Chapter 4

SMC for Mismatched Uncertain System using EID Method

The chapter explains design of SMC law for mismatched uncertain system, with state-dependent uncertainties. The issue of mismatched uncertainties is addressed by estimating EID with a low-pass filter. The design is validated for slip regulation in an anti lock braking system problem.

The chapter starts with an introduction in section 4.1 followed by problem formulation in section 4.2. Section 4.3 describes the estimation of EID using low-pass filter. A control based on SMC is explained in Section 4.4 and the proposed control in section 4.5. Section 4.6 gives the proof of boundedness using Lyapunov analysis. The performance is illustrated by an application to anti lock braking system in section 4.7. The chapter concludes with a summary in section 4.8.

4.1 Introduction

The conventional SMC design can guarantee invariance to only matched uncertainties (Drazenovic, 1969). This poses severe limitations on the applicability of SMC, as there many applications where; uncertainties and disturbances act in channels in which, a control input is not present.
A variety of control strategies like adaptive control (Wen and Cheng, 2008), LMI-based control (Choi, 2007), fuzzy (Tao et al., 2003; Zhang et al., 2010), integral sliding mode control (Liang et al., 2012) have been proposed in literature to address the problem of mismatched uncertainties. The integrator back-stepping (IB) (Kanellakopoulos et al., 1991; Krstic et al., 1995) is a widely used method for control of mismatched systems. This method has been extended to multiple surface sliding control (Won and Hedrick, 1996) and adaptive multiple-surface sliding control (Huang and Chen, 2004). A novel sliding surface that includes the estimate of unmatched disturbances obtained using DO is proposed in (Yang et al., 2013). This method is augmented with an extended DO in (Ginoya et al., 2014).

The Equivalent Input Disturbance (EID) approach is a promising technique for control of uncertain systems; both matched and mismatched. An EID is a disturbance on the control input channel that produces the same effect on the controlled output as actual disturbances do. An EID always exists for a controllable and observable plant with no zeros on imaginary axis (She et al., 2008). This technique is concerned with the estimation of equivalent input disturbance instead of actual disturbance. The fundamental premise is that; the disturbance may be acting on any channel, but it is the control signal that is used for disturbance rejection. The EID is always different from the actual disturbance imposed on plant.

The EID system is just a system, reconstructed based on the output of plant, so that the state of EID system and the actual state of original plant are usually different (She et al., 2008). The EID approach has been validated for disturbance rejection performance in linear systems (She et al., 2008). This method has been extended for MIMO (She and Xin, 2007), under-actuated (She et al., 2012) and non-minimum phase (Liu et al., 2013) systems.

In this work, SMC is combined with EID and extended to a nonlinear system with mismatched uncertainties and disturbances. The system formulation considered here covers a large class of applications, and admits a large class of uncertainties. The robustness is assured through estimate of EID mechanized through a low-pass filter, and the performance is improved by using a higher-order filter. The EID helps SMC mitigate the effect of mismatched disturbances. The composite control is applied to a representative problem of anti-lock braking system.
4.2 Problem Formulation

Consider an uncertain nonlinear system given by,

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + f(x,t) + Bu(t) + g(x,t)u(t) + B_d d(x,t) \\
y(t) &= Cx(t)
\end{align*}
\]  

(4.1)

where

- \( x(t) \in \mathbb{R}^n \) is the state vector
- \( u(t) \in \mathbb{R}^m \) is the control input
- \( y(t) \in \mathbb{R}^1 \) is the output of plant
- \( f(x,t) \) is the uncertain nonlinear plant vector
- \( g(x,t) \) is the uncertain nonlinear input vector
- \( d(x,t) \in \mathbb{R}^n \) is external unmeasurable disturbance
- \( A, B \) and \( C \) are the plant, input and output matrices representing nominal plant

It is evident from (4.1) that the disturbance may be imposed on a channel other than that of the control input, and the number of disturbances and associated input channels may be larger than one.

The uncertain nonlinear plant \( f(x,t) \) and input vector \( g(x,t) u \) can be considered as state and input dependent disturbance and is lumped with external unmeasurable disturbances \( (B_d d) \). The plant in (4.1) can be thus modified as,

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) + \{f(x,t) + g(x,t)u + B_d d(x,t)\} \\
y(t) &= Cx(t)
\end{align*}
\]  

(4.2)

As the existence of EID is guaranteed for both matched and mismatched disturbances (She et al., 2008), there always exists a signal \( d_e \) on control input channel that produces the same effect on output as \( (f(x,t) + g(x,t)u + B_d d(x,t)) \) does. The plant is now given by,

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) + Bd_e(x,u,t) \\
y(t) &= Cx(t)
\end{align*}
\]  

(4.3)

where \( d_e = f(x,t) + g(x,t)u + B_d d(x,t) \)
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Assumption 4.1 This assumption is necessary to guarantee the existence of EID

1. The pair \((A, B)\) is controllable

2. The pair \((A, C)\) is observable

3. The triplet \((A, C, B)\) has no invariant zeros, i.e. for all \(\lambda \in \mathbb{C}\)

\[
\text{Rank} \begin{bmatrix} \lambda I - A & -B \\ C & 0 \end{bmatrix} = n + 1
\]

Assumption 4.2 The EID \(d_e(x, t)\) is continuous and satisfies,

\[
\left| \frac{d^{(j)}d_e(x, u, t)}{dt^{(j)}} \right| \leq \mu \quad \text{for } j = 0, 1, 2, \ldots, r \quad (4.4)
\]

where \(\mu\) is a small positive number.

Remark 4.1 The Assumption 4.2 implies that the uncertainty \(d_e(x, u, t)\) and its derivatives up to some finite order \((r)\) be bounded but the bound is not required to be known. The assumption includes a fairly large class of uncertainties and disturbances that can be estimated by EID.

The objective is to design a control \((u)\) such that, the output \((y)\) of uncertain plant \((4.1)\) follows the desired trajectory. The control is expected to ensure robust tracking in presence of uncertainties and disturbances of varying kind.

The EID approach usually uses a state-feedback with integral as a nominal control with estimates of EID for improved disturbance rejection. The EID is normally estimated with a first-order low-pass filter.

