Chapter I

PRELIMINARIES

Definition 1.1

Let \( V = \{v_1, v_2, v_3, \ldots\} \) be a set and \( E = \{e_1, e_2, e_3, \ldots\} \) be another set such that each \( e_i \) is an unordered pair of elements of \( V \) so that \( e_i = (v_i, v_j) = (v_j, v_i) \) for some \( i \) and \( j \). Then the ordered pair \( G = (V, E) \) is called a graph. The elements of \( V \) are called the vertices or points and the elements of \( E \) are associated with vertices.\( e_i \).

Definition 1.2

Two vertices \( v_i \) and \( v_j \) of a graph \( G \) are said to be adjacent if there is an edge joining \( v_i \) and \( v_j \). Two edges \( e_i \) and \( e_j \) are said to be adjacent if they have a common end vertex.

Definition 1.3

An edge with identical ends is called a loop or self-loop.

Definition 1.4

Edges having the same end vertices are called multiple edges or parallel edges. Edges with distinct ends are called links.

Definition 1.5

A graph that has neither loops nor multiple edges is called a simple graph.
Definition 1.1

Let $V = \{ v_1, v_2, v_3, \ldots \}$ be a set and $E = \{ e_1, e_2, e_3, \ldots \}$ be another set such that each $e_k$ is an unordered pair of elements of $V$ so that $e_k = (v_i, v_j) = (v_j, v_i)$ for some $i$ and $j$. Then the ordered pair $G = (V, E)$ is called a graph. The elements of $V$ are called the vertices or points and the elements of $E$ are called the edges or lines. The vertices $v_i, v_j$ associated with $e_k$ are called the end vertices of $e_k$ or simply ends of $e_k$.

Definition 1.2

Two vertices $v_i$ and $v_j$ of a graph $G$ are said to be adjacent if there is an edge joining $v_i$ and $v_j$. Two edges $e_i$ and $e_j$ are said to be adjacent if they have a common end vertex.

Definition 1.3

An edge with identical ends is called a loop or self-loop.

Definition 1.4

Edges having the same end vertices are called multiple edges or parallel edges. Edges with distinct ends are called links.

Definition 1.5

A graph that has neither loops nor multiple edges is called a simple graph.
Definition 1.6

A graph with a finite number of vertices as well as a finite number of edges is called a \textit{finite graph} otherwise it is an \textit{infinite graph}.

Definition 1.7

If $v_i$ and $v_j$ are the ends of the edge $e_k$, then we say that $e_k$ is \textit{incident} on $v_i$ and $v_j$.

Example

Consider the graph shown in figure 1.1

![Graph Diagram](image)

\textbf{Figure 1.1}

Figure 1.1 is a graph with 6 vertices and 10 edges. $e_2$ is a loop. $e_7$ and $e_8$ are parallel edges. $e_6$ and $e_9$ are adjacent. $e_6$ is a link.

Definition 1.8

The number of edges incident on a vertex $v_i$ (with self-loop counted twice), is called the \textit{degree} of the vertex $v_i$ and is denoted by $d(v_i)$. 
Example

In figure 1.1, \( d(v_1) = 4; d(v_2) = 5 \) and so on.

Definition 1.9

A graph in which all the vertices are of equal degree is called a \textit{regular graph}.

Definition 1.10

A vertex of degree zero is called an \textit{isolated vertex}.

Definition 1.11

A vertex of degree one is called a \textit{pendant vertex} and the corresponding edge is called a \textit{leaf}.

Example

In figure 1.2, vertex \( v_2 \) is a pendant vertex. Vertex \( v_5 \) is an isolated vertex.

Definition 1.12

A graph \( G \) with \( p \) vertices and \( q \) edges is called \((p,q)\)-graph where \( p \) is called the \textit{order} of the graph and \( q \) is called the \textit{size} of the graph \( G \).
Definition 1.13

A (p,q)-graph with \( p \neq 0, q = 0 \) is called a vertex graph and is denoted by \( \varphi \).

Definition 1.14

A (p, q)-graph with \( p = q = 0 \) is called a null graph or an empty graph and is denoted by \( \phi \).

Example

![Figure 1.3](image)

A vertex graph with 5 vertices is shown in figure 1.3.

Definition 1.15

Two graphs \( G \) and \( G' \) are said to be isomorphic to each other if there is a one to one correspondence between their vertices and between their edges such that the incidence relationship is preserved.
Example

The graphs $G$ and $H$ given in figure 1.4 are isomorphic.

Definition 1.16

A graph $H$ is said to be a subgraph of a graph $G$ if all the vertices and all the edges of $H$ are in $G$.

