CHAPTER 2

PROBLEM FORMULATION

Three broad categories of techniques are utilized for the analysis of microstrip antennas [4, 18]. The simplest of these and the relatively basic analysis approach is the transmission line model. In this approach, the rectangular microstrip antenna is treated as a pair of slots based on this approximation. The slots are excited nearly 180° out-of-phase in order for their radiated fields to reinforce at boresight [4]. The effect of fringing fields is incorporated by empirical methods through measurements. The method provides a good intuitive understanding of the mechanism of radiation from the patch; also yielding reasonably accurate expressions for resonant frequency and input resistance – the two key requirements for the antenna designer. However, for electrically thicker substrates (or in an equivalent sense, at higher operating frequencies), the expressions for both these quantities may be in significant error. Hence for millimeter-wave and monolithic applications, this elementary model proves to be unsatisfactory. Also, the simple transmission line model is useful for rectangular patch shapes. Improved versions of the model can handle other patch shapes but not proximity- or aperture-coupling. It ignores variations of the field / current along the radiating edge [4] and is not amenable to include the presence of the feeding structure.

The second analysis approach is the so-called cavity model in which the region under the microstrip patch is treated as a leaky resonant microwave cavity. The field between the microstrip patch and the ground plane is represented as a series of resonant modes of the cavity constituted by the ground & patch metallizations and the four, short bounding walls at the patch margin as magnetic walls [4]. The effect of radiation and other losses are incorporated as an equivalent loss tangent. Or it may be
included by defining an impedance boundary condition at the walls – this is a more sophisticated technique. The cavity is initially solved to determine its eigenvalues under the ideal condition of no leakage. As the cavity begins to radiate, the eigenvalues acquire complex values. Yet from a perturbational standpoint, the computations are still found to be valid to a high degree of approximation. The cavity model offers a higher degree of accuracy for predicting the actual antenna behavior at a small computational penalty. However, it has problems similar to the transmission-line model. A basic problem with the cavity model is its inability to predict mutual-coupling between radiators placed adjacent [18]. In the early years, this was not a problem as single radiators were mainly of interest and the coupling level being of the order of -20 dB; it could be ignored in practical designs. However, with advances in fabrication methods and demands for arrays with low-sidelobe characteristics or large scan ranges or for wideband arrays on thick substrates, etc. this phenomenon needs more explicit consideration.

Models based on empirical data tend to become tedious, time-consuming and prohibitive in expense especially as parameters and degrees-of-freedom in the antenna geometry increase. When it is desired to obtain a more exact analysis, the third major category – those of numerical analysis methods are the best recourse. This field has seen an enormous growth in the past years and a variety of techniques have been developed. A brief overview of these several methods is provided in the following but a more detailed description may be found in the references [40 – 42].

The computational electromagnetic (CEM) methods are classified as “exact” or “low frequency” methods and “approximate” or “high frequency” methods. The former owe their designation due to the exact solution of Maxwell’s Equations that is
embodied within these methods. The solution thus obtained is exact but the size of the problem that can be analyzed is limited on account of the fine field variations in the problem geometry accounted for. As opposed to these, the so-termed “approximate” or “high frequency” methods embody ray / optical approximations (key examples being geometrical optics (GO), physical optics (PO), the geometrical theory of diffraction (GTD), the physical theory of diffraction (PTD) and their derived methods) – further reading may be found in [39].

Some of the early low-frequency methods were the unimoment-Monte Carlo method and the direct form of network analogs (DFNA) method – see references of [4]. Two of the most important methods today are contemporary with these – the Method of Moments (MOM) that is the underlying analytical method of this thesis and the finite element method (FEM.) The method of moments, also termed as the method of weighted residuals, represents unknown current distributions on electric and/or magnetic surfaces in terms of a set of known basis functions. Another set of functions, called testing functions are used to convert the original boundary-value problem into a set of matrix equations. Solving this matrix equation provides the unknown coefficients of the known basis functions, thus allowing one to write the unknown current in terms of these evaluated quantities. These currents can then be used to estimate the solution to the original problem geometry. More details of the MOM will be presented in the remaining part of this chapter.

The FEM uses a variational form of the inhomogeneous wave equation to solve a boundary-value electromagnetic problem in the frequency domain [40]. The problem geometry, including the nearby space for a radiation problem, is discretized into two- or three-dimensional elements of varying shapes. A sparse-matrix method may be
Problem Formulation

later utilized to solve for the field at the node points of the discretized mesh and an interpolation algorithm used for the rest of the region. By contrast the matrix obtained in MOM is dense [4]; and the solution process cannot exploit sparse matrix methods.

A method that overcomes the matrix size limitations of these methods is the generalized multipole technique (GMT) [41], also reported under the name multi-level fast multipole method (MLFMM) [42]. It is also a method of weighted residues and approximates the solution field as a weighted sum of the field of several sources (or poles) spread across the problem region including the imposition of boundary conditions. In fact, by selective weighting, the effect of coupling between non-significant pairs of sources is so minimized as not to affect the accuracy of the overall solution. The MOM would, on the other hand, actually compute all these couplings increasing the execution time significantly. The method has been effectively utilized for large problems like a radiator in the presence of scattering background e.g. a shipborne or satellite-mounted low-gain antenna and, in one case, to a rapidly changing problem geometry – a fuselage top-mounted antenna on a helicopter with periodic occlusions by the rotor blades. In normal circumstances, a high-frequency method based on diffraction would be necessary for such problems.

In the recent period, a number of related methods have been developed such as the finite difference time domain (FDTD) technique, the finite difference frequency domain (FDFD) method and the finite volume time domain (FVTD) method [41]. The basis of the analysis is meshing of the complete problem domain. The solution is temporal in nature and the field values at each time instant are deduced from those in the previous instant with a constraint to satisfy Maxwell’s equations. The solution is computationally intensive due to the discretization and the time-stepping but is suited
to the capacities of modern computers. Several commercial tools based on these methods are now available.

Another time-domain technique is the transmission-line matrix (TLM) method that uses circuit-like equivalent equations to Maxwell’s equations for the analysis [41]. This method is based on a simpler formulation than the FEM and is suitable for parallel execution on many machines. A possible demerit is the need for a dense mesh to accurately represent the finer details of the problem structure.

The finite integration technique (FIT), also a recent development, is based on the integral form of Maxwell’s equations as opposed to the differential form [41]. Though originally developed in frequency domain, when used in the time domain it coincides with the FDTD method.

With this brief review of the various techniques in use for analyzing microstrip patch antennas, our discussion now returns to the method of moments which is the key technique developed in this thesis.

A very important advantage of this method is the use of exact Green’s functions that are specifically derived for a grounded dielectric slab. This formulation is referred to as the ‘spectral domain solution’ or the ‘full-wave solution’ as it explicitly accounts for the effect of field fringing thereby eliminating the empirically-determined line extensions that are used in the transmission-line model. Since the antenna problem geometry (including the near-field zone) is unbounded in two or more directions, this method allows an exact expression for the Green’s function to be computed. This latter is not a closed-form expression but is instead in the transform domain and an inverse Fourier transform is necessary to obtain the final solution. Surface-wave
excitation is explicitly accounted for since the cut-off frequencies for the same are computed within the problem formulation.

A generalized moment method solution is developed in this chapter for a microstrip antenna fed by a waveguide slot through the antenna ground plane. With the specified boundary conditions, the equivalence principle is used to express the problem in terms of coupled operator equations. The method of moments procedure is used to reduce these equations to the matrix form which is suitable for solution on a digital computer.

Section 2.1 presents the general formulation and the reduction of the operator equations to a set of linear algebraic equations. Sections 2.2 & 2.3 describe the problem geometry for the particular case when the microstrip patch is of a rectangular shape and fed by a rectangular waveguide through a rectangular ground-plane coupling aperture. The coordinate system is defined and the expansion & testing functions are selected. In order to optimize the computer implementation, the maximum possible of the matrix evaluation is carried out analytically. The derived expressions for the various matrices and vectors are presented in Sections 2.4 through 2.9. Derivation of the expressions for the antenna scattering parameters, coupled power, input impedance and VSWR are given in Section 2.10. Finally, expressions for the far-field quantities are derived based on the reciprocity principle. These are presented in Section 2.11.

2.1 General Formulation

Fig. 2.1 shows a microstrip patch antenna of arbitrary shape being fed by a waveguide of any cross-section through a thin-walled slot also of arbitrary shape. This proposed antenna geometry is shown in cross-section in Fig. 2.2(a).
Fig. 2.1: Exploded View of the Microstrip Patch Antenna fed by a Waveguide Top-Wall Slot
Problem Formulation

(a)

Region a

\[ \vec{E}^{\text{inc}}, \vec{H}^{\text{inc}} \]

Region b

\[ \varepsilon_r \varepsilon_0, \text{Dielectric} \]

Waveguide

\[ \vec{J} \]

Patch

(b)

Fig. 2.2: Application of Equivalence Principle
a) Original Problem b) Equivalent Problem
The problem may be simplified by invoking the equivalence principle. The aperture is
closed using a perfect electric conductor and equivalent surface magnetic currents \( \mathbf{M} \) and \(-\mathbf{M}\) are ‘draped’ over the sides of the closed slot in a manner that

\[
\mathbf{M} = \mathbf{E} \times \mathbf{n}
\]  

(2.1)

where \( \mathbf{E} \) is the electric field in the plane of the aperture in the original problem and \( \mathbf{n} \) is the unit outward normal from the slot. Placement of these currents ensures the continuity of the tangential electric filed across the aperture. The patch is replaced by an equivalent surface electric current distribution, \( \mathbf{J} \).

