CHAPTER II

The Inductive Groupoid $G(\mathcal{S}_n)$

It follows from theorems from Chapter I that the semigroup $\mathcal{S}_n$ is an idempotent generated regular semigroup. Hence the inductive groupoid $G(\mathcal{S}_n)$ of $\mathcal{S}_n$ may be constructed from the biordered set $E_n = E(\mathcal{S}_n)$ by the procedure described in Section I.3. We proceed to apply these to $G(\mathcal{S}_n)$ which will lead to some interesting consequences for the semigroup $\mathcal{S}_n$.

1. THE BIORDERED SET AND GREEN'S RELATIONS ON $\mathcal{S}_n$

We know that $\mathcal{S}_n$ is regular. Hence $E_n = E(\mathcal{S}_n)$ is a regular biordered set. Now elements of $E_n$ are idempotent endomorphisms of $V$ and $e \in E_n$ if and only if $\mathcal{N}(e) \oplus \mathcal{R}(e) = V$. Since $e$ is singular, we have $\mathcal{N}(e) \neq 0$ (i.e., $\mathcal{N}(e) \neq \{0\}$; since no confusion is likely, in the sequel, we shall use the shorter notation). Conversely, if $U \oplus W = V$ with $U \neq 0$, there is a unique $e \in E_n$ with $U = \mathcal{N}(e)$ and $W = \mathcal{R}(e)$. We denote this unique idempotent by $e(U,W)$. Therefore, the biorder structure of $E_n$ can be described in terms of these direct sum decompositions. In particular the quasiorder relations $\omega^r$ and $\omega^l$ on $E_n$ can be obtained as follows.

**Proposition 1**   For $e, f \in E_n$, $e \omega^l f$ if and only if $\mathcal{N}(f) \subseteq \mathcal{N}(e)$ and $e \omega^r f$ if and only if $\mathcal{R}(e) \subseteq \mathcal{R}(f)$. There is a }
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**Proof**  
\[ e \omega^r f \implies fe = e. \]

But \( \mathcal{N}(f) \subseteq \mathcal{N}(fe) = \mathcal{N}(e). \)

Conversely \( \mathcal{N}(f) \subseteq \mathcal{N}(e) \implies u(1 - f) \in \mathcal{N}(e) \) for every \( u \in V. \) Hence 
\[ u(1 - f)e = 0 \text{ for every } u \in V \implies (1 - f)e = 0 \implies e = fe. \] That is \( e \omega^r f. \)

Again \( e \omega^l f \implies ef = e \)

Now \( \mathcal{N}(e) = \mathcal{N}(ef) \subseteq \mathcal{N}(f). \)

Conversely, \( \mathcal{N}(e) \subseteq \mathcal{N}(f) \implies ue \in \mathcal{N}(f) \) for every \( u \in V \implies uef = ue \) for every \( u \in V \implies ef = e. \) That is \( e \omega^l f. \)

Since \( \omega = \omega^l \cap \omega^r, \) for \( e, f \in E_n \) \( e \omega f \) if and only if \( \mathcal{N}(e) \subseteq \mathcal{N}(f) \) and \( \mathcal{N}(e) \supseteq \mathcal{N}(f). \) The basic product in \( E_n \) is the composition of endomorphisms restricted to \( DE_n. \) A more general construction of biordered sets from complemented modular lattices are in [20].

Recall that in a semigroup \( S \) we define the relations. \( \mathcal{R}, \mathcal{L} \) and \( \mathcal{J} \) by \( a \mathcal{R} b \) if and only if \( aS \cup \{a\} = bS \cup \{b\}. \) \( a \mathcal{L} b \) if and only if \( Sa \cup \{a\} = Sb \cup \{b\} \) and \( a \mathcal{J} b \) if and only if \( aS \cup Sa \cup Sa \cup \{a\} = bS \cup Sb \cup Sb \cup \{b\}. \) Also, we define \( \mathcal{D} \) to be the equivalence generated by \( \mathcal{R} \cup \mathcal{L}. \) It can be shown that \( \mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}. \)

Finally we define \( \mathcal{H} = \mathcal{R} \cap \mathcal{L}. \) These are equivalences on \( S \) called the Green's relations [1,12]. The \( \mathcal{L} \)-class (\( \mathcal{R} \)-class, \( \mathcal{J} \)-class, \( \mathcal{H} \)-class, \( \mathcal{D} \)-class) containing an element \( a \) of a semigroup \( S \) is denoted by \( La(Ra, Ja, Ha, Da). \)

We will now characterize the Green's relations on \( \mathcal{S}(V). \) Since \( \mathcal{S}(V) \) is a regular subsemigroup of \( L(V), \) two elements of \( \mathcal{S}(V) \) are \( \mathcal{R} \) or \( \mathcal{L} \) related in \( \mathcal{S}_n \) if and only if the corresponding relations holds in \( L(V). \) Hence first we characterize the Green's relations in \( L(V). \) For that we need the following result.

