B M u_m k + B^T B u_k + B^T B e_k \]

Selecting

\[ u_k = u_k^{\text{eq}} + u_k^n \]  

The equivalent control \( (u_k^{\text{eq}}) \) can be derived as,

\[ u_k^{\text{eq}} = -[L x_k - M u_m k] - (B^T B)^{-1} K s_k \]  

where \( K \) is positive constant.

The dynamics of sliding surface can be written as,

\[ \delta s_k = (B^T B) u_k^n + (B^T B) e_k - K s_k \]  

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Selecting \( u_k^n = -\hat{e}_k \) and using \( \hat{e}_k = e_k - \hat{e}_k \) with \( \hat{e}_k \approx e_k \), the dynamics in (6.26) can be written as,

\[
\delta s_k = -K s_k
\]  \hspace{1cm} (6.27)

leading to

\[
s_k \delta s_k = -K s_k^2 < 0
\]  \hspace{1cm} (6.28)

Further, if \( K < \frac{2}{T} \) then discrete sliding condition

\[
\frac{-2 s_k^2}{T} < s_k \delta s_k < 0
\]  \hspace{1cm} (6.29)

is satisfied for arbitrarily small values of \( s_k \).

### 6.4.1 Estimation of Uncertainty \((e_k)\)

The UDE algorithm is based on the assumption that a signal can be approximated and estimated using a filter of right bandwidth. The lumped uncertainty \( e(x, u, t) \) can be estimated as, \( \hat{e}(x, u, t) = G_f(s)e(x, u, t) \), where \( G_f(s) \) is a low-pass filter with unity steady state gain and sufficiently large bandwidth.

Using (6.26),

\[
e_k = (B^T B)^{-1}(\delta s_k + K s_k) - u_k^n
\]  \hspace{1cm} (6.30)

Let \( G_f(\gamma) \) be unity gain digital filter with unity steady state gain, where ‘\( \gamma \)’ is the transform variable as used in (Middleton and Goodwin, 1990).

Using the concept of UDE,

\[
\hat{e}_k = e_k G_f(\gamma)
\]  \hspace{1cm} (6.31)

Now consider a digital filter given by

\[
G_f(\gamma) = \frac{1 - e^{-T\gamma}}{1 + T\gamma - e^{-T/\tau}}
\]  \hspace{1cm} (6.32)

which is digital equivalent of continuous filter \( G_f(s) = 1/(\tau s + 1) \), in \( \delta \)-domain. Therefore,

\[
\hat{e}_k = [(B^T B)^{-1}(\delta s_k + K s_k) - u_k^n] G_f(\gamma)
\]  \hspace{1cm} (6.33)

\[
= (B^T B)^{-1}(\delta s_k + K s_k) \frac{G_f(\gamma)}{1 + G_f(\gamma)}
\]  \hspace{1cm} (6.34)

\[
= (B^T B)^{-1} (s_k + \frac{K s_k}{\gamma}) \left( \frac{1 - e^{-T/\tau}}{T} \right)
\]  \hspace{1cm} (6.35)
6.4.2 Improvement in Estimation – 2nd order UDE

The accuracy of estimation in UDE depends on the order of filter. The results with first-order filter are extended to show that error in estimation is reduced, if a second-order filter is used.

A second-order filter in delta-form Middleton and Goodwin (1990) is defined as,

\[ G_f(\gamma) = \frac{\gamma \alpha \tau - \gamma e^{-T/\tau} + \alpha^2 \tau}{\tau (\gamma + \alpha)^2} \]  \hspace{1cm} (6.36)

where \( \alpha = \frac{1 - e^{-T/\tau}}{T} \).

The lumped uncertainty can be written as,

\[ e_k = e_k G_f(\gamma) + e_k(1 - G_f(\gamma)). \]  \hspace{1cm} (6.37)

Using (6.36) and (6.37),

\[ e_k = e_k G_f(\gamma) + e_k \left( \frac{\gamma^2 \tau + \gamma \alpha \tau + \gamma e^{-T/\tau}}{\gamma^2 \tau + 2 \gamma \tau \alpha + \alpha^2 \tau} \right) \]  \hspace{1cm} (6.38)

\[ = e_k G_f(\gamma) + e_k \left( \frac{\gamma^2 \tau + \gamma \alpha \tau + \gamma e^{-T/\tau}}{\gamma^2 \tau + 2 \gamma \tau \alpha + \alpha^2 \tau} \right) \]  \hspace{1cm} (6.39)