Example

Figure 1.4

Figure 1.5 (a)
Graphs given in Figure 1.5(b) and 1.5(c) are subgraphs of the graph in Figure 1.5(a).

**Definition 1.17**

Two subgraphs of a graph are said to be *edge disjoint* if they have no edges in common. Similarly two subgraphs of a graph are said to be *vertex disjoint* if they have no vertex in common.

**Example**

The two subgraphs given in figure 1.5(b) and 1.5(c) are edge disjoint subgraphs of the graph given in figure 1.5(a).

**Definition 1.18**

Let \( G = (V, E) \) be any graph. Let \( V_1 \) be a non-empty subset of the vertex set \( V \). The subgraph of \( G \) with vertex set \( V_1 \) and edge set as the set of those edges of \( G \) have both ends in \( V_1 \) is called the subgraph of \( G \) *induced* by \( V_1 \) and it is denoted by \( G[V_1] \) or \( <V_1> \).

\(<V_1>\) is also called an *induced subgraph* of \( G \).
Graph shown in figure 1.6(b) is an induced subgraph of the graph given in 1.6(a).

**Definition 1.19**

A **spanning subgraph** of a graph $G$ is a subgraph of $G$ containing all the vertices of $G$.

**Example**

Graph given in figure 1.7(b) is a spanning subgraph of $G$, displayed in figure 1.7(a).
Definition 1.20

A walk of a graph G is an alternating sequence of vertices and edges say \( v_0 e_1 v_1 e_2 v_2 \ldots e_{n-1} v_n \) beginning and ending with vertices in which each edge is incident with two vertices immediately preceding and following it. The above walk may also be called \( v_0-v_n \) walk. The walk \( v_0-v_n \) is said to be a closed walk if \( v_0 = v_n \) and is open otherwise.

Definition 1.21

An open walk in which no vertex appears more than once is called a path. The number of edges in a path is called the length of the path. The terminal vertices of a path are of degree one, and the rest of the vertices (called intermediate vertices) are of degree two. Generally a path on \( n \) vertices is denoted by \( P_n \).

Remark:

In the above \( v_0- v_n \) walk if all the vertices are distinct it will be termed as a \( v_0- v_n \) path it may also be written as \( v_0v_1v_2\ldots v_n \) by omitting the edges.

Definition 1.22

A closed path is called a circuit. A circuit is also called a cycle, elementary chain, circular path or polygon. A cycle on \( n \) vertices is denoted by \( C_n \).
Example

Definition 1.23

A graph $G$ is said to be connected if there is at least one path between every pair of vertices in $G$. Otherwise, it is disconnected.

Example

Graph in Figure 1.8 is disconnected. A vertex graph with more than one vertex is connected.

Definition 1.24

A disconnected graph is a disjoint union of two or more connected graphs.

Figure 1.8

In figure 1.8, $v_1e_1v_2e_2v_3e_4v_4$ is a walk. $v_1$ and $v_4$ are the terminal vertices of the walk. $v_1e_1v_2e_8v_6e_7v_5$ is a path. Length of this path is 3.

Example

Definition 1.25

The union of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is another graph $G = (V_1 \cup V_2, E_1 \cup E_2)$ whose vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$.

Definition 1.26

The intersection denoted by $G_1 \cap G_2$ of two graphs $G_1$ and $G_2$ is a graph $G$ consisting of those vertices which are common to both $G_1$ and $G_2$.

Figure 1.9

For any graph $G$, then $G = G_1 \cap G_2$.
Three different circuits $C_6$, $C_1$, $C_2$ are shown in figure 1.9.

**Definition 1.23**

A graph $G$ is said to be *connected* if there is at least one path between every pair of vertices in $G$. Otherwise $G$ is *disconnected*.

**Example**

Graph in figure 1.2 is disconnected. A vertex graph with more than one vertex is disconnected. The graph in figure 1.1 is connected.

**Definition 1.24**

A disconnected graph consists of two or more connected subgraphs of the graph. Each of these connected subgraphs is called a *component*.

**Example**

The graph in figure 1.2 consists of two components.

**Definition 1.25**

The *union* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is another graph $G = (V, E)$ whose vertex set $V = V_1 \cup V_2$ and the edge set $E = E_1 \cup E_2$.

**Definition 1.26**

The *intersection* denoted by $G_1 \cap G_2$ of two graphs $G_1$ and $G_2$ is a graph $G$ consisting of those vertices and edges that are in both $G_1$ and $G_2$.

For any graph $G$, $G \cup G = G$; $G \cap G = G$. If $v_i$ is a vertex in a graph $G$, then $G - v_i$ denotes a subgraph of $G$ obtained from $G$ by
deleting $v_i$ from $G$. [Deletion of a vertex implies the deletion of all edges incident on that vertex.] If $e_j$ is an edge in $G$, then $G - e_j$ is a subgraph of $G$ obtained by deleting $e_j$ from $G$. [Deletion of an edge does not imply deletion of its end vertices.]