Let \( S \) be a semigroup. There is a partial order \( \leq \) defined on the set of
all $\mathcal{R}$--class $S/\mathcal{R}$ [$\mathcal{L}$--class $S/\mathcal{L}$] as follows

$$
R_t \leq R_s \iff t \in sS' \iff t = su \text{ for some } u \in S',
$$

$$
L_t \leq L_s \iff t \in S's \iff t = us \text{ for some } u \in S'.
$$

where

$$
S' = \begin{cases} 
S & \text{if } S \text{ contains an identity element} \\
S \cup \{1\} & \text{otherwise where } s1 = 1 = 1s \forall s \in S
\end{cases}
$$

Since $S/\mathcal{R}$ is in one--to--one correspondence with the set of right ideals, this defines a partial order on the set of all right ideals of $S$ which is clearly the inclusion. Similarly, the relation $\leq$ defined on $S/\mathcal{L}$ induces the inclusion relation on the set of all left ideals.

**Lemma 2** Let $s, t \in L(V)$. Then

1. $R_t \leq R_s \iff \mathcal{N}(t) \supseteq \mathcal{N}(s)$.
2. $L_t \leq L_s \iff \mathcal{R}(t) \subseteq \mathcal{R}(s)$.

**Proof** (1) $R_t \leq R_s \implies t = su$ for some $u \in L(V)'$. Now $x \in \mathcal{N}(s) \implies xs = 0 \implies xsu = (xs)u = 0$. That is, $xt = 0 \implies x \in \mathcal{N}(t)$. Therefore, $\mathcal{N}(s) \subseteq \mathcal{N}(t)$.

Conversely, suppose that $\mathcal{N}(s) \subseteq \mathcal{N}(t)$. Define $l$ arbitrarily on the complement of $\mathcal{R}(s)$ in $V$ and for $x \in \mathcal{R}(s)$ letting $y \in V$ be such that $ys = x$. Define $xl = yt$. The fact that $\mathcal{N}(s) \subseteq \mathcal{N}(t)$ ensures that $l$ is well defined. Clearly $l \in L(V)$ and $ysl = xl = yt$ for every $y \in V$, showing that $t = sl \implies R_t \leq R_s$. 
(2) \( L_t \leq L_s \iff t = us \) for some \( u \in L(V)' \). Now \( \mathcal{R}(t) = \mathcal{R}(us) \subseteq \mathcal{R}(s) \).
That is, \( \mathcal{R}(t) \subseteq \mathcal{R}(s) \).

Conversely, let \( \mathcal{R}(t) \subseteq \mathcal{R}(s) \). For each \( x \in V \), we may choose one \( y \in V \) such that \( ys = xt \) and declare \( xl = y \). It follows that \( l \in L(V) \) and \( xls = ys = xt \) for every \( x \in V \). Hence \( ls = t \implies L_t \leq L_s \). \( \square \)

**Theorem 3**  Let \( s, t \in L(V) \), then we have the following

1. \( tLs \) if and only if \( \mathcal{R}(t) = \mathcal{R}(s) \).
2. \( tRs \) if and only if \( \mathcal{N}(t) = \mathcal{N}(s) \).
3. \( tHs \) if and only if \( \mathcal{R}(t) = \mathcal{R}(s) \) and \( \mathcal{N}(t) = \mathcal{N}(s) \).
4. \( tDs \) if and only if \( \text{rank } t = \text{rank } s \).
5. \( \mathcal{J} = \mathcal{D} \) in \( L(V) \).

**Proof**  \( tLs \iff L_t = L_s \) and \( tRs \iff R_t = R_s \). Hence statements (1) and (2) of this theorem are consequence of the above lemma. Also \( tHs \) if and only if \( tRs \) and \( tLs \). Hence we get (3).

(4)  Assume that \( tDs \). Then there exists \( l \in L(V) \) such that \( tLl \) and \( tRs \).
By part (1) \( tLl \implies \mathcal{R}(t) = \mathcal{R}(l) \) and hence \( \text{rank } t = \text{rank } l \). Again by part (1) \( tRs \implies \mathcal{N}(l) = \mathcal{N}(s) \). Hence \( l \) and \( s \) have the same nullity and hence the same rank. Therefore, \( t \) and \( s \) have the same rank.