The limitations of integral action are to be mitigated by designing a sliding-mode law for nominal plant. The synthesis of control law is envisaged using the states of observer. The issue of chatter is addressed by using a linear and saturation gain. The observed states are also required for estimating EID, which is then combined with SMC to yield a robust control law. The performance improvement by using a higher-order filter to estimate EID and its derivatives is to be validated.
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4.3 Estimation of EID

A servo control system with an EID estimator is shown in Fig. 4.1.

Using (4.3)

\[ d_e = B^*(\hat{x} - Ax - Bu) \]  \hspace{1cm} (4.5)

Remark 4.2. The equation (4.5) requires all the states, however the channel on which EID is imposed may be different from that of actual disturbance. The available states of the plant (in the presence of disturbance) may be different from those of the plant with an EID. As such a full-order observer is imperative to estimate EID. It is important to guarantee that \( y(t) - \hat{y}(t) \) converge to zero.

The observer is given as,

\[
\begin{align*}
\dot{x} &= A\hat{x} + B_{eq}(x - \hat{x}) \\
\dot{\hat{y}} &= C\hat{x}
\end{align*}
\]  \hspace{1cm} (4.6)

where, \( L \) is the observer gain and \( u_{eq} \) is the nominal control.
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The state estimation error can be written as,

\[
\begin{align*}
\hat{x} &= x - \hat{x} \\
\hat{\dot{x}} &= \dot{x} - \hat{\dot{x}}
\end{align*}
\]  
(4.7)

The observer error dynamics can be determined by using (4.3) and (4.6) in (4.7)

\[
\begin{align*}
\hat{\dot{x}} &= A(x - \hat{x}) + B(u - u_{eq}) + Bd_e - LC(x - \hat{x}) \\
&= (A - LC)\hat{x} - B\hat{d} + Bd_e \\
&= (A - LC)\hat{x} + B\hat{d}
\end{align*}
\]  
(4.8)

where \( \hat{d} = d_e - \hat{d} \)

The EID \((d_e)\) in (4.5) can now be estimated with the observed states as,

\[
\begin{align*}
d^* &= B^+(\hat{x} - A\hat{x} - Bu) \\
&= B^+(A\hat{x} + Bu_{eq} + LC\hat{x} - A\hat{x} - Bu) \\
&= B^+LC\hat{x} + u_{eq} - u \\
&= B^+LC\hat{x} + \hat{d}
\end{align*}
\]  
(4.9)

The estimate \((\hat{d})\) is obtained by passing \((d^*)\) through a low-pass filter \(G_f(s)\)

\[
\hat{d} = G_f(s) d^*
\]  
(4.10)

where, \(\tau\) is a small positive constant.

For a first order filter, (4.10) becomes,

\[
\hat{d} = \frac{d^*}{1 + \tau s}
\]  
(4.11)

Therefore,

\[
\begin{align*}
\hat{d} + \tau \hat{\dot{d}} &= d^* \\
\hat{d} + \tau \hat{\dot{d}} &= B^+LC\hat{x} + \hat{d} \\
\hat{\dot{d}} &= \frac{B^+LC}{\tau} \hat{x}
\end{align*}
\]  
(4.12)

Thus, the estimate of EID is given by,

\[
\hat{d} = \frac{B^+LC}{\tau} \int \hat{x}
\]  
(4.13)
The estimation error dynamics can be determined as,

\[
\dot{d}_e - \dot{\hat{d}} = -\frac{B^+LC}{\tau} \ddot{x} + \dot{d}_e \tag{4.14}
\]

Therefore,

\[
\dot{\hat{d}} = -\frac{B^+LC}{\tau} \ddot{x} + \dot{d}_e \tag{4.15}
\]

Using (4.8) and (4.15)

\[
\begin{bmatrix}
\dot{\hat{x}} \\
\dot{\hat{d}}
\end{bmatrix} = \begin{bmatrix}
(A - LC) & B \\
-\frac{B^+LC}{\tau} & 0
\end{bmatrix} \begin{bmatrix}
\ddot{x} \\
\ddot{d}
\end{bmatrix} + \begin{bmatrix}
0 \\
1
\end{bmatrix} \dot{d}_e \tag{4.16}
\]

The error dynamics in (4.16) can be further simplified as,

\[
\begin{bmatrix}
\dot{\hat{x}} \\
\dot{\hat{d}}
\end{bmatrix} = \left(\begin{bmatrix}
A & B \\
0_{1\times2} & 0
\end{bmatrix} - \begin{bmatrix}
L \\
\frac{B^+LC}{\tau}
\end{bmatrix} \begin{bmatrix}
C \\
0
\end{bmatrix}\right) \begin{bmatrix}
\ddot{x} \\
\ddot{d}
\end{bmatrix} + \begin{bmatrix}
0_{2\times1} \\
1
\end{bmatrix} \dot{d}_e \tag{4.17}
\]

A compact form of (4.17) can be written as,

\[
\dot{\tilde{e}} = (H - KG) \tilde{e} + E \dot{d}_e \tag{4.18}
\]

where,

\[
\begin{align*}
\tilde{e} &= \begin{bmatrix}
\ddot{x} \\
\ddot{d}
\end{bmatrix} \\
H &= \begin{bmatrix}
A & B \\
0_{1\times2} & 0
\end{bmatrix} \\
K &= \begin{bmatrix}
L \\
\frac{B^+L}{\tau}
\end{bmatrix} \\
G &= \begin{bmatrix}
C & 0
\end{bmatrix} \\
E &= \begin{bmatrix}
0 & 0 & 1
\end{bmatrix}^T
\end{align*}
\]

**Remark 4.3** It is evident from (4.18) that, the error dynamics is driven by \(\dot{d}_e\). Thus the error dynamics is asymptotically stable if \(\dot{d}_e \approx 0\). It is obvious that for bounded \(\dot{d}_e\), bounded input-bounded output stability is assured. However, asymptotic stability for the error dynamics can always be assured if some higher derivative of the uncertainty is equal to zero.
4.3.1 Improvement in Estimation – 2\textsuperscript{nd} order filter

The EID ($d_e$) and its derivative ($\dot{d}_e$) can be estimated using a second order filter of the form,

$$G_f(s) = \frac{1 + 2\tau s}{\tau^2 s^2 + 2\tau s + 1}$$  \hspace{1cm} (4.19)

where $\tau$ is a small positive constant.