**Definition 1.27**

A graph in which there exists an edge between every pair of distinct vertices is called a *complete graph*. Generally $K_p$ denotes a complete graph in $p$ vertices.

**Example**

![Figure 1.10](image)

A complete graph on 6 vertices is displayed in figure 1.10.

**Definition 1.28**

A *tree* is a connected graph without any circuit.

**Definition 1.29**

A *spanning tree* of a connected graph $G$ is a spanning subgraph of $G$ which is also a tree.
Definition 1.30

Let $S$ be a set and $F = \{S_1, S_2, \ldots, S_p\}$ a non-empty family of distinct non-empty subsets of $S$ whose union is $S$. The intersection graph of $F$ is denoted by $\Omega(F)$ and is defined by $V(\Omega(F)) = F$, with $S_i$ and $S_j$ adjacent whenever $i \neq j$ and $|S_i \cap S_j| \neq 0$. Then a graph $G$ is an intersection graph on $S$ if there exists a family $F$ of subsets of $S$ for which $G \cong \Omega(F)$.

Theorem 1.31

Every graph is an intersection graph.

Definition 1.32

The intersection number $\omega(G)$ of a given graph $G$ is the minimum number of elements in a set $S$ such that $G$ is an intersection graph on $S$.

Definition 1.33

A graph $G$ is said to be a bipartite graph or bigraph if its vertex set $V$ can be partitioned into two subsets $V_1$ and $V_2$ such that each edge of $G$ has one end in $V_1$ and the other in $V_2$. If every vertices in $V_1$ is joined with all the vertices of $V_2$, then the bipartite graph is called as a complete bipartite graph and is denoted by $K_{m,n}$ where $m$ is the number of vertices in $V_1$ and $n$ is the number of vertices in $V_2$.

Definition 1.34

Let $G = (V, E)$ be a graph and $S \subseteq V$. $S$ is said to be an independent set if no two vertices of $S$ are adjacent in $G$. 
Next we give some important terminologies in the field of algebra.

**Definition 1.35**

A non empty set with an associative binary operation is called a **semigroup**. We write a multiplicative semigroup as \((S_g, \cdot)\) or simply as \(S_g\).

**Definition 1.36**

Let \((S_g, \cdot)\) be a semigroup. A non-empty subset \(T\) of \(S_g\) is called a **subsemigroup** of \(S_g\) if it is closed with respect to multiplication.

ie. for all \(x, y \in T\), \(xy \in T\).

**Definition 1.37**

A non empty set \(S_G\), together with a binary operation \(\ast\) is called a **group** if the following axioms are satisfied:

i) The binary operation \(\ast\) is associative on \(S_G\).

ii) There is an element \(e\) in \(S_G\) such that \(e \ast x = x \ast e = x\) for all \(x \in S_G\). (The element \(e\) is called the identity element for \(\ast\) on \(S_G\).)

iii) For each \(a\) in \(S_G\) there is an element \(a'\) in \(S_G\) such that \(a' \ast a = a \ast a' = e\) (The element \(a'\) is called the inverse of \(a\)).
Definition 1.38

If $L$ is a subset of $S_G$ closed under the group operation of $S_G$ and $L$ itself is a group under this induced operation, then $L$ is a \textit{subgroup} of $S_G$ and is denoted by $L \leq S_G$.

Definition 1.39

Let $S_G$ be a group. Then all subgroups of $S_G$ other than $S_G$ are \textit{proper subgroups} of $S_G$. Also $\{e\}$ is the \textit{trivial subgroup} of $S_G$. All other subgroups are non-trivial.

Definition 1.40

Let $S_G$ and $S_G'$ be any two groups. A mapping $\psi : S_G \rightarrow S_G'$ is said to be an isomorphism if

i) $\psi$ is bijective.

ii) $\psi(x \cdot y) = \psi(x) \cdot \psi(y)$ for all $x, y \in S_G$.

The groups $S_G$ and $S_G'$ are then \textit{isomorphic} and is denoted by $S_G \cong S_G'$.

Definition 1.41

Let $L$ be a subgroup of a group $S_G$ and let $a \in S_G$. The \textit{left coset} $aL$ of $L$ is the set $\{al : l \in L\}$. The \textit{right coset} $La$ is similarly defined.