Conversely, assume that \( s, t \in L(V) \) have the same rank. Hence \( \mathcal{R}(s) \) and \( \mathcal{R}(t) \) have the same dimension. So there exists an isomorphism \( \phi: \mathcal{R}(t) \to \mathcal{R}(s) \).
Define \( u = t\phi \). Then \( \mathcal{R}(u) = \mathcal{R}(s) \) and \( \mathcal{N}(u) = \mathcal{N}(t) \) since \( \phi \) is an isomorphism.
Hence from parts (1) and (2) we get \( uLs \) and \( uRt \implies sLuRt \implies sDt \).
First of all \( D \subseteq J \) in general. Now \( D = L \circ R = R \circ L \) is the smallest equivalence relation \( L \cup R \) containing both \( L \) and \( R \). We define \( aJb \) to mean \( S'asS' = S'hS' \). Hence we get \( L \subseteq J \) and \( R \subseteq J \). Therefore, \( D \subseteq J \).

To prove the other inequality, we first show that for \( s, t \in L(V), s \in J_t \) if and only if \( \text{rank } s \leq \text{rank } t \). If \( s \in J_t \), then \( s = \xi t \eta \) for some \( \xi \) and \( \eta \in L(V)' \). Hence we get \( \xi \sim \eta \) and \( R \sim \eta \).

Now \( R_s = R(\xi t \eta) \subseteq R(t \eta) \)

Also \( \eta \) induces a linear transformation of \( R(t) \) onto \( R(t \eta) \) and so \( \dim R(t) \geq \dim R(t \eta) \). Hence rank \( s \leq \text{rank } (t \eta) \leq \text{rank } t \).

Conversely, let rank \( s \leq \text{rank } t \). Then \( \dim R(s) \leq \dim R(t) \). Hence there exists a surjective linear map \( \eta: R(t) \to R(s) \), therefore \( R(t \eta) = R(s) \). Hence \( \eta L_s \). Now \( t \eta \in t L(V) \implies t \eta L(V) \subseteq t L(V) \implies J_t \eta \subseteq J_t \). Since \( t \eta L_s \), \( L(V)t \eta = L(V)s \).

Therefore, \( J_s = J_t \eta \subseteq J_t \implies s \in J_t \).

Thus \( s \in J_t \) if and only if \( \text{rank } s \leq \text{rank } t \). Hence \( sJt \) if and only if \( \text{rank } t = \text{rank } s \). So by part (4) \( sJt \) if and only if \( sDt \). Hence in \( L(V), J = D \).

Thus we have the following characterizations of Green’s relations on \( S_n \).

**Theorem 4** For \( f, g \in S_n \) we have the following

(a) \( fLg \iff R(f) = R(g) \).

(b) \( fRg \iff N(f) = N(g) \).

(c) \( fHg \iff R(f) = R(g) \) and \( N(f) = N(g) \).

(d) \( fDg \iff \text{rank } f = \text{rank } g \).

(e) \( D = J \).
We observe that the characterizations of the biorder structure on $E_n$ immediately leads to a characterization of the Green's relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$ given in (a), (b) and (c) above.

**Definition 1** A regular semigroup $S$ is completely semisimple if no two distinct $\mathcal{D}$-related idempotents are comparable by the natural partial order.

**Proposition 5** $\mathcal{S}_n$ is completely semisimple.

**Proof** Let $e, f \in E_n$ with $e \mathcal{D} f$ and $e \leq f$. Then $\dim \mathcal{N}(e) = \dim \mathcal{N}(f)$ and $\dim \mathcal{R}(e) = \dim \mathcal{R}(f)$. Also $e \leq f$ implies $ef = fe = e$. Therefore, $\mathcal{N}(e) \subseteq \mathcal{N}(f)$ since $\mathcal{N}(e)$ and $\mathcal{N}(f)$ are finite dimensional it follows that $\mathcal{R}(e) = \mathcal{R}(f)$. So $e \mathcal{L} f$ by Theorem 4 and $fe = f$. Thus $e = f$. This proves the result.  

2. **THE PROPER SET OF COMMUTATIVE CYCLES IN $\mathcal{S}_n$**

We shall say that $E$-cycle $\gamma$ in $\mathcal{G}(E_n)$ is commutative in $\mathcal{S}_n$ (or in $\mathcal{G}(\mathcal{S}_n)$) if it is $\epsilon_n\mathcal{S}_n$-commutative. We denote the set of all commutative $E$-cycles in $\mathcal{S}_n$ by $\Gamma_n$. Then $\Gamma_n$ is a closed proper set of $E$-cycles in $\mathcal{G}(E_n)$. Now for any $c = c(e_0e_1\cdots e_n) \in \mathcal{G}(E_n)$, $T_c = e_0e_1\cdots e_n$.