Thus the problem is decoupled into two separate regions. Region ‘a’ is an infinite
waveguide with incident fields and a magnetic current excitation. Region ‘b’ is a half-
space with a grounded dielectric slab of thickness \( d \) in which both the current \( \mathbf{J} \) in the
patch region and \(-\mathbf{M}\) in the slot region are radiating (Fig. 2.2(b).)

The total fields in the region ‘a’; \( \mathbf{E}^{a,\text{tot}}, \mathbf{H}^{a,\text{tot}} \) are the sum of the incident fields and
those excited by the magnetic current \( \mathbf{M} \) over \( S_1 \) (the slot region.) Thus the following
equations may be written:

\[
\mathbf{E}^{a,\text{tot}} = \mathbf{E}^{\text{inc}} + \mathbf{E}^{a}(\mathbf{M})
\]  

(2.2)

\[
\mathbf{H}^{a,\text{tot}} = \mathbf{H}^{\text{inc}} + \mathbf{H}^{a}(\mathbf{M})
\]  

(2.3)

where \( \mathbf{E}^{\text{inc}}, \mathbf{H}^{\text{inc}} \) denote the fields of the incident electromagnetic wave and
\( \mathbf{E}^{a}(\mathbf{M}), \mathbf{H}^{a}(\mathbf{M}) \) represent the fields in region ‘a’ due to the current \( \mathbf{M} \). Similarly,
the total fields in the region ‘b’ are those due to $-\vec{M}$ over $S_1$ and $\vec{J}$ over $S_2$ (the patch region.) These may be written as:

$$\vec{E}^{b,\text{tot}} = \vec{E}^b (-\vec{M}) + \vec{E}^b (\vec{J}) \quad (2.4)$$

$$\vec{H}^{b,\text{tot}} = \vec{H}^b (-\vec{M}) + \vec{H}^b (\vec{J}) \quad (2.5)$$

The remaining boundary conditions that need to be enforced are:

1) Tangential magnetic field must be continuous across the slot; and
2) Tangential electric field must be zero across the patch.

Using equations (2.2) and (2.3) in conjunction with the boundary conditions (1), the following operator equation may be obtained:

$$\vec{H}_t^a (\vec{M}) + \vec{H}_t^b (\vec{M}) - \vec{H}_t^b (\vec{J}) = \vec{H}_t^{\text{inc}} \quad \text{over } S_1 \quad (2.6)$$

The negative sign associated with $\vec{M}$ may be brought outside the parenthesis because $\vec{H}_t$ is an integro-differential operator and is hence linear. The subscript ‘t’ indicates that only the tangential (to slot) component of the field is considered.

Equation (2.4) and the boundary condition (2) yield the other operator equation.

$$- \vec{E}_t^a (\vec{M}) + \vec{E}_t^b (\vec{J}) = 0 \quad \text{over } S_2 \quad (2.7)$$

Equations (2.6) and (2.7) are a pair of coupled integro-differential equations that may be solved through the method of moments. As the first step, a set of expansion functions $\{ \vec{M}_m, m = 1, 2, \ldots N^a \}$ is defined over the surface of the slot $S_1$ and
another set \{ \overline{J}_l, l = 1,2, \ldots N^b \} is defined over the surface of the patch \( S_2 \). The unknown currents \( \overline{M} \) and \( \overline{J} \) are now expanded as:

\[
\overline{M} = \sum_{m=1}^{N^a} V_m \overline{M}_m
\]

and

\[
\overline{J} = \sum_{l=1}^{N^b} I_l \overline{J}_l
\]

(2.8)

(2.9)

where \( V_m \) and \( I_l \) are the unknown complex coefficients to be determined and \( N^a, N^b \) denote the number of terms in the expansion that need to be retained respectively in the two series for obtaining computational results of a certain accuracy. Substituting (2.8) and (2.9) into (2.6) and (2.7) and using the linearity property of the operators:

\[
\sum_{m=1}^{N^a} V_m \overline{H}_t^a (\overline{M}_m) + \sum_{m=1}^{N^a} V_m \overline{H}_t^b (\overline{M}_m) - \sum_{l=1}^{N^b} I_l \overline{H}_t^b (\overline{J}_l) = - \overline{H}_t^{inc} \text{ over } S_1
\]

(2.10)

\[
\sum_{m=1}^{N^a} V_m \overline{E}_t^b (\overline{M}_m) + \sum_{l=1}^{N^b} I_l \overline{E}_t^b (\overline{J}_l) = 0 \text{ over } S_2
\]

(2.11)

where \( \overline{H}_t^a (\overline{M}_m) \) denotes the tangential magnetic field in the region ‘a’ due to the \( m^{th} \) magnetic current expansion function. The other terms are also defined in a similar fashion.
In the above equations, it may be noted that the operation arguments are now the known basis functions rather than the unknown currents as was the case in (2.6) and (2.7.) So the fields $\overline{E}$, $\overline{H}$ in (2.10) and (2.11) may be determined. These equations now contain only $V_m$’s and $I_l$’s as the only unknown parameters.

Next a set of weighting functions $\{ \overline{W}_i, i = 1, 2, ... N^a \}$ over $S_1$ and another set of functions $\{ \overline{P}_s, s = 1, 2, ... N^b \}$ over $S_2$ are defined. For the present problem, a convenient inner product may be defined as:

$$< \overline{A}, \overline{B} > = \iint_{S_1 \text{ (or } S_2)} \overline{A} \cdot \overline{B} \; ds \quad (2.12)$$

Taking the inner product of $\overline{W}_i$ with (2.10) and of $\overline{P}_s$ with (2.11), we obtain:

$$\sum_{m=1}^{N^a} V_m \{ < \overline{W}_i, \overline{H}_t^a(\overline{M}_m) > + < \overline{W}_i, \overline{H}_t^b(\overline{M}_m) > \} +$$

$$\sum_{l=1}^{N^b} I_l \{ < \overline{W}_i, -\overline{H}_t^a(J_l) > \} = < \overline{W}_i, -\overline{H}_t^{inc} > \; i = 1, 2, ... N^a \text{ over } S_1 \quad (2.13)$$

$$\sum_{m=1}^{N^a} V_m \{ < \overline{P}_s, -\overline{E}_t(\overline{M}_m) > \} + \sum_{l=1}^{N^b} I_l \{ < \overline{P}_s, -\overline{E}_t(J_l) > \} = 0;$$

$$s = 1, 2, \ldots N^b \quad (2.14)$$

Equations (2.13) and (2.14) represent a set of $N^a + N^b$ algebraic equations that may be written in matrix form as overleaf.
where the various matrices and vectors are defined as:

(i) A waveguide admittance matrix:

\[
\begin{bmatrix}
Y^{a} + Y^{b} \\
C^{b}
\end{bmatrix}
\begin{bmatrix}
T^{b} \\
Z
\end{bmatrix}
= 
\begin{bmatrix}
\vec{V}_{m} \\
\vec{I}_{t}
\end{bmatrix}
= 
\begin{bmatrix}
\vec{I}_{i} \\
0
\end{bmatrix}
\]

(ii) An admittance matrix for region ‘b’:

\[
\begin{bmatrix}
Y^{b}_{im} \\
C^{b}_{rm}
\end{bmatrix}
\begin{bmatrix}
T^{b}_{nl} \\
Z_{st}
\end{bmatrix}
= 
\begin{bmatrix}
\langle \overline{W}_{i} , \overline{H}_{t}^{b} (\overline{M}_{m}) \rangle \\
\langle \overline{P}_{r} , -\overline{E}_{t}^{b} (\overline{M}_{m}) \rangle
\end{bmatrix}
\]

(iii) A matrix representing the patch-to-slot coupling:

(iv) A matrix representing the slot-to-patch coupling:

(v) An impedance matrix for the region ‘b’:

\[
Z_{st} = \begin{bmatrix}
\langle \overline{P}_{s} , \overline{E}_{t}^{b} (\overline{J}_{t}) \rangle
\end{bmatrix}_{N^{b} \times N^{b}}
\]

(vi) The coefficient vector consists of two column vectors:

\[
\vec{V}_{m} = [ V_{m} ]_{N^{a} \times 1}
\]

\[
\vec{I}_{t} = [ I_{t} ]_{N^{b} \times 1}
\]

(vii) The excitation vector consists of:

\[
\vec{I}_{i} = [ \langle \overline{W}_{i} , \overline{H}_{t}^{inc} \rangle ]_{N^{a} \times 1}
\]

\[
\vec{0} = [ 0 ]_{N^{b} \times 1}
\]
The null vector consists of a column of $N^b$ zero entries. The solution of the matrix equation (2.15) is the coefficient vector. This solution may be obtained by invoking a suitable standard method. In this thesis, the moment matrix is inverted and pre-multiplied to the excitation vector to obtain the required solution to the moment matrix equation.

### 2.2 Problem Geometry and the Coordinate System

The formulation presented in the preceding section is totally general in the sense that the patch size and shape, slot shape and dimensions or the waveguide cross-section have not been specified. Now the specific case of a rectangular microstrip patch antenna is considered that is fed by a rectangular top-wall aperture in a waveguide of rectangular cross-section (see Fig. 2.3.)

Although the solution of any problem is, in principle, independent of the coordinate system chosen; the analysis becomes simplified if the selected coordinate system is consistent with the problem geometry. In this case, it is evident that a Cartesian coordinate system is the best choice.