Therefore using (4.9), (4.10) and (4.19)

$$\dot{\hat{d}}_1 = \frac{1 + 2\tau s}{\tau^2 s^2 + 2\tau s + 1} \left[ B^+ LC \ddot{x} + \hat{d}_1 \right]$$  \hspace{1cm} (4.20)

Therefore,

\[
\begin{align*}
\tau^2 \ddot{\hat{d}}_1 + 2\tau \dot{\hat{d}}_1 + \hat{d}_1 &= B^+ LC \ddot{x} + 2\tau B^+ LC \dot{\hat{x}} + \hat{d}_1 + 2\tau \ddot{\hat{d}}_1 \\
\tau^2 \ddot{d}_1 &= B^+ LC \ddot{x} + 2\tau B^+ LC \dot{x} \\
\dot{\hat{d}}_1 &= \frac{1}{\tau^2} B^+ LC \ddot{x} + \frac{2}{\tau} B^+ LC \dot{x} \\
\ddot{\hat{d}}_1 &= \ddot{\hat{d}}_2 + \frac{2}{\tau} B^+ LC \dot{x}
\end{align*}
\]  \hspace{1cm} (4.21)

Therefore,

\[
\begin{align*}
\dot{\hat{d}}_1 &= \frac{2}{\tau} B^+ LC \ddot{x} + \hat{d}_2 \\
\dot{\hat{d}}_2 &= \frac{1}{\tau^2} B^+ LC \dot{x}
\end{align*}
\]  \hspace{1cm} (4.22)

The estimation error is written as,

\[
\begin{align*}
\dot{\hat{d}}_1 &= -\frac{2}{\tau} B^+ LC \ddot{x} + \hat{d}_2 \\
\dot{\hat{d}}_2 &= -\frac{1}{\tau^2} B^+ LC \dot{x} + \ddot{d}_e
\end{align*}
\]  \hspace{1cm} (4.23)

The error dynamics can be written using (4.8) and (4.23) as,

\[
\begin{bmatrix}
\dot{\hat{x}} \\
\dot{\hat{d}}_1 \\
\dot{\hat{d}}_2
\end{bmatrix} =
\begin{bmatrix}
A & B \\
0_{2 \times 2} & 0_{2 \times 1}
\end{bmatrix}
- \begin{bmatrix}
L \\
\frac{2}{\tau} B^+ L \\
\frac{1}{\tau^2} B^+ L
\end{bmatrix}
\begin{bmatrix}
C & 0_{1 \times 2} \\
0_{1 \times 1}
\end{bmatrix}
\begin{bmatrix}
\hat{x} \\
\hat{d}_1 \\
\hat{d}_2
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0_{1 \times 1} \\
1
\end{bmatrix} \ddot{d}_e
\]  \hspace{1cm} (4.24)
A compact form of (4.24) can be written as,

\[
\dot{\tilde{e}} = (H - KG) \tilde{e} + E \ddot{d}_e
\]  

(4.25)

where,

\[
\begin{align*}
\tilde{e} &= \begin{bmatrix} \tilde{x} \\ \tilde{d}_1 \\ \tilde{d}_2 \end{bmatrix} \\
H &= \begin{bmatrix} A & B \\ 0_{2 \times 2} & 0_{2 \times 1} \end{bmatrix} \\
K &= \begin{bmatrix} L \\ \frac{2}{\tau} B^+ L \\ \frac{1}{\tau^2} B^+ L \end{bmatrix} \\
G &= \begin{bmatrix} C & 0_{1 \times 2} \end{bmatrix} \\
E &= \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T
\end{align*}
\]

**Remark 4.4** It is evident from (4.25) that, the error dynamics is driven by \( \ddot{d}_e \). Thus the error dynamics is asymptotically stable if \( \ddot{d}_e \approx 0 \). The estimation accuracy is thus improved as both \( d_e \) and \( \dot{d}_e \) are estimated.

### 4.3.2 Generalization – \( k \)th order filter

In general, if \( k \)th derivative of the disturbance, \( \dot{d}_e^{(k)} \) is zero; the asymptotic stability for error dynamics is guaranteed, if one chooses the filter as,

\[
G_f(s) = \frac{(1 + \tau s)^k - (\tau s)^k}{(1 + \tau s)^k}
\]  

(4.26)

The equations for estimate of EID and its derivatives can be written as,

\[
\begin{align*}
\dot{\hat{d}}_1 &= \frac{k}{\tau} B^+ L C \tilde{x} + \hat{d}_2 \\
\dot{\hat{d}}_2 &= \frac{k}{\tau^2} B^+ L C \tilde{x} + \hat{d}_3 \\
&\vdots \\
\dot{\hat{d}}_{k-1} &= \frac{1}{\tau^{k-1}} B^+ L C \tilde{x} + \hat{d}_k \\
\dot{\hat{d}}_k &= \frac{1}{\tau^k} B^+ L C \tilde{x}
\end{align*}
\]

(4.27)

where \( \hat{d}_1 \) is the estimate of EID \( d_e \) and \( \hat{d}_i, i = 2, 3, \ldots, k \) are the estimates of derivatives of EID \( \dot{d}_e^{(j)} \), \( j = 1, 2, \ldots, (k-1) \).
Defining the estimate of EID and its derivatives as \( d_1 = d_e, d_2 = \dot{d}_e, \ldots, d_k = d_e^{(k-1)} \) and defining the estimation errors as, \( \tilde{d}_i = d_i - \hat{d}_i \) for \( i = 1, 2, \ldots, k \),

The dynamics of estimation error is written as,

\[
\begin{align*}
\dot{\tilde{d}}_1 &= -\frac{k}{\tau} B^+ L C \tilde{x} + \tilde{d}_2 \\
\dot{\tilde{d}}_2 &= -\frac{k}{\tau^2} B^+ L C \tilde{x} + \tilde{d}_3 \\
&\vdots \\
\dot{\tilde{d}}_{k-1} &= -\frac{k}{\tau^{k-1}} B^+ L C \tilde{x} + \tilde{d}_k \\
\dot{\tilde{d}}_k &= -\frac{1}{\tau^k} B^+ L C \tilde{x} + d_e^{(k)}
\end{align*}
\] (4.28)