is a singular endomorphism of the vector space $V$; since $\mathcal{S}_n$ is idempotent generated, every element of $\mathcal{S}_n$ is of this form by Theorem I.2. Note that, if $c$ is reduced, then we have

$$T_c = \begin{cases} 
  e_0e_2\cdots e_n, & \text{if } e_0\mathcal{L}e_1, e_{n-1}\mathcal{R}e_n \\
  e_1e_3\cdots e_n, & \text{if } e_0\mathcal{R}e_1, e_{n-1}\mathcal{R}e_n \\
  e_0e_2\cdots e_n, & \text{if } e_0\mathcal{L}e_1, e_{n-1}\mathcal{L}e_n \\
  e_1e_3\cdots e_n, & \text{if } e_0\mathcal{R}e_1, e_{n-1}\mathcal{L}e_n 
\end{cases}$$
We shall say that $T_e$ has the *normal form* if the first case above holds. By introducing trivial edges into $c$, if necessary, we can always ensure that the first case above holds. To avoid ambiguity, we shall adopt the convention that, unless otherwise indicated, the expression $T_e$ is in the normal form. This in particular, means that the number $n$ of vertices of $c$ is odd and the number of edges is even. Moreover, $T_e = T_{e'}$ if and only if $c(e_{e'}, e_{e''})c(f_{e'}, f_{e''})c^{-1} \in \Gamma_n$.

Notice that an $E$-cycle $\gamma$ is commutative in $\mathcal{G}_n$ if and only if $T_{\gamma} = e_{\gamma}$. Thus

$$\Gamma_n = \{ \gamma : \gamma \text{ is an } E\text{-cycle and } T_\gamma = e_\gamma \}.$$

We therefore, have a mapping $T: \mathcal{G}(E_n) \to \mathcal{G}_n$ which, in view of the isomorphism between $S(G_{\Gamma_n}) = S(G(\mathcal{G}_n))$ and $\mathcal{G}_n$ is equivalent to the map $W_{\Gamma_n}$ defined by Equation $W_{\Gamma_n} = \epsilon_{\Gamma_n}p_{\Gamma^n}$ where $\epsilon_{\Gamma_n}$ is the surjective evaluation on $G_{\Gamma_n}$ and $p_{\Gamma^n}$ denotes the canonical mapping of $G_{\Gamma_n}$ onto $G_{\Gamma_n}/p(G_{\Gamma_n})$. Next lemma characterizes commutative $E$-squares.

**Lemma 6** For $e, f \in E_n$, the following statements are equivalent.

(a) There exists $g, h \in E_n$ such that $\delta = \left( \begin{smallmatrix} e & g \\ h & f \end{smallmatrix} \right)$ is a commutative $E$-square.

(b) $\mathcal{R}(f) \subseteq \mathcal{R}(e) \oplus N$ where $N = \mathcal{N}(e) \cap \mathcal{N}(f)$.

(c) $(e - f)^2 | \mathcal{R}(e) = 0$.

**Proof** (a) $\implies$ (b): Since $\delta$ is an $E$-square, either $eRg$ or $eLg$; to fix notation we may assume that $eRg$. Since $\delta$ is commutative we have $efe = e$ and $gh = e$. Therefore, the map $\tilde{g}: v \to vg$ is a linear isomorphism of $\mathcal{R}(e)$ onto $\mathcal{R}(f)$ and the map $\tilde{h}: w \to wh$ is its inverse. For, $v\tilde{g} = vg$, $v\tilde{g}\tilde{h} = vgh = ve = v$. It is easy to see that for each $v \in \mathcal{R}(e)$, $vg = v + n$ where $n \in \mathcal{N}(e)$. Similarly, for $w \in \mathcal{R}(f)$, $wh = w + m$ for $m \in \mathcal{N}(f)$. Hence for each $w \in \mathcal{R}(f)$, there is a unique $v \in \mathcal{R}(e), n \in \mathcal{N}(e)$ and $m \in \mathcal{N}(f)$ such that $w = v + n$ and
\[ v + n + m = v. \] This implies that \( n + m = 0 \) so that \( n = -m \in \mathcal{N}(e) \cap \mathcal{N}(f) = N. \) Therefore, (b) holds.