Some of the geometrical parameters of the proposed antenna are illustrated in the perspective view in Fig. 2.3. The input waveguide has the dimensions $a \times b$. The centre of the feeding slot is located at $(x_s, b, 0)$; where $x_s$ is the offset of the slot from the narrow wall of the waveguide. Fig. 2.4 illustrates the geometrical details of the region ‘b’. The coupling-slot dimensions are $L_s \times W_s$. The patch itself is of size $L_p \times W_p$ and is located in the plane of the substrate i.e. $z = d$. The slot centre may be offset from the slot centre. This offset is denoted by the patch position $(x_p, y_p)$ in the slot-centred coordinate system.
Problem Formulation

Fig. 2.3: Perspective View of the Proposed Waveguide-fed Microstrip Patch Antenna illustrating the Coordinate Systems employed
Problem Formulation

Fig. 2.4: Illustrating the Patch Dimensional Parameters
   a) Side View      b) Top View
2.3 Choice of Expansion and Testing Functions

Two kinds of functions are usually preferred as basis functions. If the current distribution to be computed is absolutely unknown, it is preferable to use sub-sectional basis functions. The advantage is that by reducing the size of the sub-sections to a sufficiently small value i.e. by using a large number of basis functions, a good approximation of any unknown current distribution may be obtained. However, the resulting matrix size will be large and would require greater data storage and computer execution time. If an *a priori* estimate of the nature of current distribution is known, the second class of functions, called entire-domain basis functions can be profitably employed. This requires much fewer functions and thus cuts down the execution time by drastically reducing the matrix size for the same problem size.

From earlier results on similar geometries, it is expected that the distribution in the present case will be sinusoidal [29, 54] both on the slot as well as the patch. As a result, for this thesis, entire-domain basis functions with sinusoidal distribution are utilized.

In general, both the unknown currents have $x$- as well as $\hat{y}$-directed components. However, since the coupling aperture is electrically narrow, the electric field may safely be assumed to be constant along its width. This implies that the $\hat{y}$-component of the slot magnetic current may be neglected. For this reason, the basis function for the slot magnetic current is chosen as:

$$
\bar{M}_m (x, y) = \hat{x} \sin \left[ \frac{m \pi (x + \frac{L_s}{2})}{L_s} \right] \quad \text{over} \quad S_1
$$

\hspace{1cm} (2.25)

$$
-\frac{L_s}{2} \leq x \leq \frac{L_s}{2} ; \quad -\frac{W_s}{2} \leq y \leq \frac{W_s}{2}
$$
In the case of the patch, neither of the two components may be assumed to be negligible under all the excitation conditions. However, for the TM$_{0n}$-modes of operation, there are $n$-half wavelengths of sinusoidal current along the $\hat{y}$-direction and no variation along the $\hat{x}$-direction. As a result, the $\hat{x}$-component of the patch electric current is neglected and the expansion function used for $\mathbf{J}$ is:

$$
\tilde{J}_t(x,y) = \hat{y} \sin \left[ \frac{t \pi (y - y_p + \frac{W_p}{2})}{\frac{W_p}{2}} \right] \quad \text{over } S_2
$$

(2.26)

$$
x_p - \frac{L_p}{2} \leq x \leq x_p + \frac{L_p}{2}; \quad y_p - \frac{W_p}{2} \leq y \leq y_p + \frac{W_p}{2}
$$

Equations (2.25) and (2.26) in conjunction with (2.8) and (2.9) represent the complete mathematical model of the proposed problem and their solution the unknown currents.

The same functions as presented above are also used as the weighting functions also over the problem domains $S_1$ and $S_2$. This choice is referred to as “Galerkin’s Procedure.” An advantage of this approach is that since the same functions are being used, the Fourier transforms need not be calculated again for the basis and testing functions. The convergence of the solution is also faster with this choice.

2.4 Evaluation of $[Y^a]$

From equation (2.16), the $pt$th element of $[Y^a]$ may be written as:

$$
[Y^a_{pt}] = \left[ <\bar{W}_p, \bar{H}_t^a(\bar{M}_t) > \right]
$$

(2.27)

$\bar{M}_p$ and $\bar{M}_t$ are given by (2.25) but a coordinate transformation needs to be carried out to express these functions in the waveguide system.
The expansion function $\overline{M}_t$ may thus be written as:

$$
\overline{M}_t (x_1, z_1) = \hat{z}_1 \sin \left[ \frac{t \pi (z_1 + \frac{L_s}{2})}{L_s} \right]
$$

over $S_1$ \hspace{1cm} (2.28)

$$
-\frac{L_s}{2} \leq z_1 \leq \frac{L_s}{2}; \quad x_s - \frac{W_s}{2} \leq x_1 \leq x_s + \frac{W_s}{2}
$$

The magnetic field $\overline{H}_t^a (\overline{M}_t)$ can be determined by considering the Green’s function for the electric vector potential in a waveguide which is given by [43]:

$$
\overline{G}_m = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\epsilon_{0n} \epsilon_{0m}}{2ab} \Gamma_{nm} \left[ e^{-\Gamma_{nm}} | z_1 - z'_1 | \cos \frac{n \pi x_1}{a} \cos \frac{n \pi x'_1}{a} \right. \times \\
\left. \cos \frac{m \pi y_1}{b} \cos \frac{m \pi y'_1}{b} \right] \hat{z}_1 \hat{z}_1 \hspace{1cm} (2.29)
$$

where $\Gamma_{nm} = \sqrt{\left( \frac{n \pi}{a} \right)^2 + \left( \frac{m \pi}{b} \right)^2 - \omega^2 \mu \epsilon}$ \hspace{1cm} (2.30)

$\epsilon_{0q}$ is the Neumann’s number defined as:

$$
\epsilon_{0q} = \begin{cases} 
1 & ; \quad q = 0 \\
2 & ; \quad q > 0
\end{cases} \hspace{1cm} (2.31)
$$

In equation (2.29), the primed and unprimed variables represent the source coordinates and the field coordinates respectively. The electric vector potential is now given by:

$$
\overline{F} = \epsilon_0 \int_{S_1} \overline{G}_m \cdot \overline{M}_t \; ds \hspace{1cm} (2.32)
$$
and the magnetic field can be obtained from:

$$
\vec{H} = -j \omega \vec{F} + \frac{\nabla \times \vec{F}}{j \omega \mu_0 \varepsilon_0}
$$

(2.33)

Using this procedure, the waveguide admittance matrix \([Y^a]\) has been obtained by Lyon and Sangster [44] in the course of their work on a radiating shunt-slot from a rectangular waveguide. This is given by:

$$
Y_{pt}^a = \frac{j \omega \varepsilon_0}{2a b} \sum_{n=0}^{\infty} \varepsilon_{0n} \left\{ \left( \frac{a}{n \pi} \right) \left( \frac{n \pi}{a} \left( x_s + \frac{W_s}{2} \right) \right) - 
\left\{ \left( \sin \left( \frac{n \pi (x_s - \frac{W_s}{2})}{a} \right) \right) \right\}^2 \left\{ \left( \frac{\delta_{n0}}{k^2} + \left( 1 - \left( \frac{t \pi}{k L_s} \right)^2 \right) \left( \frac{b}{\pi} \right)^2 \right) \times \right\} \right\} 
\gamma (n, t) \left\{ L_s \delta_{pt} + (tp) \left( \frac{\pi}{k L_s} \right)^2 \left( 1 + (-1)^p + t \right) \right\}
$$

$$
\sum_{m=0}^{\infty} \varepsilon_{0m} \left\{ 1 - (-1)^p e^{-\gamma_{nm} L_s} \right\} \left\{ 1 + \left( \frac{k}{\gamma_{nm}} \right)^2 \right\}
\gamma_{nm}^3 \left\{ 1 + \left( \frac{tp}{\gamma_{nm} L_s} \right)^2 \right\} \left\{ 1 + \left( \frac{p \pi}{\gamma_{nm} L_s} \right)^2 \right\}
$$

(2.34)

$$
n \neq m = 0
$$

where

$$
\delta_{is} = 0 ; \ i \neq s
$$

$$
1 ; \ i = s
$$

$$
\gamma (n, t) = \frac{-\pi^2 \cos b \sqrt{k^2 - \left\{ \left( \frac{t \pi}{L_s} \right)^2 + \left( \frac{n \pi}{a} \right)^2 \right\}}}{b \sqrt{k^2 - \left\{ \left( \frac{t \pi}{L_s} \right)^2 + \left( \frac{n \pi}{a} \right)^2 \right\}} \sin b \sqrt{k^2 - \left\{ \left( \frac{t \pi}{L_s} \right)^2 + \left( \frac{n \pi}{a} \right)^2 \right\}}
$$

(2.36)
2.5 Evaluation of [ \( Y^b \) ]

From equation (2.17), an element of [ \( Y^b \) ] may be written as:

\[
\begin{bmatrix} Y^b_{lm} \end{bmatrix} = \left[ < M^b_l, \vec{H}^b_l (\vec{M}_m) > \right]
\]

(2.37)

It may be observed in the above equation that \( \vec{M} \) is purely \( \hat{x} \)-directed. Consequently, only the \( \hat{x} \)-component of the magnetic field need be evaluated. Stated alternately, the required Green’s function is the \( \hat{x} \)-directed magnetic field over the slot region due to an infinitesimal \( \hat{x} \)-directed magnetic current dipole also on the slot, which is given by [45]:

\[
G_{HMxx} (x, y, 0 | x_0, y_0, 0) = \int_{-\infty}^{\infty} Q_{HMxx}(k_x, k_y) e^{jk_x(x-x_0)} e^{jk_y(y-y_0)} \, dk_x \, dk_y
\]

(2.38)

where

\[
Q_{HMxx}(k_x, k_y) = \frac{-j}{4 \pi^2 k_0 Z_0} \frac{1}{k_1 T_e T_m} \left[ j k_x^2 k_y^2 (\varepsilon_r - 1) + (\varepsilon_r k_0^2 - k_x^2) \times \left\{ k_1 k_2 (\varepsilon_r + 1) \sin(k_1 d) \cos(k_1 d) + j (\varepsilon_r k_2^2 \sin^2(k_1 d) \right. \right.