Simplifying the above equation leads to,

\[ e_k = e_k G_f(\gamma) + e_k \left( \frac{\gamma \alpha \tau + \gamma e^{-T/\tau} + \gamma^2 \tau}{\gamma \alpha \tau - \gamma e^{-T/\tau} + \alpha^2 \tau} \right) G_f(\gamma) \]  \hspace{1cm} (6.40)

The estimation of lumped uncertainty is,

\[ \hat{e}_k = e_k G_f(\gamma) \left( 1 + \frac{\gamma \alpha \tau + \gamma e^{-T/\tau} + \gamma^2 \tau}{\gamma \alpha \tau - \gamma e^{-T/\tau} + \alpha^2 \tau} \right) \]  \hspace{1cm} (6.41)

Using (6.30) and (6.41),

\[ \hat{e}_k = \left( 1 + \frac{\gamma \alpha \tau + \gamma e^{-T/\tau} + \gamma^2 \tau}{\gamma \alpha \tau - \gamma e^{-T/\tau} + \alpha^2 \tau} \right) G_f(\gamma)((B^T B)^{-1}(\gamma + k)s_k - u_k^e) \]  \hspace{1cm} (6.42)

Using (6.40) and (6.41),

\[ \tilde{e}_k = e_k \left( \frac{\gamma^2 \tau}{\gamma \alpha \tau - \gamma e^{-T/\tau} + \alpha^2 \tau} \right) G_f(\gamma) \]  \hspace{1cm} (6.43)
6.4.3 Control Design

The unified controller design method using a second order filter is presented here.

Selecting,

\[ u^n_k = -\hat{e}_k \]  \hspace{1cm} (6.44)

Using (6.42),

\[ u^n_k = -\left(1 + \frac{\gamma \alpha \tau + \gamma e^{-T/\tau}}{\gamma \alpha \tau - \gamma e^{-T/\tau} + \alpha^2 \tau}\right) G_f(\gamma) ((B^T B)^{-1}(\gamma + k)s_k - u^n_k) \]  \hspace{1cm} (6.45)

solving for left-hand side,

\[ u^n_k \left[ 1 - \left(1 + \frac{\gamma \alpha \tau + \gamma e^{-T/\tau}}{\gamma \alpha \tau - \gamma e^{-T/\tau} + \alpha^2 \tau}\right) G_f(\gamma) \right] \]  \hspace{1cm} (6.46)

simplifies to,

\[ = u^n_k \left[ 1 - \frac{2\gamma \alpha \tau + \alpha^2 \tau}{\gamma^2 \tau + 2\gamma \alpha \tau + \alpha^2 \tau} \right] \]  \hspace{1cm} (6.47)

\[ = u^n_k \left[ \frac{\gamma^2 \tau}{\gamma^2 \tau + 2\gamma \alpha \tau + \alpha^2 \tau} \right] \]  \hspace{1cm} (6.48)

Simplifying right-hand side of (6.45)

\[ = -G_f(\gamma) \left(\frac{2\gamma \alpha \tau + \alpha^2 \tau}{\gamma \alpha \tau - \gamma e^{-T/\tau} + \alpha^2 \tau}\right) ((B^T B)^{-1}(\gamma + k)s_k) \]  \hspace{1cm} (6.49)

with \( G_f(\gamma) \)

\[ = -\left(\frac{2\gamma \alpha \tau + \alpha^2 \tau}{\gamma^2 \tau + 2\gamma \alpha \tau + \alpha^2 \tau}\right) ((B^T B)^{-1}(\gamma + k)s_k) \]  \hspace{1cm} (6.50)

Equating (6.47) and (6.50)

\[ u^n_k(\gamma^2 \tau) = -(2\gamma \alpha \tau + \alpha^2 \tau)(\gamma + k)((B^T B)^{-1}s_k) \]  \hspace{1cm} (6.51)

simplifies to control law,

\[ u^n_k = -(B^T B)^{-1} \left[ 2\alpha + \left(\frac{2K\alpha + \alpha^2}{\tau} + \frac{\alpha^2 K}{\gamma^2}\right) \right] (s_k) \]  \hspace{1cm} (6.52)
6.5 Design of Observer

A discrete observer to simultaneously estimate the states and uncertainties is designed here using UDE.