The error dynamics can now be written using (4.8) and (4.28) as,

\[
\begin{bmatrix}
\dot{\tilde{x}} \\
\dot{\tilde{d}}_1 \\
\dot{\tilde{d}}_2 \\
\vdots \\
\dot{\tilde{d}}_{k-1} \\
\dot{\tilde{d}}_k
\end{bmatrix}
= \begin{bmatrix} A & B \\ 0_{k \times 2} & 0_{k \times 1} \end{bmatrix}
- \begin{bmatrix} L \\ \frac{k}{\tau} B^+ L \\ \vdots \\ \frac{k}{\tau^{k-1}} B^+ L \\ \frac{1}{\tau^k} B^+ L \end{bmatrix}
\begin{bmatrix} C & 0_{1 \times k} \\ \vdots & \vdots \\ C & 0_{1 \times k} \end{bmatrix}
\begin{bmatrix}
\tilde{x} \\
\tilde{d}_1 \\
\tilde{d}_2 \\
\vdots \\
\tilde{d}_{k-1} \\
\tilde{d}_k
\end{bmatrix}
+ \begin{bmatrix} 0 \\ 0_{(k-1) \times 1} \\ 1 \end{bmatrix} d_e^{(k)}
\] (4.29)

A compact form of (4.29) can be written as,

\[
\dot{\hat{e}} = (H - KG) \hat{e} + E d_e^{(k)}
\] (4.30)

where,

\[
\hat{e} = \begin{bmatrix} \dot{x} \\ \dot{d}_1 \\ \dot{d}_2 \\ \vdots \\ \dot{d}_{k-1} \\ \dot{d}_k \end{bmatrix}, \quad H = \begin{bmatrix} A & B \\ 0_{k \times 2} & 0_{k \times 1} \end{bmatrix}, \quad K = \begin{bmatrix} L \\ \frac{k}{\tau} B^+ L \\ \vdots \\ \frac{k}{\tau^{k-1}} B^+ L \\ \frac{1}{\tau^k} B^+ L \end{bmatrix}, \quad G = \begin{bmatrix} C & 0_{1 \times k} \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 0 & 0_{(k-1) \times 1} & 1 \end{bmatrix}^T
\]
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4.4 Design of Control

The EID can be combined with any control design to improve the performance of system. A SMC combined with EID is proposed here to address the issue of matched and/or mismatched uncertainty. The objective is to control the output \( y \) in the presence of parametric uncertainties and external disturbances.

The control \( (u) \) in (4.3) can be written as,

\[
 u = u_{eq} + u_n \tag{4.31}
\]

where \( u_{eq} \) is the nominal control designed using sliding-mode and \( u_n = -\hat{d} \) is the control for uncertain part with \( \hat{d} \) as the estimate of \( d \).

4.4.1 Sliding Surface

The sliding surface is defined as,

\[
 \sigma = y - y_d \tag{4.32}
\]

where \( y_d \) is the desired trajectory of output.

Using (4.3) in (4.32),

\[
 \sigma = Cx - y_d \tag{4.33}
\]

The estimation of EID is based on observed states, as such the control \( (u_{eq}) \) is also designed using estimated states \( (\hat{x}) \). Therefore, the sliding surface is written as,

\[
 \hat{\sigma} = C\hat{x} - y_d \tag{4.34}
\]

The sliding surface (4.34) is modified as,

\[
 \hat{\sigma}^* = \hat{\sigma} - \hat{\sigma}(0) e^{-\alpha t} \tag{4.35}
\]

where \( \alpha \) is a user chosen positive constant.

Remark 4.5 The sliding surface of (4.35) eliminates reaching phase and also aids in mitigating chatter (Deshpande and Phadke, 2012). The modified sliding variable \( \hat{\sigma}^* \) is zero at \( t = 0 \), thus precluding the initial control from taking large values. Additionally, \( \hat{\sigma}^* \to \hat{\sigma} \) as \( t \to \infty \).
4.4.2 Sliding Mode Control

A control is designed such that, the sliding condition is satisfied and plant follows the desired trajectory.

Differentiating (4.35) and using (4.34) and (4.6),

\[
\dot{\hat{\sigma}}^* = CA\dot{x} + CB\dot{u}_{eq} + CLC\ddot{x} + \alpha\dot{\hat{\sigma}}(0)e^{-\alpha t} - \dot{y}_d
\]  

(4.36)

The control that ensures sliding can be written as,

\[
u_{eq} = -(CB)^{-1}\{CA\dot{x} + \alpha\dot{\hat{\sigma}}(0)e^{-\alpha t} - \dot{y}_d + k_l\hat{\sigma}^* + k_s\text{sat}(\hat{\sigma}^*)\}
\]

(4.37)

where \(k_l > 0\) is the linear gain and \(k_s > 0\) is the switching gain to be designed and,

\[
\text{sat}(\hat{\sigma}^*) = \begin{cases} 
\text{sgn}(\hat{\sigma}^*) & \text{if } |\hat{\sigma}^*| > \epsilon, \epsilon > 0 \\
\hat{\sigma}^*/\epsilon & \text{if } |\hat{\sigma}^*| \leq \epsilon
\end{cases}
\]

With the control in (4.37), the dynamics of sliding surface can be written as,

\[
\dot{\hat{\sigma}}^* = -k_l\hat{\sigma}^* - k_s\text{sat}(\hat{\sigma}^*) + CLC\ddot{x}
\]  

(4.38)

Remark 4.6 It is seen from (4.38) that, as \(\ddot{x} \to 0\), sliding condition is satisfied and \(\hat{\sigma}^*\) will asymptotically approach 0, if \(k_l > 0\) and \(k_s > 0\).

4.5 Proposed Control

The proposed SMC law combined with EID can be written as,

\[
\begin{align*}
\dot{u}_{eq} &= -(CB)^{-1}\{CA\dot{x} + \alpha\dot{\hat{\sigma}}(0)e^{-\alpha t} - \dot{y}_d + k_l\hat{\sigma}^* + k_s\text{sat}(\hat{\sigma}^*)\} \\
u_n &= -\dot{\hat{d}} \\
\dot{\hat{d}} &= \frac{B^+LC}{\tau}\int \ddot{x}
\end{align*}
\]

(4.39)

It may be noted that the EID in (4.39) is estimated with a first-order filter.