\]

\[
\left. \left. \left. - k_1^2 \cos^2(k_1 d) \right\} \right] \right)
\]

(2.39)

where \( k_x, k_y \) are Fourier transform variables and

\[
k_1 = \sqrt{\varepsilon_r k_0^2 - \beta^2} ; \quad Re \{k_1\} \geq 0; \quad Im \{k_1\} \leq 0
\]

(2.40)

\[
k_2 = \sqrt{k_0^2 - \beta^2} ; \quad Re \{k_2\} \geq 0; \quad Im \{k_2\} \leq 0
\]

(2.41)

\[
T_e = k_1 \cos(k_1 d) + j k_2 \sin(k_1 d)
\]

(2.42)

\[
T_m = \varepsilon_r k_2 \cos(k_1 d) + j k_1 \sin(k_1 d)
\]

(2.43)
Problem Formulation

\[ k_0^2 = \omega^2 \mu_0 \varepsilon_0 \]  \hspace{1cm} (2.44)

\[ \beta^2 = k_x^2 + k_y^2 \]  \hspace{1cm} (2.45)

\[ Z_0 = \sqrt{\mu_0 / \varepsilon_0} \]  \hspace{1cm} (2.46)

The magnetic field \( H_t^b (M_m) \) is the convolution of (2.38) with (2.25) as:

\[
H_t^b (M_m) = \hat{e} \int_{x_0} \int_{y_0} M_m (x_0, y_0) \int_{-\infty}^{\infty} Q_{HMxx} (k_x, k_y) e^{jk_x (x - x_0) + jk_y (y - y_0)}
\]

\[ dk_x \, dk_y \, dx_0 \, dy_0 \]  \hspace{1cm} (2.47)

Using equation (2.47), equation (2.37) may be rewritten as:

\[
Y_{lm}^b = \int_{x} \int_{y \, \bar{s}_1} M_l (x, y) \int_{x_0} \int_{y_0 \, \bar{s}_1} M_m (x_0, y_0) \int_{-\infty}^{\infty} Q_{HMxx} (k_x, k_y) \times
\]

\[ e^{jk_x (x - x_0) + jk_y (y - y_0)} \, dk_x \, dk_y \, dx_0 \, dy_0 \]  \hspace{1cm} (2.48)

The above six-fold integral may be converted to a simpler double integral by a rearrangement of the integrand,

\[
Y_{lm}^b = \int_{-\infty}^{\infty} Q_{HMxx} (k_x, k_y) \left[ \int_{x_0} \int_{y_0 \, \bar{s}_1} M_m (x_0, y_0) e^{jk_x x_0 - jk_y y_0} \, dx_0 \, dy_0 \right] \times
\]

\[ \left[ \int_{x} \int_{y \, \bar{s}_1} M_l (x, y) e^{jk_x x + jk_y y} \, dx \, dy \right] \, dk_x \, dk_y \]  \hspace{1cm} (2.49)

The first term in square brackets is the Fourier transform of \( M_m (x_0, y_0) \).
The second pair of square brackets contains the complex conjugate Fourier transform of \( M_l (x, y) \). Thus,

\[
\int_{x_0}^{y_0} \int_{S_1} M_m (x_0, y_0) e^{j k_x x_0 - j k_y y_0} dx_0 dy_0 = F \left[ M_m (x_0, y_0) \right]
\]

\[
= F_{M_m}(k_x, k_y)
\]  (2.50)

where \( F \left[ \cdot \right] \) denotes the Fourier transform operation. Since, \( M_m (x_0, y_0) \) can be written as:

\[
M_m (x_0, y_0) = M_m (x_0) \ M_m (y_0)
\]  (2.51)

The equation (2.50) becomes

\[
F \left[ M_m (x_0, y_0) \right] = \left\{ \int_{x_0} M_m (x_0) e^{-j k_x x_0} dx_0 \right\} \left\{ \int_{y_0} M_m (y_0) e^{-j k_y y_0} dy_0 \right\}
\]

\[
= F_{M_m}(k_x) \ F_{M_m}(k_y)
\]  (2.52)

where

\[
F_{M_m}(k_x) = \int_{x_0} M_m (x_0) e^{-j k_x x_0} dx_0
\]  (2.53)

\[
F_{M_m}(k_y) = \int_{y_0} M_m (y_0) e^{-j k_y y_0} dy_0
\]  (2.54)

In a similar manner,
\[
\int \int_{S_1} M_i(x, y) \ e^{j k_x x - j k_y y} \ dx \ dy = F^* \left[ M_i(x, y) \right]
\]

\[
= F^*_{M_i}(k_x) \ F^*_{M_i}(k_y) \quad (2.55)
\]

where * denotes the complex conjugate quantity.

Substituting (2.52) and (2.53) in (2.49), we obtain

\[
Y_{im}^b = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q_{HMx} (k_x, k_y) \ F_{M_m}(k_x) \ F_{M_m}(k_y) \ F^*_{M^*_i}(k_x) \ F^*_{M^*_i}(k_y) \ dk_x \ dk_y \quad (2.56)
\]

Since the Fourier transforms in this expression can be evaluated analytically, only the double integral needs to be evaluated numerically. The required Fourier transforms are obtained as:

\[
F_{M_m}(k_x) = \frac{2 \ m \ \pi \ L_s}{m^2 \ \pi^2 - k_x^2 \ L_s^2} \begin{cases} \cos \left( k_x L_s / 2 \right); & m \ \text{odd} \\ j \sin \left( k_x L_s / 2 \right); & m \ \text{even} \end{cases} \quad (2.57)
\]

\[
F_{M_m}(k_y) = \frac{2}{k_y} \sin \left( k_y W_s / 2 \right) \quad (2.58)
\]

\[
F^*_{M_i}(k_x) = \frac{2 \ i \ \pi \ L_s}{i^2 \ \pi^2 - k_x^2 \ L_s^2} \begin{cases} \cos \left( k_x L_s / 2 \right); & m \ \text{odd} \\ -j \sin \left( k_x L_s / 2 \right); & m \ \text{even} \end{cases} \quad (2.59)
\]
Problem Formulation

\[ F_{M_1}(k_y) = \frac{2}{k_y} \sin \left( k_y \frac{W_s}{2} \right) \]  

(2.60)

It may be noted that the conjugate transform may be obtained by substituting \( j \) by \( -j \), wherever it occurs in the expressions. Now, by substituting (2.57) through (2.60) into (2.56), we obtain,

\[
Y_{im}^b = \iint_{-\infty}^{\infty} Q_{HMXx}(k_x, k_y) \frac{16 i m \pi^2 L_s^2 \sin^2 (k_y \frac{W_s}{2})}{k_y^2 \left( m^2 \pi^2 - k_x^2 L_s^2 \right) \left( i^2 \pi^2 - k_y^2 L_s^2 \right)} \times \\
\begin{cases}
\cos^2 \left( k_x \frac{L_s}{2} \right); & m, i \text{ odd} \\
- \frac{j}{2} \sin \left( k_x L_s \right); & m \text{ odd}, i \text{ even} \\
\frac{j}{2} \sin \left( k_x L_s \right); & m \text{ even}, i \text{ odd} \\
\sin^2 \left( k_x \frac{L_s}{2} \right); & m, i \text{ even}
\end{cases} \, dk_x \, dk_y
\]

(2.61)

This expression involves a doubly-infinite integration. This may be simplified by a transformation to polar coordinates. We employ the substitution \( k_x = \beta \cos \alpha \), \( k_y = \beta \sin \alpha \), and the equation (2.61) may be put in the form:

\[
Y_{im}^b = \int_{0}^{2\pi} \int_{0}^{\infty} Q_{HMXx} (\alpha, \beta) \ F1 (\alpha, \beta) \times \\
\begin{cases}
\cos^2 \left( \beta \cos \alpha \frac{L_s}{2} \right); & m, i \text{ odd} \\
- \frac{j}{2} \sin \left( \beta \cos \alpha L_s \right); & m \text{ odd}, i \text{ even} \\
\frac{j}{2} \sin \left( \beta \cos \alpha L_s \right); & m \text{ even}, i \text{ odd} \\
\sin^2 \left( \beta \cos \alpha \frac{L_s}{2} \right); & m, i \text{ even}
\end{cases} \, \beta \, d\beta \, d\alpha
\]