The discrete plant and model as defined in (6.7) and (6.22) are rewritten as,

\[ \begin{align*}
\delta x_k &= A x_k + B u_k + B e_k \\
y_k &= C x_k
\end{align*} \]  
(6.53)

\[ \begin{align*}
\delta x_{mk} &= A_m x_{mk} + B_m u_{mk} \\
y_{mk} &= C_m x_{mk}
\end{align*} \]  
(6.54)

An observer is defined as,

\[ \begin{align*}
\delta \hat{x}_k &= A \hat{x}_k + B u_k + B \hat{e}_k + J (y_k - \hat{y}_k) \\
\hat{y}_k &= C \hat{x}_k
\end{align*} \]  
(6.55)

where, \( \hat{x}, J, \hat{e}_k \) are observer state vector, observer gain matrix and estimate of the lumped uncertainty respectively.

The observation error \( \tilde{x}_k = x_k - \hat{x}_k \) has an exponentially convergent dynamics as,

\[ \delta \tilde{x}_k = (A - JC) \tilde{x}_k + B \hat{e}_k \]  
(6.56)

where, \( \hat{e}_k = e_k - \hat{e}_k \).

**Assumption 6.4** This assumption is necessary to guarantee the asymptotic stability of observer.

1. The pair \((A, B)\) is controllable
2. The pair \((A, C)\) is observable
3. The triplet \((A, C, B)\) has no invariant zeros, i.e. for all \( \lambda \in C \)

\[
\text{Rank} \begin{bmatrix} \lambda I - A & -B \\ C & 0 \end{bmatrix} = n + 1
\]
The state estimation error dynamics are derived from (6.53), (6.55) as,

\[ \delta \tilde{x}_k = (A - JC) \tilde{x}_k + B \tilde{e}_k \]  
\[ \tilde{y}_k = C \tilde{x}_k \]  

(6.57)

The estimation of uncertainty is defined in (6.31) as,

\[ \hat{e}_k = e_k G_f(\gamma) = e_k \frac{1 - e^{-T/\tau}}{1 + T\gamma - e^{-T/\tau}} \]  

(6.58)

where \( \hat{e}_k \) is estimate of uncertainty and \( G_f(\gamma) \) is discrete first order low pass filter.

Therefore,

\[ \hat{e}_k + \delta \hat{e}_k - e^{-T/\tau} \hat{e}_k = e_k - e^{-T/\tau} e_k \]  
\[ T \delta \hat{e}_k = (e_k - \hat{e}_k) - (e_k - \hat{e}_k) e^{-T/\tau} \]  

(6.59)

(6.60)

From (6.53) the lumped uncertainty is written as,

\[ e_k = B^+ (\delta \tilde{x}_k - A \tilde{x}_k - Bu_k) \]  
\[ = B^+ [J(y_k - \hat{y}_k) + B \hat{e}_k] \]  
\[ = \hat{e}_k + B^+ JC \tilde{x}_k \]  

(6.61)

(6.62)

(6.63)

Using (6.60) and (6.61),

\[ \delta \hat{e}_k = \frac{1}{T} [B^+ JC \tilde{x}_k (1 - e^{-T/\tau})] \]  

(6.64)

Subtracting both the sides of above equation from \( \delta e_k \) and with Assumption 6.2,

\[ \delta \tilde{e}_k = - \frac{1}{T} [B^+ JC \tilde{x}_k (1 - e^{-T/\tau})] + \delta e_k \]  

(6.65)

Therefore, combining (6.58) and (6.65)

\[
\begin{bmatrix}
\delta \tilde{x}_k \\
\delta \tilde{e}_k
\end{bmatrix} =
\begin{bmatrix}
(A - JC) & B \\
-B^+ JC \frac{1}{T} (1 - e^{-T/\tau}) & 0
\end{bmatrix}
\begin{bmatrix}
\tilde{x}_k \\
\tilde{e}_k
\end{bmatrix} +
\begin{bmatrix}
0 \\
1
\end{bmatrix} \delta e_k
\]

(6.66)

The observer dynamics can be stabilized with proper choice of \( J \) and \( T \), if the pair \( (A, C) \) is observable. Thus \( \tilde{x}_k \rightarrow 0 \) and \( \tilde{e}_k \rightarrow 0 \) under the Assumption 6.4 and 6.2.
6.6 Stability

The error dynamics in (6.66) can be written in a compact form as,

\[ \delta \hat{d}_k = D \hat{d}_k + E \delta \epsilon_k \]  

(6.67)

where, \( \delta \hat{d}_k = [\delta \hat{x}_k \quad \delta \hat{\epsilon}_k] \) and \( \hat{d}_k = [\hat{x}_k \quad \hat{\epsilon}_k] \).