Remark 4.7 The control proposed here (4.39) is different from the one developed in (She et al., 2008). The nominal control i.e. state-feedback with integral is replaced with a sliding-mode here. The estimation of EID is generalized to a \(k^{th}\) order filter to include a large class of uncertainties and disturbances.
4.6 Stability

The error dynamics are rewritten using (4.16) as,

\[
\begin{bmatrix}
\dot{\tilde{x}} \\
\dot{\tilde{d}}
\end{bmatrix} = \begin{bmatrix}
(A - LC) & B \\
-B^+ LC & 0
\end{bmatrix} \begin{bmatrix}
\tilde{x} \\
\tilde{d}
\end{bmatrix} + \begin{bmatrix}
0 \\
1
\end{bmatrix} \dot{d}_e
\] (4.40)

A compact form of (4.40) can be written as,

\[
\dot{\tilde{e}} = D \tilde{e} + E \dot{d}_e \quad \text{where} \quad \tilde{e} = \begin{bmatrix} \tilde{x} \\ \tilde{d}\end{bmatrix}^T
\] (4.41)

It is evident from (4.41) that eigen values of \( D \) can be placed arbitrarily. If the observer gain \( L \) is selected such that all eigen values of \( D \) have negative real parts, one can always find a positive definite matrix \( P \) such that,

\[
PD + D^T P = -Q
\] (4.42)

for a given positive definite matrix \( Q \). Let \( \lambda_e \) be the smallest eigen value of \( Q \).

Defining a Lyapunov function as,

\[
V(\tilde{e}) = \tilde{e}^T P \tilde{e}
\] (4.43)

Taking derivative of \( V(\tilde{e}) \) along (4.41)

\[
\dot{V}(\tilde{e}) = \tilde{e}^T P \dot{\tilde{e}} + \dot{\tilde{e}}^T P \tilde{e}
\]

\[
\leq -\lambda_e \| \tilde{e} \|^2 + 2 \| \tilde{e} \| \| P E \| \mu
\] (4.47)

It is seen from (4.48) that, \( \| \tilde{e} \| \) is ultimately bounded and remains in a ball of radius \( \frac{2 \| P E \| \mu}{\lambda_e} \), which guarantees that,

\[
\| \tilde{x} \| \leq \lambda_1 = \frac{2 \| P E \| \mu}{\lambda_e}
\] (4.49)

\[
\| \dot{d} \| \leq \lambda_2 = \frac{2 \| P E \| \mu}{\lambda_e}
\] (4.50)
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The dynamics of $\dot{\sigma}^*$ can be written using (4.38) as,

$$\dot{\sigma}^* = -k_l\dot{\sigma}^* - k_s\text{sat}(\dot{\sigma}^*) + CLC\ddot{x}$$

(4.51)

Therefore,

$$\dot{\sigma}^*\dot{\sigma}^* = -k_l|\dot{\sigma}^*|^2 - k_s|\dot{\sigma}^*|\text{sat}(\dot{\sigma}^*) + |\dot{\sigma}^*|CLC\ddot{x}$$

(4.52)

$$\leq -k_l|\dot{\sigma}^*|^2 - k_s|\dot{\sigma}^*|\text{sat}(\dot{\sigma}^*) + |\dot{\sigma}^*| |CLC|\|\ddot{x}\|$$

(4.53)

$$\leq -k_l|\dot{\sigma}^*|^2 - k_s|\dot{\sigma}^*|\text{sat}(\dot{\sigma}^*) + |\dot{\sigma}^*| |CLC|\|\lambda_1$$

(4.54)

Define,

$$\zeta = |CLC|\lambda_1$$

(4.55)

Therefore,

$$\dot{\sigma}^*\dot{\sigma}^* = -|\dot{\sigma}^*|(k_l|\dot{\sigma}^*| + k_s - \zeta)$$

(4.56)

if $|\dot{\sigma}^*| > \epsilon$

$$\dot{\sigma}^*\dot{\sigma}^* = -|\dot{\sigma}^*|(k_l|\dot{\sigma}^*| + k_s - \zeta)$$

(4.57)

Therefore,

$$|\dot{\sigma}^*| \leq \frac{\zeta - k_s}{k_l}$$

(4.58)

if $|\dot{\sigma}^*| \leq \epsilon$

$$\dot{\sigma}^*\dot{\sigma}^* = -|\dot{\sigma}^*|(k_l|\dot{\sigma}^*| + k_s - \zeta)$$

(4.59)

Therefore,

$$|\dot{\sigma}^*| \leq \frac{\zeta}{k_l + \frac{k_s}{\epsilon}}$$

(4.60)

Thus, the sliding variable ($\dot{\sigma}^*$) is ultimately bounded by,

$$|\dot{\sigma}^*| \leq \lambda_3 = \max\left(\frac{\zeta - k_s}{k_l}, \frac{\zeta}{k_l + \frac{k_s}{\epsilon}}\right)$$

(4.61)

In view of (4.61), it is evident that $||\ddot{y} = y - y_d||$ shall be bounded.

It is seen from (4.49), (4.50) and (4.61) that, $||\ddot{x}||$, $||\ddot{d}||$ and $|\dot{\sigma}^*|$ are ultimately bounded. The bounds can be lowered by appropriate choice of control parameters $k_l$, $k_s$ and $L$.

The practical stability is thus proved in the sense of Corless and Leitmann (1981)
4.7 Application: Anti Lock Braking System

The Antilock Braking System (ABS) is an integral component of safety in modern cars. The prime objective of ABS is to prevent locking of wheels and reduce the stopping distance. The dynamics of ABS is complex due to the presence of nonlinearities and uncertainties. The tire-road friction coefficient is a crucial information required for design of control (Pacejka, 2005; Kiencke and Nielsen, 2005). The uncertainties in vehicle mass and changes in road gradient are also challenges in efficient ABS control.