(b) \( \implies \) (c): Let \( B_f = \{ w_i : i = 1, 2, \ldots, r \} \) be a basis of \( \mathfrak{R}(f). \) It follows from (b) that for each \( w_i \) there is a unique \( v_i \in \mathfrak{R}(e) \) and \( n_i \in N \) such that \( w_i = v_i + n_i. \) Then \( B_e = \{ v_i : 1 \leq i \leq r \} \) is a linearly independent set. For, if \( \sum_i \alpha_i v_i = 0, \) then \( \sum_i \alpha_i (w_i - n_i) = 0 \) and so \( \sum_i \alpha_i w_i = \sum_i \alpha_i n_i. \) This implies that \( \sum_i \alpha_i w_i \) is a vector in \( \mathfrak{R}(f) \cap N \subseteq \mathfrak{R}(f) \cap \mathfrak{N}(f) = \{ 0 \} \) and so \( \sum_i \alpha_i w_i = 0. \) Since \( B_f \) is a basis, we have \( \alpha_i = 0 \) for all \( i. \) Also \( \dim B_e = r. \) Hence \( B_e \) is a basis of \( \mathfrak{R}(e). \) Now for any \( i, \) \( v_i(e - f) = v_i - w_i = -n_i \) and since \( n_i \in N = \mathfrak{N}(e) \cap \mathfrak{N}(f), \) it follows that \( v_i(e - f)^2 = (-n_i)(e - f) = 0. \) Since \( B_e \) is a basis of \( \mathfrak{R}(e), \) (c) holds.

(c) \( \implies \) (a): Let, as above, \( B_e = \{ v_i : 1 \leq i \leq r \} \) be a basis of \( \mathfrak{R}(e) \) and let \( w_i = v_i f \) for each \( i. \) Then there exists \( v_i' \in \mathfrak{R}(e) \) and \( n_i' \in \mathfrak{N}(e) \) such that \( v_i f = w_i = v_i' + n_i'. \) For \( v_i \in B_e, \) we have from (c),

\[ v_i(e - f)^2 = v_i e - v_i f - v_if e + v_if = v_i - v_if e = 0 \]

since \( v_if e = v_i' \); it follows that \( w_i = v_i + n_i' \) for any \( i. \) Since \( f \) is an idempotent, we have \( n_i' f = w_i f - v_i f = 0 \) and so \( n_i' \in \mathfrak{N}(e) \cap \mathfrak{N}(f) = N. \) Now if \( \sum_i \alpha_i w_i = 0, \) then we get using the equality \( w_i = n_i' + v_i \) that \( \sum_i \alpha_i v_i = - \sum_i \alpha_i n_i' \in N, \) we get \( \alpha_i = 0 \) for all \( i. \) Thus \( B_f = \{ w_i : 1 \leq i \leq r \} \) is a basis of \( \mathfrak{R}(f). \) Also \( \mathfrak{R}(f) \oplus \mathfrak{N}(e) = V. \) For if, \( u \in \mathfrak{R}(f) \cap \mathfrak{N}(e) \) then \( u = \sum \alpha_i w_i \) for some \( \alpha_i \in K. \) Hence \( u = \sum \alpha_i v_if \) and also we have \( ue = 0 = \sum \alpha_i v_if e. \) This implies that \( \sum \alpha_i v_i = 0. \) Hence \( \alpha_i = 0 \) for all \( i. \) Thus we have \( u = 0. \) Hence \( \mathfrak{R}(f) \cap \mathfrak{N}(e) = \{ 0 \}. \) So there exists an idempotent \( g \) with \( \mathfrak{N}(g) = \mathfrak{N}(e) \) and \( \mathfrak{R}(g) = \mathfrak{R}(f). \) Hence by Theorem 4, we have \( eRgLf. \) Now for each \( i, \) we have observed that \( n_i' \in N \) from which it follows that \( w_i(e - f)^2 = 0. \) Thus
(e - f)² | \mathcal{R}(f) = 0 and so, as above there exists an idempotent h with 
\epsilon \mathcal{L} h \mathcal{R} f. Then by Clifford–Miller Theorem (see [1]) since g ∈ L_f ∩ R_e the
product fe ∈ H_k. Since \nu_i fe = \nu_i e = \nu_i for all i, we conclude that fe = h.
So efe = eh = e. Hence (a) holds. □

For each \alpha ∈ S_n, let \alpha^* = 1 - \alpha. Clearly, for any \alpha ∈ S_n, \alpha^* ∈ E_n if and
only if \alpha ∈ E_n. Further, for \epsilon ∈ E_n, \mathcal{N}(\epsilon^*) = \mathcal{N}(\epsilon) and \mathcal{R}(\epsilon^*) = \mathcal{R}(\epsilon). Hence
\delta = \left( \begin{smallmatrix} \epsilon & \epsilon^* \\ h & f \end{smallmatrix} \right) is an E-square if and only if \delta^* = \left( \begin{smallmatrix} \epsilon^* & h^* \\ \epsilon & f^* \end{smallmatrix} \right) is an E-square. Notice
that (e - f)² = (\epsilon^* - f^*)². Since, for \epsilon ∈ E_n, \mathcal{R}(\epsilon) ⊕ \mathcal{N}(\epsilon) = V, (e - f)² = 0
if and only if (e - f)² | \mathcal{R}(\epsilon) = 0 and (e - f)² | \mathcal{N}(\epsilon) = 0. The following
observation is an immediate consequence of the lemma above.