(2.62)
where

\[ Q_{H_{Mxx}} (\alpha, \beta) = \frac{-j}{4 \pi^2 k_0 Z_0} \frac{1}{k_1 T_e T_m} \left[ j \beta^2 \cos^2 \alpha \ k_1 (\varepsilon_r - 1) + \right. \]

\[ (\varepsilon_r k_0^2 - \beta^2 \cos^2 \alpha \ k_1) \{ \ k_1 k_2 (\varepsilon_r + 1) \sin(k_1 d) \cos(k_1 d) + j (\varepsilon_r k_2^2 \times \]

\[ \sin^2 (k_1 d) - k_1^2 \cos^2 (k_1 d) \} \right] \]  (2.63)

\[ F_1 (\alpha, \beta) = \frac{16 i m \pi^2 L_s^2 \sin^2 (\beta \sin \alpha \ W_s/2 \ )}{\beta^2 \sin^2 \alpha \ (m^2 \pi^2 - \beta^2 \cos^2 \alpha \ L_s^2 \ ) (i^2 \pi^2 - \beta^2 \cos^2 \alpha \ L_s^2 \ )} \]  (2.64)

By making the substitution \( \theta = \alpha - \pi \) in (2.63), the \( \alpha \)-limits may be changed to \( -\pi \) to \( \pi \). The even and odd properties of the integrand may be utilized to reduce equation (2.63) to the following form:

\[ Y_{im}^b = 4 \int_0^\infty \int_0^{\pi/2} \frac{Q_{H_{Mxx}} (\alpha, \beta) F_1 (\alpha, \beta)}{} \begin{cases} \cos^2 \left( \beta \cos \alpha \frac{L_s}{2} \right); & m, \ i \ odd \\ \sin^2 \left( \beta \cos \alpha \frac{L_s}{2} \right); & m, \ i \ even \end{cases} \]

\[ \beta \ d\beta \ d\alpha \]  (2.65)
2.6 Evaluation of $[ T^b ]$

From equation (2.18), an element of $[ T^b ]$ is given by:

$$[ T^b_{n\ell} ] = [ \langle \mathcal{W}_n , -\mathcal{H}_t^b (\bar{J}_t) \rangle ]$$

(2.66)

The required Green’s function is the $\hat{x}$–directed magnetic field over the aperture due to an infinitesimal $\hat{y}$–directed electric current dipole over the patch. This is given by [45]:

$$G_{HJxy}(x,y,0| x_0, y_0, d) = \int_{-\infty}^{\infty} Q_{HJxy}(k_x, k_y) e^{j k_x (x-x_0)} e^{j k_y (y-y_0)} \, dk_x dk_y$$

(2.67)

where

$$Q_{HJxy}(k_x, k_y) = \frac{1}{4 \pi^2} \frac{-\epsilon_r k_1 k_2 \cos(k_1 d) + j \{ k_2^2 (\epsilon_r - 1) - k_1^2 \} \sin(k_1 d)}{\epsilon T_m}$$

(2.68)

The matrix element may be written in a manner similar to that adopted for (2.56) as:

$$T^b_{n\ell} = \int_{-\infty}^{\infty} Q_{HJxy}(k_x, k_y) F_{j_1}(k_x) F_{j_1}(k_y) F_{M_n}^*(k_x) F_{M_n}^*(k_y) \, dk_x \, dk_y$$

(2.69)

The two conjugate transforms are given by the equations (2.59) and (2.60). The transforms of the patch electric current are:

$$F_{j_1}(k_x) = \frac{2}{k_x} e^{-j k_x x_p} \sin \left( k_x L_p / 2 \right)$$

(2.70)
Problem Formulation

\[ F_{1l}(k_y) = \frac{2l \pi W_p e^{-j k_y y_p}}{m^2 \pi^2 - k_x^2 L_s^2} \begin{cases} \cos \left( k_y \frac{W_p}{2} \right); & l \text{ odd} \\ j \sin \left( k_y \frac{W_p}{2} \right); & l \text{ even} \end{cases} \quad (2.71) \]

The exponential terms in these expressions arise due to the patch centre being offset with respect to the centre of the slot. Thus equation (2.69) will contain a term \( e^{-j (k_x x_p + k_y y_p)} \) that may be expanded as follows.

\[
e^{-j (k_x x_p + k_y y_p)} = e^{-j \beta (x_p \cos \alpha + y_p \sin \alpha)}
\]

\[
= \cos (\beta x_p \cos \alpha + \beta y_p \sin \alpha) - j \sin (\beta x_p \cos \alpha + \beta y_p \sin \alpha)
\]

\[
= \cos (\beta x_p \cos \alpha) \cos (\beta y_p \sin \alpha) - \sin (\beta x_p \cos \alpha) \sin (\beta y_p \sin \alpha)
- j [\sin (\beta x_p \cos \alpha) \cos (\beta y_p \sin \alpha) + \cos (\beta x_p \cos \alpha) \sin (\beta y_p \sin \alpha)]
\]

\( (2.72) \)

By using the even and odd property of the integrand and by manipulations similar to those performed in the evaluation of \( [Y^b] \), the final expression may be written as:

\[
T_{nl}^b = -4 \int_{\beta=0}^{\infty} \int_{\alpha=0}^{\pi/2} Q_{Hjxy} (\alpha, \beta) F_2 (\alpha, \beta) \times
\]

\[
\begin{cases}
\cos \{\beta(x_p \cos \alpha + y_p \sin \alpha)\} \cos(\beta \cos \alpha \ L_s/2) \cos(\beta \sin \alpha \ W_p/2); \ l, n \ odd \\
- \sin \{\beta(x_p \cos \alpha + y_p \sin \alpha)\} \sin(\beta \cos \alpha \ L_s/2) \cos(\beta \sin \alpha \ W_p/2); \ l \ odd, n \ even \\
\sin \{\beta(x_p \cos \alpha + y_p \sin \alpha)\} \cos(\beta \cos \alpha \ L_s/2) \sin(\beta \sin \alpha \ W_p/2); \ l \ even, n \ odd \\
\cos \{\beta(x_p \cos \alpha + y_p \sin \alpha)\} \sin(\beta \cos \alpha \ L_s/2) \sin(\beta \sin \alpha \ W_p/2); \ l, n \ even
\end{cases}
\]

\[
\beta \ d\beta \ d\alpha \quad (2.73)
\]

54
where

\[
Q_{Hxy}(\alpha, \beta) = \frac{1}{4\pi^2} \frac{-\varepsilon k_1 k_2 \cos(k_1 d) + j \{\beta^2 \sin^2 \alpha (\varepsilon - 1) - k_1^2\} \sin(k_1 d)}{T_e T_m}
\]

\tag{2.74}

\[
F2(\alpha, \beta) = \frac{16 \pi l \pi^2 L_s W_p \sin\left(\beta \cos\alpha \frac{L_p}{2}\right) \sin\left(\beta \sin\alpha \frac{W_s}{2}\right)}{\beta^2 \sin\alpha \cos\alpha \left(l^2 \pi^2 - \beta^2 \sin^2 \alpha W_p^2\right) \left(n^2 \pi^2 - \beta^2 \cos^2 \alpha L_s^2\right)}
\]

\tag{2.75}

### 2.7 Evaluation of \([ C^b ]\)

From (2.19), an element of \([ C^b ]\) may be written as:

\[
[C^b_{rm}] = \langle \mathbf{J}_r, -\mathbf{E}^b_t (\mathbf{M}_m) \rangle = -\iiint_{S_2} \mathbf{J}_r \cdot \mathbf{E}^b_t (\mathbf{M}_m) \, ds_0 \tag{2.76}
\]

and can be evaluated in a manner similar to that adopted for \(T^b_{mi}\). However, it is not necessary to compute \([ C^b ]\) directly. It can be shown with the help of the reciprocity theorem that

\[
[C^b] = -[T^b]^T \tag{2.77}
\]

where the superscript ‘T’ denotes the transpose operation upon the matrix.

In its most general form, the reciprocity theorem may be stated as:

\[
\iiint_S \left( \mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1 \right) ds = \iiint_V \left( \mathbf{H}_1 \cdot \mathbf{M}_2 - \mathbf{E}_1 \cdot \mathbf{J}_2 - \mathbf{H}_2 \cdot \mathbf{M}_1 + \mathbf{E}_2 \cdot \mathbf{J}_1 \right) dV \tag{2.78}
\]
where $\overrightarrow{E_1}, \overrightarrow{H_1}$ and $\overrightarrow{E_2}, \overrightarrow{H_2}$ denote the fields produced by the sources $\overrightarrow{J_1}, \overrightarrow{M_1}$ and $\overrightarrow{J_2}, \overrightarrow{M_2}$ respectively. S is an arbitrary surface enclosing the volume V. If the surface S encloses all the sources completely, the L.H.S. of (2.78) vanishes [46] and equation (2.78) reduces to:

$$\iint_V (\overrightarrow{H_1} \cdot \overrightarrow{M_2} - \overrightarrow{E_1} \cdot \overrightarrow{J_2} - \overrightarrow{H_2} \cdot \overrightarrow{M_1} + \overrightarrow{E_2} \cdot \overrightarrow{J_1}) \, dV = 0$$

(2.79)

For the present problem, this reduces to:

$$\iint_V \overrightarrow{E_2} \cdot \overrightarrow{J_1} \, dV = \iint_V \overrightarrow{H_1} \cdot \overrightarrow{M_2} \, dV$$

(2.80)

Since $\overrightarrow{J_1}$ exists only on the surface $S_2$ and is $\hat{y}$–directed, only the $\hat{y}$–directed component of $\overrightarrow{E_2}$ will survive in the dot product and the L.H.S. of equation (2.80) reduces to a surface integral taken over $S_2$. By a similar reasoning, the R.H. S. of (2.80) reduces to a surface integral over $S_1$. Thus, (2.80) reduces to

$$\iint_{S_2} \overrightarrow{J} \cdot \overrightarrow{E}^b \left( \overrightarrow{M}^b \right) \, ds_0 = - \iint_{S_1} \overrightarrow{M} \cdot \overrightarrow{H}^b \left( \overrightarrow{J} \right) \, ds_0$$