It is possible to select the observer gains in such a way that the eigen values of \( D \) can be placed arbitrarily in a circle with center \((-1/T, 0)\) and radius \(1/T\). If the observer gains are selected such that all eigen values of \( D \) have negative real parts, one can always find a positive definite matrix \( P \) such that,

\[ D^T P + PD = -Q \]  

(6.68)

where \( Q \) is a given positive definite matrix. Let \( \lambda_d \) be the smallest eigen value of \( Q \).

Defining a Lyapunov function

\[ V_1(\hat{d}_k) = \hat{d}_k^T P \hat{d}_k \]  

(6.69)

and calculating \( \delta V_1(\hat{d}_k) \) along (6.67)

\[ \delta V_1(\hat{d}_k) = \hat{d}_k^T (D^T P + PD) \hat{d}_k + 2 \hat{d}_k^T P E \delta \epsilon_k \]

\[ \leq -\hat{d}_k^T Q \hat{d}_k + 2 \| PE \| \cdot \| \hat{d}_k \| \mu \]

\[ \leq -\lambda_d \| \hat{d}_k \|^2 + 2 \| PE \| \cdot \| \hat{d}_k \| \mu \]

\[ \leq -\| \hat{d}_k \| (\lambda_d \| \hat{d}_k \| - 2 \| PE \| \mu) \]  

(6.70)

Thus the estimation error \( \| \hat{d}_k \| \) is bounded by \( \frac{2 \| PE \| \mu}{\lambda_d} \). This implies,

\[ \| \hat{x}_k \| \leq \frac{2 \| PT \| \mu}{\lambda_d} \quad \text{and} \quad \| \hat{\epsilon}_k \| \leq \frac{2 \| PT \| \mu}{\lambda_d} \]  

(6.71)

The ratio \( \frac{\| P \|}{\lambda_d} \) depends on the choice of the eigenvalues of \( D \). The bounds on the disturbance estimation errors can be lowered by selecting larger values for the observer gains which in turn increases the sensitivity to measurement noise. Consequently, the choice of observer gains is a matter of trade-off between the desired accuracy and the quality of measurement.

The practical stability is thus proved in the sense of Corless and Leitmann (1981).
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6.7 Application : Motion Control

Motion control is a vital requirement in many practical applications. The critical requirement in motion control is robustness; which is concerned with tracking performance in the presence of uncertainties and disturbance. The control is expected to ensure trajectory tracking, with fast convergence.

A two axis motion control using sliding mode control and neural network is reported in (Lin and Shen, 2006). A variety of other strategies like extended state observer (Talole et al., 2010b), adaptive back-stepping (Lin et al., 2008), adaptive friction compensation (Jing et al., 2014) have been proposed for robust motion control. The robustness is a major concern due to backlash, coulomb friction, uneven load distribution (Gerdes and Kumar, 1995; Kolnik and Agranovich, 2012; Aldrich and Skelton, 2006).

6.7.1 Dynamic Model

An industrial motion control test set-up is used to validate the designed algorithm. The set-up, industrial plant emulator (ECP220, 2004) includes a DC brushless servo system with a PC based control platform. The system consists of two motors, one as a drive, and other as a source of disturbance, a power amplifier and an encoder for position feedback. The inertia, friction and backlash are all adjustable. A schematic is shown in Fig. 6.1.

The drive motor is coupled via a timing belt to a drive disk with variable inertia. Another timing belt connects the drive disk to the speed reduction (SR) assembly while a third belt completes the drive train to the load disk. The load and drive disks have variable inertia which may be adjusted by moving or removing brass weights. Speed reduction is adjusted by interchangeable belt pulleys in the SR assembly. Backlash may be introduced through a mechanism incorporated in the SR assembly. A disturbance motor connects to the load disk via a 4:1 speed reduction and is used to emulate viscous friction and disturbances at the plant output. A brake below the load disk may be used to introduce coulomb friction.
In this work, a typical case is considered in which 4 brass weight, each of 500 gm is added on disturbance motor and no weight on drive motor. The gear ratio is chosen by selecting top and bottom pulley in SR assembly. In the present case the pulleys selected are with $n_{pl}$ as 18 and $n_{pd}$ as 72.
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The dynamics of industrial control test-bed can be written as in \((ECP220, 2004)\),

\[
J_r \ddot{\theta} + C_r \dot{\theta} = T_d
\]  
(6.72)

where \(J_r\) is reflected inertia at drive and \(C_r\) is reflected damping to drive. The parameter \(T_d\) is the desired torque which can be achieved suitably by selecting appropriate control voltage \((u)\) and hardware gain \((k_{hw})\).