The problem of slip regulation in ABS control is addressed using diverse control strategies as PID control (Song et al., 2007), feedback linearization (Pour-samad, 2009), fuzzy control (Lin and Hsu, 2003), observer based adaptive fuzzy-neural (Wang et al., 2009), model based control (Shi et al., 2010) and optimal control (Mirzaei et al., 2006) and SMC (Kayacan et al., 2009; Wu and Shih, 2001).

4.7.1 Dynamic Model

The dynamic model of ABS is derived from free body diagram of quarter car model shown in Figure 4.2. The setup of quarter car model used is Inteco (Inteco, 2013).

Figure 4.2: Free body diagram of ABS setup
Chapter 4. SMC for Mismatched Uncertain System using EID Method

The description of ABS parameters is given in Table 4.1.

Table 4.1: Parameters of ABS

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r_1 )</td>
<td>radius of upper wheel</td>
<td>9.95e-2</td>
<td>m</td>
</tr>
<tr>
<td>( r_2 )</td>
<td>radius of lower wheel</td>
<td>9.90e-2</td>
<td>m</td>
</tr>
<tr>
<td>( J_1 )</td>
<td>moment of inertia of upper wheel</td>
<td>7.53e-3</td>
<td>kgm(^2)</td>
</tr>
<tr>
<td>( J_2 )</td>
<td>moment of inertia of lower wheel</td>
<td>2.56e-2</td>
<td>kgm(^2)</td>
</tr>
<tr>
<td>( q_1 )</td>
<td>viscous friction coeff. of upper wheel</td>
<td>1.18e-4</td>
<td>kgm(^2)/s</td>
</tr>
<tr>
<td>( q_2 )</td>
<td>viscous friction coeff. of lower wheel</td>
<td>2.14e-4</td>
<td>kgm(^2)/s</td>
</tr>
<tr>
<td>( M_{10} )</td>
<td>static friction of upper wheel</td>
<td>3.2e-3</td>
<td>Nm</td>
</tr>
<tr>
<td>( M_{20} )</td>
<td>static friction of lower wheel</td>
<td>9.25e-2</td>
<td>Nm</td>
</tr>
<tr>
<td>( M_g )</td>
<td>gravitational and shock absorber torques acting on the balance lever</td>
<td>19.62</td>
<td>Nm</td>
</tr>
<tr>
<td>( L )</td>
<td>distance between the contact point of wheels and the rotational axis of the balance lever</td>
<td>0.370</td>
<td>m</td>
</tr>
<tr>
<td>( \phi )</td>
<td>Angle between the normal in the contact point and line ( L )</td>
<td>65.61</td>
<td>degrees</td>
</tr>
</tbody>
</table>

There are three torques acting on the upper wheel; the braking torque \( (T_b) \), the friction torque in upper wheel bearing and the friction torque among wheels. The equation for upper wheel dynamics can be written as,

\[
J_1 \ddot{z}_1 = F_n r_1 S \mu(\lambda) - q_1 z_1 - s_1 M_{10} - s_1 T_b \tag{4.62}
\]

There are two torques acting on the lower wheel; the friction torque in lower bearing and the friction torque among wheels. The equation for lower wheel dynamics can be written as,

\[
J_2 \ddot{z}_2 = -F_n r_2 S \mu(\lambda) - q_2 z_2 - s_2 M_{20} \tag{4.63}
\]
The normal force \( F_n \) is sum of the torques at point A and it is calculated as,

\[
F_n = \frac{M_g + s_1 T_b + s_1 M_{10} + q_1 z_1}{L \left( \sin(\phi) - S\mu(\lambda) \cos(\phi) \right)} \quad (4.64)
\]

The auxiliary variables are defined as, \( S = \text{sgn}(r_2 z_2 - r_1 z_1) \) and \( s_1 = \text{sgn}(z_1) \), with \( \text{sgn} \) as the signum function.

\( z_1 \) is the angular velocity of the upper wheel and \( z_2 \) is the angular velocity of the lower wheel.

Define \( S(\lambda) \) as,

\[
S(\lambda) = \frac{S\mu(\lambda)}{L \left( \sin(\phi) - S\mu(\lambda) \cos(\phi) \right)} \quad (4.65)
\]

Using (4.64) and (4.65) in (4.62) and (4.63)

\[
\dot{z}_1 = S(\lambda) \left( c_{11} z_1 + c_{12} + c_{13} z_1 + c_{14} + s_1 T_b (c_{15} S(\lambda) + c_{16}) \right) \quad (4.66)
\]

\[
\dot{z}_2 = S(\lambda) \left( c_{21} z_1 + c_{22} + c_{23} z_2 + c_{24} + S(\lambda) c_{25} s_1 T_b \right) \quad (4.67)
\]

The constants in (4.66) and (4.67) are described in Table 4.2.

<table>
<thead>
<tr>
<th>Name</th>
<th>Description</th>
<th>Value</th>
<th>Name</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_{11} )</td>
<td>( \frac{r_1 q_1}{J_1} )</td>
<td>0.0015</td>
<td>( c_{21} )</td>
<td>( -\frac{r_2 q_1}{J_2} )</td>
<td>0.000464</td>
</tr>
<tr>
<td>( c_{12} )</td>
<td>( \frac{(s_1 M_{10} + M_g) r_1}{J_1} )</td>
<td>2.5933e2</td>
<td>( c_{22} )</td>
<td>( \frac{(s_1 M_{10} + M_g) r_2}{J_2} )</td>
<td>75.8696</td>
</tr>
<tr>
<td>( c_{13} )</td>
<td>( -\frac{q_1}{J_1} )</td>
<td>0.0159</td>
<td>( c_{23} )</td>
<td>( -\frac{q_2}{J_2} )</td>
<td>0.00878</td>
</tr>
<tr>
<td>( c_{14} )</td>
<td>( -\frac{s_1 M_{10}}{J_1} )</td>
<td>0.3985</td>
<td>( c_{24} )</td>
<td>( -\frac{s_2 M_{20}}{J_2} )</td>
<td>3.6323</td>
</tr>
<tr>
<td>( c_{15} )</td>
<td>( \frac{r_1}{J_1} )</td>
<td>13.2171</td>
<td>( c_{25} )</td>
<td>( -\frac{r_2}{J_2} )</td>
<td>3.8667</td>
</tr>
<tr>
<td>( c_{16} )</td>
<td>( -\frac{1}{J_1} )</td>
<td>132.8356</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The road adhesion friction coefficient is a nonlinear function of slip, given as,

\[ \mu(\lambda) = \frac{w_4 \lambda^p}{a + \lambda^p} + w_3 \lambda^3 + w_2 \lambda^2 + w_1 \lambda \]  \hspace{1cm} (4.68)

where \( \lambda \) is the wheel slip. The other constants are, \( a = 2.5 \times 10^{-4}, \) \( p = 2.099, \) \( w_1 = -0.042, \) \( w_2 = 0.29 \times 10^{-9}, \) \( w_3 = 0.035 \) and \( w_4 = 0.406. \)