**Corollary 7**  
Let \delta = \left( \begin{smallmatrix} \epsilon & \epsilon^* \\ h & f \end{smallmatrix} \right) be an E-square in \mathcal{G}(E_n). Then \delta and \delta^* are
commutative E-squares in S_n if and only if (e - f)² = 0. □

Since \Gamma_n is a closed and proper set of E-cycles, by (P1) we have \Gamma_0 ⊆ \Gamma_0 ⊆ \Gamma_n. This implies that every singular E-square is commutative in S_n.
The next lemma shows that every E-square in \Gamma_n belongs to \Gamma_0.

**Lemma 8**  
Let \delta = \left( \begin{smallmatrix} \epsilon & \epsilon^* \\ h & k \end{smallmatrix} \right) be an E-square in \mathcal{G}(E_n). Then \delta is commutative
in \mathcal{G}(E_n) if and only if it is degenerate or \delta is both row and column singular.

**Proof**  
In view of the observation above, it is sufficient to show that if \delta is
commutative and not degenerate, then \delta is both row and column singular.
Suppose that \delta is commutative. Then by Lemma 6, we have

\begin{equation}
\mathcal{R}(k) ⊆ \mathcal{R}(f) ⊕ N \text{ where } N = \mathcal{N}(f) \cap \mathcal{N}(k)
\end{equation}

If \mathcal{N}(f) = \mathcal{N}(k), we have fRK by Lemma 4(b) and then g = k and f = h. So
\delta is degenerate. Similarly, if N = 0, then by (*) \mathcal{N}(k) = \mathcal{N}(f) and so kLF
by Theorem 4(b). In this case \( f = g \) and \( h = k \) and so \( \delta \) is again degenerate. So we may, in the following assume that \( N \) is a proper non-zero subspace of \( \mathfrak{N}(f) \) and \( \mathfrak{N}(k) \).

Let \( N' \) be a complement of \( N \) in \( \mathfrak{N}(f) \). Then by hypothesis, \( N' \neq 0 \). Let 
\[
 e = e(N, \mathfrak{R}(f) \oplus N').
\]
By Proposition 1, \( g, k \in \omega'(e) \). Also, if \( \{u_i; 1 \leq i \leq r\} \) is a basis of \( \mathfrak{R}(f) = \mathfrak{R}(h) \), by (\*), there is a basis \( \{w_i; 1 \leq i \leq r\} \) of \( \mathfrak{R}(k) = \mathfrak{R}(g) \) such that
\[
 w_i = u_i + n_i \text{ where } n_i \in N \text{ for all } 1 \leq i \leq r
\]
Since \( \mathfrak{N}(f) = \mathfrak{N}(g) \), we have \( v_i g = w_i g = w_i \) for all \( i \). Therefore, \( v_i g e = w_i e = v_i \) for all \( i \). Since, clearly, \( \mathfrak{N}(ge) = \mathfrak{N}(g) = \mathfrak{N}(f) \), we have \( ge = f \).

Similarly, \( v_i k = (v_i + n_i)k = w_i k = w_i \) so that \( v_i ke = w_i e = v_i \). Also \( \mathfrak{N}(ke) = \mathfrak{N}(k) = \mathfrak{N}(h) \) so that \( ke = h \). Thus \( \delta = \left( \begin{array}{cc} e & 0 \\ 0 & k \end{array} \right) \) and hence \( \delta \) is row singular.

Suppose that \( e' = e(N'; \mathfrak{R}(f) \oplus N) \). From (\*) it follows that \( \mathfrak{R}(k) \subseteq \mathfrak{R}(e') \). Also \( \mathfrak{R}(f) = \mathfrak{R}(h) \subseteq \mathfrak{R}(e') \). Hence by Proposition 1, \( k, h \in \omega'(e') \).

Then \( e' h \mathcal{L} h \mathcal{L} f \) and so by Theorem 4(a) \( \mathfrak{R}(e'h) = \mathfrak{R}(f) \). Since \( \mathfrak{N}(f) = N \oplus N' \), if \( n \in \mathfrak{N}(f) \), then \( ne' \in N \) and so \( ne'h = 0 \). Hence \( \mathfrak{N}(f) \subseteq \mathfrak{N}(e'h) \). Since \( \dim \mathfrak{N}(f) = \dim \mathfrak{N}(e'h) \) it follows that \( \mathfrak{N}(f) = \mathfrak{N}(e'h) \) and so \( f = e'h \).

Similarly, \( g = e'k \) and so \( \delta = \left( \begin{array}{cc} e'h & e'k \\ h & k \end{array} \right) \). Hence \( \delta \) is column singular.