(2.81)

or

$$C^b_{rm} = - T^b_{mr}$$

(2.82)

which proves the desired result shown by equation (2.77.)
2.8 Evaluation of \([ Z ]\)

From (2.20), an element of \([ Z ]\) may be written as:

\[
[ Z_{st} ] = \left[ < \bar{J}_s , -\bar{E}_t (\vec{J}_t) > \right]
\] (2.83)

The Green’s function required for evaluating \(\bar{E}_t (\vec{J}_t)\) is the \(\hat{y}\)–directed electric field over the patch surface due to an infinitesimal \(\hat{y}\)–directed electric current dipole also over the patch, which is given by [45]:

\[
G_{EJJy}(x, y, d \mid x_0, y_0, d) = \int_{-\infty}^{\infty} Q_{EJJy}(k_x, k_y) e^{ik_x(x-x_0)} e^{ik_y(y-y_0)} dk_x dk_y
\] (2.84)

where

\[
Q_{EJJy}(k_x, k_y) = -\frac{j \epsilon_0}{4 \pi^2 k_0} (\epsilon_0^2 k_0^2 - k_y^2) k_2 \cos(k_1 d) + j (k_0^2 - k_y^2) k_1 \sin(k_1 d)
\]

\[
\times \sin(k_1 d)
\] (2.85)

Proceeding in the same manner as earlier, we can show that

\[
Z_{st} = 4 \int_{\beta=0}^{\pi/2} \int_{\alpha=0}^{\pi/2} Q_{EJJy}(\alpha, \beta) \ F_4(\alpha, \beta) \left\{ \begin{array}{ll}
\cos^2 \left( \beta \ \sin \alpha \ \frac{W_0}{2} \right) ; & s, l \ odd \\
0 & s + l \ odd \\
\sin^2 \left( \beta \ \sin \alpha \ \frac{W_0}{2} \right) ; & s, l \ even
\end{array} \right\} \ d\beta \ d\alpha
\] (2.86)
where

\[
Q_{Ejyy}(\alpha, \beta) = \frac{j Z_0}{4 \pi^2 k_0} \left( \varepsilon J_0^2 - \beta^2 \sin^2 \alpha \right) \frac{k_2 \cos( k_1 d ) + j (k_0^2 - \beta^2 \sin^2 \alpha) k_1 \sin(k_1 d)}{T_e T_m} \sin(k_1 d)
\]

(2.87)

\[
F_{4}(\alpha, \beta) = \frac{16 s l \pi^2 W_p^2 \sin^2 \left( \beta \cos \alpha \frac{L_p}{2} \right)}{\beta^2 \cos^2 \alpha \left( l^2 \pi^2 - \beta^2 \sin^2 \alpha W_p^2 \right) \left( s^2 \pi^2 - \beta^2 \sin^2 \alpha W_p^2 \right)}
\]

(2.88)

2.9 Evaluation of \([ I_i ]\)

From (2.23), an element of \([ I_i ]\) may be written as:

\[
I_i = \langle \overline{M}_i , - \overline{H}^\text{inc}_t \rangle
\]

(2.89)

Assuming dominant mode incidence in the waveguide, the e.m. fields in of the incident wave may be written as [44]:

\[
E^+_y = - j \omega \mu_0 \varphi \sin \left( \frac{\pi x_1}{a} \right) e^{- j \beta_g z_1} \hat{y}_1
\]

(2.90)

\[
H^+_y = j \beta_g \varphi \sin \left( \frac{\pi x_1}{a} \right) e^{- j \beta_g z_1} \hat{x}_1
\]

(2.91)

\[
H^+_z = \frac{\pi}{a} \varphi \cos \left( \frac{\pi x_1}{a} \right) e^{- j \beta_g z_1} \hat{z}_1
\]

(2.92)

where \( \varphi = j \sqrt{2} \left( k_0 Z_0 \beta_g a b \right)^{-1/2} \)

(2.93)
\[ \beta_g = \sqrt{\omega^2 \mu \varepsilon - \left(\frac{\pi}{a}\right)^2} \]  

(2.94)

Since \( \overline{M}_1 \) is \( \hat{z}_1 \)-directed, only (2.92) needs to be substituted in equation (2.89). After performing direct integration, the result obtained is:

\[ l_i = -\varphi \left[ \sin \left\{ \frac{\pi}{a} \left( x_s + \frac{W_s}{2} \right) \right\} - \sin \left\{ \frac{\pi}{a} \left( x_s - \frac{W_s}{2} \right) \right\} \right] \times \]

\[ \frac{i \pi}{L_s} \left( \frac{i \pi}{L_s} \right)^2 - \beta_g^2 \]

\[ \begin{align*}
2 \cos \left( \beta_g \frac{L_s}{2} \right) ; & \text{ i odd} \\
2 j \sin \left( \beta_g \frac{L_s}{2} \right) ; & \text{ i even}
\end{align*} \]

(2.95)

### 2.10 Antenna Input Characteristics

The input characteristics of the antenna can be studied with the help of its scattering matrix. For deriving the s-parameters, a pair of reference planes must be selected. These are chosen as \( T_1 \) at \( z_1 = -\lambda_g \) and \( T_2 \) at \( z_1 = \lambda_g \). With this choice, the scattering matrix for these planes will be identical to that for the terminal planes \( z_1 = 0^- \) and \( z_1 = 0^+ \) because

\[ e^{\pm j \beta_g z_{T_1}} = e^{\pm j \beta_g z_{T_1}} = 1 \]

(2.96)

The field incident at \( T_1 \) is the impressed field while the outgoing field at \( T_1 \) is the field due to the current \( \overline{M} \). The outgoing field at \( T_2 \) is the sum of the incident field at \( T_2 \) and the field due to the current \( \overline{M} \). Therefore \( s_{11} \) may be written as:
The negative sign arises as the modal vector for the transverse component of $\overline{H}$ changes sign for a wave travelling in the $-\hat{z}$-direction.

The parameter $s_{21}$ is the ratio of the field in the outgoing wave at T2 to that incident at T1, that is:

$$s_{21} = \frac{\overline{H}_t (\overline{M}) \big|_{T_2} + \overline{H}_{t \text{ inc}} \big|_{T_2}}{\overline{H}_{t \text{ inc}} \big|_{T_1}}$$  \hspace{1cm} (2.98)

Since the amplitude of the incident fields at T1 and T2 are the same, we may write:

$$s_{21} = 1 + \frac{\overline{H}_t (\overline{M}) \big|_{T_2}}{\overline{H}_{t \text{ inc}} \big|_{T_1}}$$  \hspace{1cm} (2.99)

Substituting (2.8) in the above equations and by using the linearity of the operators, we obtain:

$$\overline{H}_t (\overline{M}) = \overline{H}_t \left\{ \sum_{s=1}^{N_\text{a}} V_s \overline{M}_s \right\} = \sum_{s=1}^{N_\text{a}} V_s \overline{H}_t (\overline{M}_s)$$  \hspace{1cm} (2.100)

The expression for $\overline{H}_t (\overline{M}_s)$ is obtained by using the relations (2.29), (2.32) and (2.33). The modulus term in (2.29) is handled by considering it separately at T1 and T2:

$$e^{-\Gamma_{nm}|z_1-z'_1|} = \begin{cases} e^{-\Gamma_{nm}(z_1-z'_1)} ; & \text{if } z_1 > z'_1 \text{ (at plane T1)} \\ e^{-\Gamma_{nm}(z'_1-z_1)} ; & \text{if } z_1 < z'_1 \text{ (at plane T2)} \end{cases}$$  \hspace{1cm} (2.101)
The integral occurring in (2.32) is evaluated separately at T1 and T2 due to the exponential term given in (2.101). The incident field component to be used is given by (2.91) because it is the component transverse to the waveguide axis and is, therefore, responsible for carrying power. The following final expressions are obtained for the s-parameters.

\[
s_{11} = \frac{-1}{j\sqrt{2\omega}\mu_0\beta_g a b} \left[ \sin \left\{ \frac{\pi}{a} \left( x_s + \frac{W_s}{2} \right) \right\} - \sin \left\{ \frac{\pi}{a} \left( x_s - \frac{W_s}{2} \right) \right\} \right] \times
\]

\[
\sum_{s=1}^{N^a} V_s \frac{s \pi}{L_s} \left\{ \begin{array}{ll}
2 \cos \left( \beta_g \frac{L_s}{2} \right) &; s \text{ odd} \\
2j \sin \left( \beta_g \frac{L_s}{2} \right) &; s \text{ even}
\end{array} \right.
\]

\[
s_{21} = 1 - \frac{1}{j\sqrt{2\omega}\mu_0\beta_g a b} \left[ \sin \left\{ \frac{\pi}{a} \left( x_s + \frac{W_s}{2} \right) \right\} - \sin \left\{ \frac{\pi}{a} \left( x_s - \frac{W_s}{2} \right) \right\} \right] \times
\]

\[
\sum_{s=1}^{N^a} V_s \frac{s \pi}{L_s} \left\{ \begin{array}{ll}
2 \cos \left( \beta_g \frac{L_s}{2} \right) &; s \text{ odd} \\
-2j \sin \left( \beta_g \frac{L_s}{2} \right) &; s \text{ even}
\end{array} \right.
\]

In the summation for \( s_{21} \), the terms are identical to the corresponding terms of \( s_{11} \) except that a sign reversal occurs whenever \( s \) is even.