The wheel slip is defined as,

\[ \lambda = \frac{r_2 z_2 - r_1 z_1}{r_2 z_2} \]  \hspace{1cm} (4.69)

Differentiating (4.69) and using (4.66) and (4.67), the dynamics of the wheel slip can be written as,

\[ \dot{\lambda} = f(\lambda, z_2) + g(\lambda, z_2) T_b \]  \hspace{1cm} (4.70)

where the nonlinear function \( f(\lambda, z_2) \) and \( g(\lambda, z_2) \) are given as,

\[ f(\lambda, z_2) = - (S(\lambda)c_{11} + c_{13})(1 - \lambda) - \left( \frac{r_1}{r_2 z_2} \right) (S(\lambda)c_{12} + c_{14}) \]
\[ + \frac{(1 - \lambda)}{z_2} \left( \left( S(\lambda)(1 - \lambda) \frac{r_2}{r_1} c_{21} + c_{23} \right) z_2 \right) \]
\[ + \frac{(1 - \lambda)}{z_2} (S(\lambda)c_{22} + c_{24}) \]  \hspace{1cm} (4.71)

\[ g(\lambda, z_2) = - \frac{r_1}{r_2 z_2} s_1 \left( S(\lambda)c_{15} - c_{16} + (1 - \lambda) \frac{r_2}{r_1} S(\lambda)c_{25} \right) \]  \hspace{1cm} (4.72)

The braking torque is generated by an actuator; the dynamics are represented as,

\[ \dot{T}_b = c_{31} (b(u) - T_b) \]  \hspace{1cm} (4.73)

where,

\[ b(u) = \begin{cases} b_1 u + b_2, & u \geq u_0 \\ 0, & u < u_0 \end{cases} \]  \hspace{1cm} (4.74)

with, \( c_{31} = 20.37, \) \( b_1 = 15.264, \) \( b_2 = -6.21 \) and \( u_0 = 0.415 \) are constants.

The objective of control is to achieve the desired slip in presence of matched and mismatched disturbances and parametric uncertainties. The plant dynamics can be written using (4.70) and (4.73) as,

\[ \dot{\lambda} = f(\lambda, z_2) + g(\lambda, z_2) T_b \]  \hspace{1cm} (4.75)

\[ \dot{T}_b = -c_{31} T_b + c_{31} b_1 u + c_{31} b_2 \]  \hspace{1cm} (4.76)
4.7.2 Control Design

The dynamics in (4.75) and (4.76) can be rewritten as,

\[ \lambda &= T_b + d_1 \quad (4.77) \\
\dot{T}_b &= bu + d_2 \quad (4.78) \]

where \( b = c_{31} b_1 \) is a constant and \( d_1, d_2 \) are mismatched and matched uncertainty respectively, with

\[ d_1 = f(\lambda, z_2) + g(\lambda, z_2) T_b - T_b \quad (4.79) \]
\[ d_2 = c_{31} b_2 - c_{31} T_b \quad (4.80) \]

The system in (4.77) and (4.78) can be written in a compact form as,

\[ \dot{x} = Ax + Bu + B_d d \quad (4.81) \]

where,

\[
\begin{bmatrix}
\lambda \\
T_b
\end{bmatrix}
= 
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\lambda \\
T_b
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
d_1 \\
d_2
\end{bmatrix}
\]

An observer is designed as,

\[
\begin{aligned}
\dot{\hat{x}} &= A\hat{x} + Bu_{eq} + LC (x - \hat{x}) \\
\hat{y} &= C\hat{x}
\end{aligned}
\quad (4.82)
\]

where, \( L \) is the observer gain and \( u_{eq} \) is the nominal control.

The sliding surface is selected as,

\[
\sigma = C (\hat{x} - [\lambda_d, 0]^T) \\
\sigma^* = \sigma - \sigma(0)e^{-\alpha t} \\
\alpha > 1
\]

The control is designed to enforce sliding along (4.84)

The proposed SMC law combined with EID estimate can be written as,

\[
\begin{aligned}
u &= -(CB)^{-1} \left\{ CA\hat{x} + \alpha \hat{\sigma}(0)e^{-\alpha t} - \dot{\lambda}_d + k_t \hat{\sigma}^* + k_s \text{sat}(\hat{\sigma}^*) \right\} - \hat{d} \\
\hat{d} &= \frac{B^+ LC}{\tau} \int \tilde{x}
\end{aligned}
\]

(4.85)
4.7.3 Results

The slip (\( \lambda \)) is controlled using SMC supplemented with estimate of EID. The control strategy is tested for different cases without changing the controller parameters. The following cases are considered:

- Case 1: uncertain plant with nominal parameters
- Case 2: uncertainty in mass \( (m) \)
- Case 3: Uncertainty in road gradient \( (\theta) \)
- Case 4: different friction models

The observer poles are located at \([-20 - 30 - 50] \) and controller gains are set at \( k_l = 10 \) and \( k_s = 2 \). The value of \( \tau \) is derived from the observer poles. The reference slip is \( \lambda_d = 0.2 \). The initial value of \( z_1 \) and \( z_2 \) is set to 1700 rpm.

**Case 1: Nominal plant**

The control performance for nominal plant with parameters as in Table 4.1 is shown in Fig. 4.3. The state-dependent non-linearities are estimated by EID. The slip ratio (\( \lambda \)) and control effort (\( u \)) is depicted in Fig. 4.3a and 4.3b.