Definition 1.42

A subgroup $H$ of a group $S_G$ is said to be \textit{normal} if $g^{-1}Lg = L$ for all $g \in S_G$. 
Definition 1.43

If $N$ is a normal subgroup of a group $S_G$, the group of cosets of $N$ under the induced operation is called the \textit{factor group} of $S_G$ modulo $N$ and is denoted by $S_G / N$. The cosets are residue classes of $S_G$ modulo $N$.

Definition 1.44

A \textit{normal series} of a group $S_G$ is a finite sequence $L_0, L_1, \ldots, L_n$ of normal subgroups of $S_G$ such that $L_i < L_{i+1}$; $L_0 = \{e\}$ and $L_n = S_G$.

Example

Consider the group $\mathbb{Z}$ under addition then $\{0\} < 8\mathbb{Z} < 4\mathbb{Z} < \mathbb{Z}$ are two normal series of $\mathbb{Z}$.

Definition 1.45

A normal series $\{K_j\}$ is a \textit{refinement} of a normal series $\{L_i\}$ of a group $S_G$ if $\{L_i\} \subseteq \{K_j\}$; i.e. if each $L_i$ is one of the $K_j$.

Example

The series $\{0\} < 72\mathbb{Z} < 24\mathbb{Z} < 8\mathbb{Z} < 4\mathbb{Z} < \mathbb{Z}$ is a refinement of $\{0\} < 72\mathbb{Z} < 8\mathbb{Z} < \mathbb{Z}$. 
Definition 1.46

Two normal series \( \{L_i\} \) and \( \{K_j\} \) of the same group \( G \) are \textit{isomorphic} if there is one to one correspondence between the collections of factor groups \( \{L_{i+1} / L_i\} \) and \( \{K_{j+1} / K_j\} \) such that the corresponding factor groups are isomorphic.

Two isomorphic normal series must have the same number of groups.

Example

Consider \( \mathbb{Z}_{15} \)

The two series \( \{0\} < 5 < \mathbb{Z}_{15} \) and \( \{0\} < 3 < \mathbb{Z}_{15} \) are isomorphic.

Since \( \mathbb{Z}_{15} / 5 \) and \( 3 / \{0\} \) are isomorphic to \( \mathbb{Z}_5 \), \( \mathbb{Z}_{15} / 3 \) is isomorphic to \( 5 / \{0\} \) or to \( \mathbb{Z}_3 \).

Definition 1.47

A \textit{lattice} is a partially ordered set \( S \) in which each pair of elements has greatest lower bound and least upper bound. If \( x, y \in S \) then greatest lower bound is denoted by \( x \wedge y \) and least upper bound is denoted by \( x \vee y \).

Definition 1.48

A lattice \( S \) is said to be \textit{complete} if every non-empty subset of \( S \) has a greatest lower bound and least upper bound.
Now we just present two basic definitions one in algebra and the other related to topology.

**Definition 1.49**

Let $\mathcal{F}$ be a collection of subsets of a set $S$. Then $\mathcal{F}$ is called a *field* iff $S \in \mathcal{F}$ and $\mathcal{F}$ is closed under complementation and finite union.

(a) $S \in \mathcal{F}$

(b) If $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$

(c) If $A_1, A_2, \ldots, A_n \in \mathcal{F}$ then $\bigcup_{i=1}^{n} A_i \in \mathcal{F}$

It follows that $\mathcal{F}$ is closed under finite intersection. For, if $A_1, A_2, \ldots, A_n \in \mathcal{F}$, then

$$\bigcap_{i=1}^{n} A_i = \left(\bigcup_{i=1}^{n} A_i^c\right)^c \in \mathcal{F}$$

If (c) is replaced by closure under countable union, ie,

(d) If $A_1, A_2, A_3, \ldots$, $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$ then $\mathcal{F}$ is called a *σ-field*. $\mathcal{F}$ is also closed under countable intersection.

If $\mathcal{F}$ is a field, a countable union of sets in $\mathcal{F}$ can be expressed as the limit of an increasing sequence of sets in $\mathcal{F}$ and conversely. ie.

if $A = \bigcup_{i=1}^{\infty} A_i$, then $\bigcup_{i=1}^{n} A_i \uparrow A$; conversely if $A_n \uparrow A$, then $A = \bigcup_{i=1}^{\infty} A_i$. 
This shows that σ-field is a field that is closed under limits of increasing sequence.

**Definition 1.50**

Let $X$ be a non empty set and $\tau$ be a collection of subset of $X$. $\tau$ is said to be a *topology* on $X$ if it satisfies

i) $X, \emptyset \in \tau$

ii) The union of the elements of any subcollection of $\tau$ is in $\tau$.

iii) The intersection of the elements of any finite subcollection of $\tau$ is in $\tau$.

Then pair $(X, \tau)$ is called a *topological space*. 