In [18] it is shown how a partial order can be defined on a regular semigroup. The relevant result is given below. If \( S \) is semigroup for \( T \subseteq S \) we write \( E(T) \) to denote the set of idempotents for \( T \subseteq S \).

**Proposition 9** Let \( S \) be a regular semigroup and \( x, y \in S \). Then the following are equivalent.

II.2 THE PROPER SET OF COMMUTATIVE CYCLES IN $\mathcal{S}_n$

(1) $xS \subseteq yS$ and there exists $f \in E(R_x)$ with $x = fy$.

(2) For each $f \in E(R_y)$ there exists $e \in E(R_x)$ with $e\omega f$ and $x = ey$.

(3) For each $f \in E(L_y)$ there exists $e \in E(L_x)$ with $e\omega f$ and $x = ye$. □

For proof see Propositions 1.1 and 1.3 of [18]. If $x, y$ satisfy any (and so all) of the above conditions then we write $x \preceq y$. In [18] it is shown that this defines a partial order on $S$ and is named the natural partial order on $S$.

PROPOSITION 10 If $\gamma \leq \gamma'$ with respect to the partial order in the ordered groupoid $\mathcal{G}(E_n)$ then $T_\gamma \leq T_{\gamma'}$ with respect to the natural partial order on the semigroup $\mathcal{S}_n$. Moreover, if $\gamma \leq \gamma'$ in $\mathcal{G}(E_n)$ and if $\gamma'$ is commutative so is $\gamma$.

PROOF Let $\gamma = (h_0, h_1, \ldots, h_n)$ and $\gamma' = (e_0, e_1, \ldots, e_n)$ be in $\mathcal{G}(E_n)$ such that $T_\gamma$ and $T_{\gamma'}$ are in the normal form. Now $\gamma \leq \gamma'$ implies $h_0 \omega e_0$ and $\gamma = h_0 \ast \gamma'$ by Equation (2) in §I.2. Then

$$h_i = e_i h_{i-1} e_i \text{ for } i = 1, 2, \ldots, n.$$  

$$T_\gamma = h_0 h_1 h_2 \cdots h_n = h_0 h_2 \cdots h_n = h_0 e_1 e_2 \cdots e_n e_{n-1} e_{n-2} \cdots e_2 e_1 h_0 e_1 e_2 \cdots e_n = h_0 e_1 e_2 \cdots e_n,$$  

since $e_i L e_{i+1}$ or $e_i R e_{i+1}$ for each $i$.

$$T_{\gamma'} = h_0 T_{\gamma'},$$

Therefore, $T_\gamma = h_0 T_{\gamma'}$ where $h_0 \omega e_0 \mathcal{R} T_{\gamma'}$ since $T_{\gamma'}$ is in the normal form. Hence by Proposition 9 $T_\gamma \leq T_{\gamma'}$ with respect to the natural partial order on $\mathcal{S}_n$.

$\gamma'$ commutative means $e_n = e_0$ and $T_{\gamma'} = e_0$. Hence $h_n = h_0$ and $T_\gamma = h_0 e_0 = h_0$, which implies that $\gamma$ is commutative. Hence the last statement follows. □

We have the following characterizations of commutative $E$-cycles in $\mathcal{S}_n$. 
Proposition 11. An \( E \)-square \( \delta \) is commutative in \( S_n \) if and only if \( \delta \) is singular. Moreover, if \( n \geq 2 \), an \( E \)-cycle \( \gamma \) based at \( e \) is commutative if and only if \( \gamma \in \Gamma_r \) and satisfies the condition that whenever \( \delta \) is an \( E \)-square based at \( e' \omega e \) with \( T_\delta \leq T_\gamma \), then \( \delta \) is commutative.

Proof. The first statement follows from Lemma 8. Now by definition, the \( E \)-cycle \( \gamma \) based at \( e \) is commutative if and only if \( T_\gamma = e \). Hence for an \( E \)-cycle \( \delta \) based at \( e' \omega e \), \( T_\delta \leq T_\gamma \) implies that \( T_\delta = e' \) and so \( \delta \) is commutative.