![Figure 4.3: Slip regulation for nominal plant with uncertainty](image)

(a) slip ratio  
(b) control
The plot of other variables for case 1 is shown in Fig. 4.4. The braking torque ($T_b$) is shown in Fig. 4.4a and the corresponding car velocity and wheel velocity is shown in Fig. 4.4b. The sliding variable ($\sigma$) is shown in Fig. 4.4d.

Figure 4.4: Control performance for nominal plant with uncertainty
Case 2: Uncertainty in mass ($m$)

The nominal plant is modified by changing vehicle mass ($m$) in the defined range over its nominal value. The vehicle mass is changed in the range of ±40%. The Figure 4.5a shows the behavior of slip ($\lambda$) and Figure 4.5b shows the corresponding control effort with nominal (solid), +30% (dashed) and −30% (dashed-dot).

![Figure 4.5: Control performance with uncertainty in mass](image)

The Table 4.3 depicts the estimation and control performance for robust tracking of slip for different mass uncertainties.

<table>
<thead>
<tr>
<th>Uncertainty in mass (%)</th>
<th>$\bar{\lambda}$ ($\lambda - \lambda_{ref}$)</th>
<th>$u$</th>
<th>$\sigma$</th>
<th>$\tilde{y}$ ($y - \hat{y}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>+20</td>
<td>0.0455</td>
<td>0.7296</td>
<td>0.1115</td>
<td>0.0016</td>
</tr>
<tr>
<td>−20</td>
<td>0.0440</td>
<td>0.6924</td>
<td>0.1094</td>
<td>0.0013</td>
</tr>
<tr>
<td>+30</td>
<td>0.0459</td>
<td>0.7394</td>
<td>0.1121</td>
<td>0.0017</td>
</tr>
<tr>
<td>−30</td>
<td>0.0436</td>
<td>0.6834</td>
<td>0.1091</td>
<td>0.0013</td>
</tr>
<tr>
<td>+40</td>
<td>0.0462</td>
<td>0.7493</td>
<td>0.1127</td>
<td>0.0017</td>
</tr>
<tr>
<td>−40</td>
<td>0.0433</td>
<td>0.6745</td>
<td>0.1088</td>
<td>0.0012</td>
</tr>
</tbody>
</table>
Case 3: Uncertainty in road gradient ($\theta$)

The slip regulation is tested for different road inclination angle ($\theta$). The relative performance for different angles is shown in Table 4.4.

Table 4.4: Performance analysis for road inclination angle

<table>
<thead>
<tr>
<th>Road inclination angle (deg.)</th>
<th>$\tilde{\lambda}$</th>
<th>$u$</th>
<th>$\sigma$</th>
<th>$\tilde{y}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$+10$</td>
<td>0.0446</td>
<td>0.7064</td>
<td>0.1103</td>
<td>0.0015</td>
</tr>
<tr>
<td>$+20$</td>
<td>0.0445</td>
<td>0.6940</td>
<td>0.1101</td>
<td>0.0014</td>
</tr>
<tr>
<td>$+30$</td>
<td>0.0442</td>
<td>0.6735</td>
<td>0.1097</td>
<td>0.0014</td>
</tr>
<tr>
<td>$+40$</td>
<td>0.0439</td>
<td>0.6462</td>
<td>0.1092</td>
<td>0.0014</td>
</tr>
<tr>
<td>$+50$</td>
<td>0.0434</td>
<td>0.6122</td>
<td>0.1086</td>
<td>0.0013</td>
</tr>
</tbody>
</table>

Case 4: Different friction models ($\mu$)

The control performance is assessed for variation in $\mu(\lambda)$. The values of coefficients are changed by $\pm 25\%$ to test the effectiveness of estimation. The results are shown in Fig. 4.6 with nominal (solid), $+25\%$ (dashed) and $-25\%$ (dashed-dot).

Figure 4.6: Control performance with variation in $\mu(\lambda)$
Chapter 4. SMC for Mismatched Uncertain System using EID Method

The robustness is further validated by testing the designed control for different friction models. The 3 friction models used are,

1. Inteco formula

\[
\mu(\lambda) = \frac{w_4\lambda^p}{a + \lambda^p} + w_3\lambda^3 + w_2\lambda^2 + w_1\lambda \tag{4.86}
\]

where \(a = 2.5 \times 10^{-4}, p = 2.099, w_1 = -0.042, w_2 = 0.29 \times 10^{-9}, w_3 = 0.035\)

and \(w_4 = 0.406\)

2. Magic formula

\[
\mu = D_m \sin\left\{C_m \tan^{-1}(B_m\lambda - E_m \tan^{-1}(B_m\lambda))\right\} \tag{4.87}
\]

where \(D_m = 0.5, C_m = 1.65, B_m = 10.38\) and \(E_m = 0.65663\)

3. Burckhardt formula

\[
\mu = c_1(1 - e^{-c_2\lambda}) - c_3\lambda \tag{4.88}
\]

where \(c_1 = 0.4004, c_2 = 33.7080, c_3 = 0.1204\)

The results are shown in Fig. 4.7.

![Figure 4.7: Control performance for different friction models](image)

The control performance is robust for all friction models, as is evident in Fig. 4.7 for Inteco (solid), Magic formula (dashed) and Burckhardt formula (dashed-dot).
4.8 Summary

A SMC is combined with EID and extended to a nonlinear system with mismatched uncertainties. The use of SMC for nominal control mitigates the need of integral action used in conventional EID based control. The designed control enforces sliding with no chatter. The SMC law is supplemented with estimate of EID to ensure robustness. The EID aids SMC in mitigating the effects of mismatched disturbances. The control performance is further improved by using a higher-order filter to estimate EID and its derivatives. A generalization to $k^{th}$-order filter is presented.

It is proved that the ultimate boundedness of state estimation error, uncertainty estimation error and sliding variable is guaranteed; and the bounds can be lowered by appropriate choice of design parameters. The efficacy of composite control is tested for slip regulation in anti lock braking system. The design is validated for varied disturbances and parametric uncertainties. The actuator dynamics is also included in the model. The results demonstrate robustness of the proposed scheme for uncertainties in vehicle mass, road gradient and friction, for regulating the vehicle slip and provide short stopping distance.