Therefore (6.72) can be rewritten as,

\[
J_r \ddot{\theta} + C_r \dot{\theta} = k_{hw} u
\]  
(6.73)

The plant dynamics can be modeled in state space notation as,

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
0 & -\frac{C_r}{J_r}
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
0 \\
\frac{k_{hw}}{J_r}
\end{bmatrix} u
\]  
(6.74)

\[
y = \begin{bmatrix}
1 & 0
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]  
(6.75)

where, \([x_1 \quad x_2]^T\) are the states - position \((\theta)\) and velocity \((\dot{\theta})\), \(u\) is the control signal in volts and \(y\) is the output position in degrees.

The other parameters in (6.74) and (6.75) are,

\[
C_r = C_1 + C_2 (gr)^{-2}
\]  
(6.76)

\[
gr = 6 \frac{n_{pd}}{n_{pl}}
\]  
(6.77)

\[
J_r = J_d + J_p (gr_{prime})^{-2} + J_l (gr)^{-2}
\]  
(6.78)

\[
J_d = J_{dd} + m_{wd} (r_{wd})^2 + J_{wd0}
\]  
(6.79)

\[
J_p = J_{pd} + J_{pl} + J_{pbl}
\]  
(6.80)

\[
gr_{prime} = \frac{n_{pd}}{12}
\]  
(6.81)

\[
J_l = J_{dl} + m_{wl} (r_{wl})^2 + J_{wld}
\]  
(6.82)

\[
J_{wd0} = \frac{1}{2} m_{wd} (r_{wd0})^2
\]  
(6.83)

The details of various plant parameters are stated in Table 6.1.
Table 6.1: Parameters of industrial motion control

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_r$</td>
<td>Reflected damping to drive</td>
<td>$4.08 \times 10^{-3}$</td>
</tr>
<tr>
<td>$C_1$</td>
<td>Rotary damping at load disk</td>
<td>0.004</td>
</tr>
<tr>
<td>$C_2$</td>
<td>Rotary damping at drive disk</td>
<td>0.005</td>
</tr>
<tr>
<td>$g_r$</td>
<td>Drive train gear ratio</td>
<td>24</td>
</tr>
<tr>
<td>$J_r$</td>
<td>Reflected gear ratio</td>
<td>$4.63 \times 10^{-4}$ kg-m$^2$</td>
</tr>
<tr>
<td>$J_d$</td>
<td>Drive inertia</td>
<td>$4 \times 10^{-4}$ kg-m$^2$</td>
</tr>
<tr>
<td>$J_p$</td>
<td>Inertia associated with idler pulley in SR-assembly</td>
<td>$5.84 \times 10^{-4}$ kg-m$^2$</td>
</tr>
<tr>
<td>$J_l$</td>
<td>Load inertia</td>
<td>0.027125 kg-m$^2$</td>
</tr>
<tr>
<td>$g_r^{\text{prime}}$</td>
<td>Drive to SR pulley gear ratio</td>
<td>6</td>
</tr>
<tr>
<td>$J_{dd}$</td>
<td>Inertia of bare drive disk plus drive motor, encoder, drive disk/ motor belt and pulleys</td>
<td>$4 \times 10^{-4}$ kg-m$^2$</td>
</tr>
<tr>
<td>$m_{wd}$</td>
<td>Weight on drive inertia</td>
<td>0 kg</td>
</tr>
<tr>
<td>$r_{wd}$</td>
<td>Radius of weight from middle axis of drive disk</td>
<td>0 m</td>
</tr>
<tr>
<td>$J_{wdo}$</td>
<td>Inertia associated with brass weights at drive disk</td>
<td>0 kg-m$^2$</td>
</tr>
<tr>
<td>$J_{pd}$</td>
<td>Drive pulley inertia</td>
<td>$5.5 \times 10^{-4}$ kg-m$^2$</td>
</tr>
<tr>
<td>$J_{pl}$</td>
<td>Load pulley inertia</td>
<td>$0.03 \times 10^{-4}$ kg-m$^2$</td>
</tr>
<tr>
<td>$J_{pbl}$</td>
<td>Inertia associated with backlash</td>
<td>$0.31 \times 10^{-4}$ kg-m$^2$</td>
</tr>
<tr>
<td>$n_{pd}$</td>
<td>Number of teeth on bottom pulley of SR-assembly</td>
<td>72</td>
</tr>
<tr>
<td>$n_{pl}$</td>
<td>Number of teeth on top pulley of SR-assembly</td>
<td>18</td>
</tr>
<tr>
<td>$J_{dl}$</td>
<td>Inertia of bare load disk plus disturbance motor, encoder, load disk/ motor belt and pulleys</td>
<td>$65 \times 10^{-4}$ kg-m$^2$</td>
</tr>
<tr>
<td>$m_{wl}$</td>
<td>Weight on load inertia</td>
<td>2 kg</td>
</tr>
<tr>
<td>$r_{wl}$</td>
<td>Radius of weight from middle axis of load disk</td>
<td>0.1 m</td>
</tr>
<tr>
<td>$J_{wlo}$</td>
<td>Inertia associated with brass weights at load disk</td>
<td>$6.25 \times 10^{-4}$kg-m$^2$</td>
</tr>
<tr>
<td>$r_{wl0}$</td>
<td>Radius of larger brass weight</td>
<td>0.025 m</td>
</tr>
<tr>
<td>$k_{hw}$</td>
<td>Hardware gain</td>
<td>5.81</td>
</tr>
</tbody>
</table>
6.7.2 Results