Conversely, assume that \( \gamma \) satisfies the given conditions. Since \( \gamma \in \Gamma_r \), \( T_\gamma \) is a scalar transformation, say \( T_\gamma = \lambda e \). Then \( R(T_\gamma) = R(e) \) and \( N(T_\gamma) = N(e) \). Let \( 0 \neq v \in R(T_\gamma) \) and let \( N' \) be a complement of \( \langle v \rangle \) containing \( N(T_\gamma) \). Define \( T' = e'T_\gamma \) where \( e' = e(N', \langle v \rangle) \). Then \( R(e') = \langle v \rangle \subseteq R(e) \) which implies \( e' \omega e \) and \( N(e') = N' \supseteq N(e) \) which implies \( e' \omega e \). Hence \( e' \omega e \). Then \( T' = e'T_\gamma = e'\lambda e = \lambda e'e = \lambda e' \) and since \( n \geq 2 \), \( \text{rank} \, T' = 1 \leq \text{dim} \, N' = \text{nullity} \, T' \). Therefore, \( T' = \lambda e' \) is a scalar matrix having rank \( \leq \) nullity and hence by Proposition 4.8 \( T' = T' \), for some \( E \)-square \( \delta \) based at \( e' \). Now \( T' = e'T_\gamma \) where \( e' \omega e \in \Gamma_r \). Hence by Proposition 2.8 \( T' \leq T_\gamma \) in the natural partial order in \( S_n \). Then by the hypothesis, \( \delta \) must be commutative which implies that \( \lambda = 1 \). Therefore, \( T_\gamma = e \). This proves that \( \gamma \) is commutative. \( \square \)

The proposition above shows that the proper set of \( E \)-cycles \( \Gamma_n \) (the set of those \( E \)-cycles which are commutative in \( S_n \)) is, in some sense “generated” by the proper set \( \Gamma_0 \) of singular \( E \)-squares. Hence it is natural to ask whether \( \Gamma_n \) is the closure of \( \Gamma_0 \) in the sense of §6, [19]. The question is equivalent to: is it true that all commutative \( E \)-cycles in \( S_n \) can be obtained as combination of commutative squares. We therefore, formulate the following.

Is it true that \( S_n \) is the free idempotent generated regular semigroup generated by \( E_n \).
The answer is No. $\mathcal{S}_n$ is not the free idempotent generated regular semigroup generated by $E_n$. We can show it by an example.

Consider the $\mathcal{D}$-class of rank$(n - 1)$ idempotents in $\mathcal{S}_n$. We know that there are so many commutative $E$-cycles in that particular $\mathcal{D}$-class. In the free idempotent generated regular semigroup generated by $E_n$, the $E$-cycles are all generated by singular $E$-squares. But there are no singular $E$-squares of rank$(n - 1)$ except the identity square. Hence there are no commutative $E$-cycles of rank$(n - 1)$ in the free idempotent generated regular semigroup generated by $E_n$ except the identity squares. Hence $\mathcal{S}_n$ is not isomorphic to the free idempotent generated regular semigroup generated by $E_n$.

We will give an example of a commutative $E$-cycle in the $\mathcal{D}$-class of rank$(n - 1)$ idempotents in $\mathcal{S}_n$.

**Example 1**

Consider the semigroup of all singular $3 \times 3$ matrices over the $\mathbb{R}$. We construct a commutative $E$-cycle of length 6 in the $\mathcal{D}$-class of rank 2 idempotents in this particular semigroup. The matrices $e_0, e_1, e_2, e_3, e_4, e_5$ given below are all idempotents of rank 2.

$$
\begin{align*}
  e_0 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
  e_1 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
  e_2 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \\
  e_3 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
  e_4 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \\
  e_5 &= \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{align*}
$$

Let $u_1, u_2, u_3$ denote the vectors $(1,0,0), (0,1,0)$ and $(0,0,1)$ respectively.
Then

\[ R(e_0) = \langle u_1, u_2 \rangle \]
\[ R(e_1) = \langle u_1, u_2 \rangle \]
\[ R(e_2) = \langle u_1, u_3 \rangle \]
\[ R(e_3) = \langle u_1, u_3 \rangle \]
\[ R(e_4) = \langle u_1, u_2 + u_3 \rangle \]
\[ R(e_5) = \langle u_1, u_2 + u_3 \rangle \]
\[ \mathcal{N}(e_0) = \langle u_3 \rangle \]
\[ \mathcal{N}(e_1) = \langle u_2 - u_3 \rangle \]
\[ \mathcal{N}(e_2) = \langle u_2 - u_3 \rangle \]
\[ \mathcal{N}(e_3) = \langle u_2 \rangle \]
\[ \mathcal{N}(e_4) = \langle u_2 \rangle \]
\[ \mathcal{N}(e_5) = \langle u_3 \rangle \]

Then by Theorem \( e_0 L e_1 R e_2 L e_3 R e_4 L e_5 R e_0 \). Hence \( \gamma = c(e_0, e_1, e_2, e_3, e_4, e_5, e_0) \) is an \( E \)-cycle based at \( e_0 \). Moreover, \( T_\gamma = e_0 e_1 e_2 e_3 e_4 e_5 e_0 = e_0 \). Hence \( \gamma \) is a commutative \( E \)-cycle of length 6 based at \( e_0 \) in the \( D \)-class of rank 2 idempotents in \( G_3 \).