The fraction of power coupled from the waveguide is given by:

\[
P_{\text{out}} = 1 - |s_{11}|^2 - |s_{21}|^2 \]
Problem Formulation

The input impedance is obtained from the s-parameters depending on whether the slot behaves like a series impedance or a shunt impedance. A longitudinal slot normally exhibits the latter behavior. This may also be confirmed later after the s-parameters have been computed by checking that

\[ s_{21} \approx 1 + s_{11} \]  

(2.105)

Consequently, the input impedance and VSWR are given by

\[ Z_{in} = \frac{1 + s_{11}}{1 - s_{11}} \]  

(2.106)

\[ VSWR = \frac{1 + |s_{11}|}{1 - |s_{11}|} \]  

(2.107)

2.11 Antenna Far-Field Characteristics

The far-field characteristics of the proposed microstrip patch antenna may be determined using the coefficients obtained from the system of equations represented by the method-of-moments matrix. The problem is essentially the radiation from a waveguide aperture into free space with (or without) a rectangular plate (patch) placed in its near-field. As a result, the particular case of a air-dielectric patch antenna is considered in this work.

The geometry for the computation of the component \( E_m \) of the electric field at a far-field point, \( \vec{r}_m \) in the region \( z > 0 \) is shown in Fig. 2.5. To carry out this measurement, an elementary electric dipole \( I \delta (\vec{r} - \vec{r}_m) \) is placed at the location of \( \vec{r}_m \) and the reciprocity theorem is invoked between its field and that of the actual antenna.
Fig. 2.5: Illustrating the Geometry of the Measurement Vectors for the Proposed Antenna
For this purpose, one source is the dipole at \( \bar{r}_m \) and one of the following two sources taken in turn: the electric current \( \mathbf{j} \) over the patch and the magnetic current \(-\mathbf{M}\) across the surface of the slot, both radiating in the presence of the (infinite) ground plane placed at \( z = 0 \). The image theory may be applied to remove the ground plane and the two sets of sources may be considered as

a) The electric dipole at \( \bar{r}_m \); and

b) The magnetic current \(-2\mathbf{M}\) over the slot, the electric current \( \mathbf{j} \) over the patch and the electric current \(-\mathbf{j}\) over the image of the patch (at \( z = -d \)).

Using the reciprocity theorem, the component of the electric field in the direction of \( \hat{e}_1 \) at \( \bar{r}_m \) due to the second set of sources may be expressed as:

\[
I_l m E_m = 2 \iint_{\text{slot}} \mathbf{M} \cdot \mathbf{H}^m \, ds + \iint_{\text{patch}} \mathbf{j} \cdot \mathbf{E}^m_1 \, ds - \iint_{\text{image}} \mathbf{j} \cdot \mathbf{E}^m_2 \, ds \quad (2.108)
\]

where \( \mathbf{E}^m_1 \) and \( \mathbf{E}^m_2 \) are the electric fields at the patch and its image respectively produced by the dipole \( \bar{r}_m \) while \( E_m \) is the component of \( \mathbf{E} \) in the direction of the dipole. As the patch is parallel to the X-Y plane, (2.108) reduces to

\[
I_l m E_m = 2 \iint_{\text{slot}} \mathbf{M} \cdot \mathbf{H}^m \, ds + \iint_{\text{patch}} \mathbf{j} \cdot (\mathbf{E}^m_1 - \mathbf{E}^m_2) \, ds \quad (2.109)
\]

Substituting from (2.8) and (2.9) into (2.109), we obtain

\[
I_l m E_m = \sum_l I_l \iint_{\text{patch}} \mathbf{j}_l \cdot (\mathbf{E}^m_1 - \mathbf{E}^m_2) \, ds + 2 \sum_m V_m \iint_{\text{slot}} \mathbf{M} \cdot \mathbf{H}^m \, ds \quad (2.110)
\]
Since the scalar products occurring in the integrands of (2.110) only involve the tangential components of the fields, we may rewrite this equation as

\[ I_l E_m = \sum_i l_i < \vec{J}_l , \left( \overline{E_{1t}^m} - \overline{E_{2t}^m} \right) > + 2 \sum_m V_m < \overline{M_m} , \overline{H_t}^m > \quad (2.111) \]

or

\[ I_l E_m = \begin{bmatrix} \vec{p}^{m1} \\ \vec{p}^{m2} \end{bmatrix} \begin{bmatrix} \vec{I} \\ \vec{V} \end{bmatrix} \quad (2.112) \]

where

\[ \vec{p}^{m1} = \left[ < \vec{J}_l , \left( \overline{E_{1t}^m} - \overline{E_{2t}^m} \right) > \right] \quad (2.113) \]

\[ \vec{p}^{m2} = \left[ 2 < \overline{M_m} , \overline{H_t}^m > \right] \quad (2.114) \]

To obtain a component \( \overline{E_m} \) on the radiation sphere, the dipole \( I_l \) is oriented perpendicular to \( \vec{r}_m \) and we allow \( \vec{r}_m \) to tend to infinity. Simultaneously, \( I_l \) is adjusted so that it produces a unit plane wave in the vicinity of the origin. The required dipole moment is given by:

\[ \frac{1}{I_l} = -\frac{j \omega \mu}{4 \pi r_m} e^{-j k_0 r_m} \quad (2.115) \]

and the plane wave produced in the vicinity of the origin is given by:

\[ \overline{E}^m = \hat{u}_m e^{-j \vec{k}_m \cdot \vec{r}} \quad (2.116) \]

\[ \overline{H}^m = \frac{1}{\eta} \left( \hat{k}_m \times \hat{u}_m \right) e^{-j \vec{k}_m \cdot \vec{r}} \quad (2.117) \]
where \( \hat{u}_m \) is the unit vector specifying the polarization of the wave, \( \vec{k}_m \) is the wave number vector pointing in the direction of wave propagation and \( \vec{r} \) is the position vector of any general point on the slot or patch. Substituting (2.115) into (2.112), we obtain

\[
E_m = -\frac{j \omega \mu_0}{4 \pi r_m} e^{-j k_0 r_m} \begin{bmatrix} \hat{p} m^1 \\ \hat{p} m^2 \end{bmatrix} \begin{bmatrix} \vec{i} \\ \vec{v} \end{bmatrix}
\]

(2.118)

\[
= -\frac{j k_0 \eta}{4 \pi r_m} e^{-j k_0 r_m} \begin{bmatrix} \hat{p} m^1 \\ \hat{p} m^2 \end{bmatrix} \begin{bmatrix} \vec{i} \\ \vec{v} \end{bmatrix}
\]

(2.119)

Once the measurement vectors \( \hat{p} m^1 \) and \( \hat{p} m^2 \) are determined, the far-field electric field can be obtained using (2.119)

### 2.11.1 Determination of Measurement Vectors

To determine \( \hat{p} m^1 \) given as in equation (2.113), first we determine \( \vec{E}_{1t}^{-m} \) and \( \vec{E}_{2t}^{-m} \). \( \vec{E}_{1t}^{-m} \) is the tangential component of the electric field at any point on the patch. The position vector of any point on the patch is given by:

\[
\vec{r} = x \hat{x} + y \hat{y} + z \hat{z}
\]

(2.120)

For the geometry illustrated in Fig. 2.5,

\[
\vec{k}_m = -k_0 \hat{r}_m = -k_0 \left[ \cos \theta \hat{x} + \sin \theta \cos \varphi \hat{y} + \sin \theta \sin \varphi \hat{z} \right]
\]

(2.121)

Using (2.120) and (2.121), we may write
\[ \vec{k}_m \cdot \vec{r} = -k_0 \left[ x \cos \theta + y \sin \theta \cos \varphi + d \sin \theta \sin \varphi \right] \quad (2.122) \]

Similarly, \( \vec{E}_{1t}^{-m} \) is the tangential component of the electric field at any point on the image of the patch. The position vector at such a point anywhere on the image of the patch is given by

\[ \vec{r} = x \hat{x} + y \hat{y} - d \hat{z} \quad (2.123) \]

Using (2.121) and (2.123), we may write

\[ \vec{k}_m \cdot \vec{r} = -k_0 \left[ x \cos \theta + y \sin \theta \cos \varphi - d \sin \theta \sin \varphi \right] \quad (2.124) \]

Let \( (P_{lm}^1)^y_{qx} \) be the measurement vector due to the y-component of the current across the patch for a \( \varphi \)-polarized wave in the \( x = 0 \) plane (\( \theta = 90^\circ \)), which may be written as:

\[ (P_{lm}^1)^y_{qx} = \iint_{\text{patch}} \gamma_y \hat{\phi} \left\{ e^{j k_0 (y \cos \varphi + d \sin \varphi)} - e^{j k_0 (y \cos \varphi - d \sin \varphi)} \right\} dx \, dy \quad (2.125) \]

This can be directly integrated to obtain

\[ (P_{lm}^1)^y_{qx} = 2 j \sin (k_0 d \sin \varphi) \frac{L_p \left( \frac{i \pi}{W_p} \right)}{-k_0^2 \cos^2 \varphi + \left( \frac{i \pi}{W_p} \right)^2} \times \]

\[ e^{j k_0 y_p \cos \varphi} \left\{ \begin{array}{ll}
2 \cos \left( k_0 \frac{W_p}{2} \cos \varphi \right) ; & l \text{ odd} \\
-2j \sin \left( k_0 \frac{W_p}{2} \cos \varphi \right) ; & l \text{ even}
\end{array} \right. \quad (2.126) \]
Let \((P_l^{m1})_{yy}^y\) be the measurement vector due to the \(y\)-directed patch current for a \(y\)-polarized wave in the \(y = 0\) plane \((\varphi = 90^\circ)\). This may be written as:

\[
(P_l^{m1})_{yy}^y = \int \int_{\text{patch}} \nabla \cdot \mathbf{E} \{ e^{jk_0(x \cos \theta + d \sin \theta)} - e^{jk_0(y \cos \theta - d \sin \theta)} \} \, dx \, dy \tag{2.127}
\]