The control law is tested for model-following strategy on an industrial motion control case-study (ECP220, 2004). The plant dynamics are as in (6.74) with the parameters as in Table 6.1. The plant is discretized to a form as in (6.7). The structure of model to be followed is as in (6.22) with,

\[
A_m = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta \omega_n \end{bmatrix}, \quad b_m = \begin{bmatrix} 0 \\ -\omega_n^2 \end{bmatrix} \quad (6.84)
\]

The initial conditions for the plant and model are,

\[
x(0) = [0 \ 1]^T \quad x_m(0) = [0 \ 0]^T \quad (6.85)
\]

The results are verified for model-following control with uncertainty estimated by UDE. The model parameters are \( \zeta = 1 \) and \( \omega_n = 5 \) in (6.84). The plant and model have an initial condition mismatch (6.85). The control gain is \( k = 2 \) and observer poles are located at \([-10 \ -20 \ -30]\). The reference input is a square wave of amplitude 1 and frequency 0.3 rad/sec.

Case 1: Nominal plant

The accuracy of tracking and estimation of states is illustrated in Fig. 6.2.

![Figure 6.2: Model following for nominal plant](image-url)
The control performance for model-following is illustrated in Fig. 6.3. The plant and model states are shown in Fig. 6.3a and 6.3a. The corresponding control effort (Fig. 6.3c) and sigma (Fig. 6.3e) are also shown.

Figure 6.3: Model following performance for nominal plant
Case 2: Effect of sampling time

The effect of sampling time is illustrated in Fig. 6.4. It is observed that the estimation and sliding width is consistent even with increase in sampling time.

![Graph](image.png)

(a) estimate of uncertainty  (b) sliding variable

Figure 6.4: Effect of different sampling time (10ms (solid) and 20 ms (dotted))

Case 3: Effect of filter order

The effect of filter order on estimation is illustrated in Fig. 6.5. The estimation is improved with a second order filter.

![Graph](image.png)

(a) estimate of uncertainty  (b) sliding variable

Figure 6.5: Effect of filter order (second order (solid) and first order (dotted))
6.8 Summary

A SMC combined with UDE is extended to the discrete-time case of an uncertain system. The control law is made implementable by designing a observer to give a robust controller-observer structure in discrete domain. A notable feature of the proposed design is that, it affords control over the magnitude of the quasi-sliding for a given sampling period. The UDE enables a reduction of quasi-sliding band for a given sampling period. The sliding width is significantly reduced by using a second-order filter.

The use of $\delta$-operator in conjunction with a new sliding condition enables complete and seamless unification of the sliding-condition, control law and UDE. It is proved that the ultimate boundedness of state estimation error, uncertainty estimation error and sliding variable is guaranteed; and the bounds can be lowered by appropriate choice of design parameters. The efficacy of design is confirmed on an application to motion control system.