Upon integration, this yields

\[
(P_l^{m1})_{yy}^y = 8j \frac{e^{jk_0x_p \cos \theta} \sin(k_0d \sin \theta) \sin \left( k_0 \frac{L_p}{2} \cos \theta \right)}{k_0 \cos \theta \left( \frac{\pi}{W_p} \right)}; \ l \ odd \tag{2.128}
\]

\[
= 0; \quad l \ even
\]

Now to obtain \(\vec{P}^{m2}\) (as per Equation 2.112), we need to first determine \(\vec{H}^{m}_t\), which is the tangential component of the magnetic field at any point \(\vec{r}\) on the slot. For this case, the positions vector \(\vec{r}\) is given by:

\[
\vec{r} = x \hat{x} + y \hat{y}\tag{2.129}
\]

Let \((P_m^{m2})_{xy}\) be the measurement vector due to the \(x\)-component of the current across the coupling aperture for a \(y\)-polarized wave in the \(y = 0\) plane \((\varphi = 90^\circ)\). The \(y\)-polarized wave is given by equation (2.117) with \(\hat{u}_m = \hat{y}\). Using equations (2.117), (2.121) and (2.129) we may write

\[
\hat{k}_m \times \hat{u}_m = - \cos \theta \hat{z} + \sin \theta \hat{x}\tag{2.130}
\]

\[
\vec{k}_m \cdot \vec{r} = -k_0 \left[ x \cos \theta + y \sin \theta \right]\tag{2.131}
\]
Thus the measurement vector may be written as

\[ (P^m_{z}^{2})_{yy}^x = \frac{2 \sin \theta}{\eta} \iint_{\text{slot}} \vec{M}_m \cdot \hat{x} \ e^{jk_0x \cos \theta} \ dx \ dy \]  

(2.132)

This may be integrated readily to obtain

\[ (P^m_{z}^{2})_{yy}^x = \frac{2 \sin \theta \ W_s}{\eta} \left( \frac{m \pi / l_s}{-k_0^2 \cos^2 \varphi + \left( \frac{m \pi}{l_s} \right)^2} \right) \times \]

\[ \begin{cases} 
2 \ \cos \left( \frac{k_0 l_s}{2} \cos \varphi \right) ; & m \text{ odd} \\
-2j \ \sin \left( \frac{k_0 l_s}{2} \cos \varphi \right) ; & m \text{ even} 
\end{cases} \]  

(2.133)

Now let \((P^m_{z}^{2})_{yx}^x\) be the measurement vector due to the x-component of the current across the coupling slot for a \(\varphi\)-polarized wave in the \(x = 0\) plane (\(\theta = 90^\circ\)). The \(\varphi\)-polarized wave is given by (2.117) with \(\hat{u}_m = \hat{\varphi}\). Thus, we may write

\[ (P^m_{z}^{2})_{yx}^x = \frac{2}{\eta} \iint_{\text{slot}} \vec{M}_m \cdot \hat{\varphi} \ e^{jk_0y \cos \varphi} \ dx \ dy \]  

(2.134)

Upon carrying out integration, we obtain

\[ (P^m_{z}^{2})_{yx}^x \begin{cases} 
= \frac{8}{\eta} \ \frac{\sin \left( \frac{k_0 W_s}{2} \cos \varphi \right)}{k_0 \cos \varphi \left( \frac{m \pi}{l_s} \right)} ; & m \text{ odd} \\
= 0 ; & m \text{ even} 
\end{cases} \]  

(2.135)
All components of the radiation from the slot as well as the aperture are now available. By substituting (2.126), (2.128), (2.133) and (2.135) into (2.119), the radiation pattern of the proposed antenna may be completely obtained.

### 2.11.2 Determination of Pattern Gain

The directive gain, $G (\theta, \varphi)$, of a radiating structure in a given direction is the ratio of the radiation intensity in that direction to the average radiated intensity in all directions (for instance, by an isotropic radiator). Thus

$$G (\theta, \varphi) = \frac{\Psi(\theta, \varphi)}{\Psi_{av}} \quad (2.136)$$

If $P_t$ is the total complex power radiated, the average power radiated will be

$$\Psi_{av} = \frac{\text{Real} (P_t)}{4 \pi} \quad (2.137)$$

Also, the radiation intensity in a given direction is

$$\Psi(\theta, \varphi) = \frac{r_m^2 |\mathbf{E}_m|^2}{\eta} \quad (2.138)$$

where $\mathbf{E}_m$ is the electric field intensity at the measurement point and $r_m$ is the distance of this point from the origin. Substituting (2.137) & (2.138) in (2.136) and using (2.119), we obtain

$$G (\theta, \varphi) = -\frac{k^2_{\theta} \eta}{4 \pi \text{Real} (P_t)} \left[ \begin{array}{c} \mathbf{p}^{m1} \\ \mathbf{p}^{m2} \end{array} \right] \left[ \begin{array}{c} \mathbf{i} \\ \mathbf{v} \end{array} \right]^2$$

$$= -\frac{k^2_{\theta} \eta}{4 \pi \text{Real} (P_t)} \left[ \begin{array}{c} \mathbf{p}^{m1} \\ \mathbf{p}^{m2} \end{array} \right] \left[ \begin{array}{cc} \mathbf{i} \cdot \mathbf{i} \\ \mathbf{i} \cdot \mathbf{v} \end{array} \right]$$

$$= -\frac{k^2_{\theta} \eta}{4 \pi \text{Real} (P_t)} \left[ \begin{array}{c} \mathbf{p}^{m1} \\ \mathbf{p}^{m2} \end{array} \right] \left[ \begin{array}{cc} \mathbf{i} \\ \mathbf{v} \end{array} \right]^2$$

$$= -\frac{k^2_{\theta} \eta}{4 \pi \text{Real} (P_t)} \left[ \begin{array}{c} \mathbf{p}^{m1} \\ \mathbf{p}^{m2} \end{array} \right] \left[ \begin{array}{cc} \mathbf{i} \\ \mathbf{v} \end{array} \right]^2$$

$$= -\frac{k^2_{\theta} \eta}{4 \pi \text{Real} (P_t)} \left[ \begin{array}{c} \mathbf{p}^{m1} \\ \mathbf{p}^{m2} \end{array} \right] \left[ \begin{array}{cc} \mathbf{i} \\ \mathbf{v} \end{array} \right]^2$$

$$= -\frac{k^2_{\theta} \eta}{4 \pi \text{Real} (P_t)} \left[ \begin{array}{c} \mathbf{p}^{m1} \\ \mathbf{p}^{m2} \end{array} \right] \left[ \begin{array}{cc} \mathbf{i} \\ \mathbf{v} \end{array} \right]^2$$
2.12 Summary

In this chapter, first the analysis techniques employed for microstrip antennas are briefly reviewed. Approximate models like the transmission-line and cavity-model are discussed that achieve a fair degree of accuracy using empirical data. More exact analysis is possible with computational electromagnetic (CEM) methods. High-frequency methods based on optics and diffraction are used for electrically-large structures. Full-wave methods, M-o-M and FEM are discussed followed by time-domain computational methods using the finite difference approach. Use of the Green’s function in M-o-M is discussed. Next, a generalized moment method formulation has been developed for an arbitrary waveguide cross-section, slot shape and patch. The problem geometry is decoupled into two separate but linked sub-problems by sealing the coupling aperture and using the equivalence principle. Boundary conditions for the slot and patch are invoked to obtain the required operator equations. Generic summation expressions for the basis and testing functions are introduced. Using the M-o-M approach, the operator equations are converted to a single matrix equation whose solution yields the desired unknown currents. Then the case of the presently proposed rectangular microstrip patch antenna fed by a rectangular waveguide through a rectangular broad-wall aperture is introduced. Entire-domain, sinusoidal basis functions are used to represent both slot magnetic current and patch electric current. Subsequently, expressions for the various matrix and vector terms involved in the moment method solution have been derived and analytically simplified. Integrations across the source regions are evaluated analytically as Fourier transforms. The slot-to-patch coupling term is not evaluated but obtained directly from the patch-to-slot term through a matrix sorting operation. Symmetry properties of the integrands are exploited to minimize computational
requirements. Integration in the polar argument is only performed in one quadrant. The excitation vector is obtained from the incident modal fields of the waveguide. Solving the matrix equation yields the values of the unknown coefficients thereby the unknown currents. In terms of these, the expressions for the antenna s-parameters and input characteristics have been also derived using a pair of reference-planes before and ahead of the slot. The computed M-o-M coefficients are substituted into these expressions to obtain the s-parameters. The expressions for computing the far-field quantities and the directive gain have also been derived for the specific case of the air-dielectric patch antenna. The reciprocity principle is used to determine these quantities from knowledge of the field incident on the antenna from an elementary dipole radiator of appropriate polarization located at the far-field observation point. These represent the basic measurement vectors, which when matrix-multiplied to the determined M-o-M coefficients, yield the net radiation pattern of the antenna.