Chapter-IV

n-c-PURE EXACT SEQUENCES

(A Generalization of Cyclic Purity)

INTRODUCTION

In this chapter, we introduce and study the concept of n-c-purity, a generalization of cyclic purity. Actually, this is a special type of $\varphi$-purity, where $\varphi$ is the family of all cyclic $R$-modules of projective dimension $\leq n$.

In Section 4.1, we introduce the concept of n-c-purity and study properties relative to this. We compare this concept with purity. We show that in the case of commutative integral domains, the concept of 1-c-purity is actually the generalization of purity. Also we prove that for Prüfer domains 1-c-purity implies purity. We show that in case of commutative integral domains the concept of 1-c-purity will be weaker than purity and stronger than $RD$-purity.

In Section 4.2, we introduce the concept of absolutely n-c-purity and study its properties. As, absolutely c-purity is nothing but injectivity (see Theorem 3.1.3), this concept will become a generalization of injectivity.

In Section 4.3, we introduce and study the concept of n-c-flat. We show that in case of commutative integral domains, the concept of 1-c-flat and torsion-free are same.
In the last Section 4.4, we study the concept of \( n \)-c-regularity.

### 4.1 \( n \)-CYCLIC PURE EXACT SEQUENCES

Let \( R \) be a ring with left global dimension at most \( m \), where \( m \) is any nonnegative integer. For any nonnegative integer \( n \leq m \), we define

**4.1.1 Definition** : A short exact sequence \( e : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \) of \( R \)-modules is said to be \( n \)-Cyclic Pure (in short, \( n \)-c-pure) if every cyclic \( R \)-module of projective dimension (in short, \( \text{p.dim} \)) \( < n \) is \( e \)-projective.

**4.1.2 Remark** :

i) Every short exact sequence of \( R \)-modules is 0-c-pure.

ii) Cyclic purity implies \( n \)-c-purity for every \( n \).

**4.1.3 Proposition** : Over a right Noetherian ring \( R \), every pure short exact sequence of \( R \)-modules is \( n \)-c-pure for every \( n \).

**Proof**: Follows from Proposition 2.2.4.

**4.1.4 Proposition** : Let \( N \) be a submodule of an \( R \)-module \( M \). Then for the canonical short exact sequence \( e : 0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0 \) and a positive integer \( n \leq m \), the following conditions are equivalent.

i) \( e \) is \( n \)-c-pure.

ii) For any \( a \in M \) and any right ideal \( I \) of \( R \) with \( \text{p.dim} \leq n - 1 \) and \( aI \subseteq N \), there exists \( b \in N \) such that \( (a - b)I = (0) \).

**Proof**: i) \( \Rightarrow \) ii) Let \( a \in M \) and \( I \) be any right ideal of \( R \) with \( \text{p.dim} \leq n - 1 \) and \( aI \subseteq N \). There exists \( b \in N \) such that \( (a - b)I = (0) \).
n-1 and aI ⊆ N. Then \( p \dim(R/I) \leq n \). Define a map \( f : R/I \rightarrow M/N \)
by \( f(\tilde{r}) = ar + N \). Since \( aI \subseteq N \), \( f \) is a well-defined \( R \)-homomorphism.

By hypothesis (i), there exists \( g \in \text{Hom}(R/I, M) \) such that \( \eta g = f \).

Now \( (\eta g)(\tilde{1}) = f(\tilde{1}) \). This implies that \( g(\tilde{1}) + N = a + N \) and hence
\( g(\tilde{1}) - a \in N \). Let \( b = a - g(\tilde{1}) \). Then, \( b \in N \) and \( (a - b)I = g(\tilde{1})I = (\bar{o}) \).

Hence the result.

\( \text{ii) } \Rightarrow \text{i) } \) Let \( R/I \) be any cyclic \( R \)-module of \( p \dim \leq n \) and let \( f \in \text{Hom}(R/I, M/N) \). Since \( p \dim(R/I) \leq n \), \( p \dim I \leq n - 1 \). Let \( f(\tilde{1}) = a + N \) for some \( a \in M \). Then \( aI + N = (a + N)I = f(\tilde{1})I = (\bar{o}) \). Hence
\( aI \subseteq N \). By hypothesis (ii), there exists \( b \in N \) such that \( (a - b)I = (o) \).

Now define \( g : R/I \rightarrow M \) by \( g(\tilde{r}) = (a - b)r \). Clearly, \( g \) is well-defined homomorphism and
\( (\eta g)(\tilde{r}) = \eta((a - b)r) = (a - b)r + N = ar + N = f(\tilde{r}) \)
for every \( \tilde{r} \in R/I \). Thus \( \eta g = f \) and hence \( R/I \) is \( \epsilon \)-projective. So \( \epsilon \) is
\( n \)-c-pure.

4.1.5 Remark : Clearly every \( n \)-c-pure short exact sequence is \( n' \)-c-pure
for every \( n' \leq n \). But if an exact sequence is \( n \)-c-pure, then it need not
be \( n'' \)-c-pure for \( n'' > n \).

Example: Let \( R \) be any Dedekind Domain which is not a field and \( I \)
be any nontrivial ideal of \( R \). Then the canonical short exact sequence
\( \epsilon : 0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0 \) is trivially 0-c-pure. Suppose \( \epsilon \) is 1-c-
pure. Since \( R \) is Dedekind Domain, \( g.dimR \leq 1 \) and hence \( p.dimR/I \leq 1 \).
So \( R/I \) is \( \epsilon \)-projective. Hence \( \epsilon \) splits. Therefore \( I \) is a direct summand of \( R \) which is impossible in a commutative integral domain. So \( \epsilon \) cannot be 1-c-pure.

4.1.6 Remark: A pure exact sequence need not be \( n \)-c-pure. In particular, it need not be 1-c-pure.

Example: Consider the example given after Remark 2.2.1. In that example \( R = \prod_{\alpha \in A} R_\alpha \) and \( S = \bigoplus_{\alpha \in A} R_\alpha \) where \( \{R_\alpha\}_{\alpha \in A} \) is a family of fields. Since each \( R_\alpha \) is projective \( S \) is also projective as an \( R \)-module. Hence \( R/S \) is of projective dimension \( \leq 1 \). If the canonical short exact sequence \( 0 \rightarrow S \rightarrow R \rightarrow R/S \rightarrow 0 \) is 1-c-pure, then \( R/S \) is projective with respect to it. Therefore \( S \) is a direct summand of \( R \). So \( S \) cannot be 1-c-pure in \( R \). Since \( R \) is von Neumann regular ring, it follows that \( S \) is pure in \( R \).

4.1.7 Proposition: Over a commutative integral domain \( R \), every pure exact sequence is 1-c-pure.

Proof: Let \( 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \) be a pure exact sequence of \( R \)-modules. Let \( M = R/I \) be a cyclic \( R \)-module of \( p.dim \leq 1 \). Since \( p.dim(R/I) \leq 1 \), \( I \) is projective. Since \( R \) is a commutative integral domain, \( I \) is invertible and hence is finitely generated. So, \( R/I \) is finitely
presented. Since $\epsilon$ is pure exact, $M = R/I$ is $\epsilon$-projective. Hence the result.

4.1.8 Proposition: Over a commutative integral domain $R$ every 1-c-pure exact sequence of $R$-modules is $RD$-pure.

Proof: Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be a 1-c-pure exact sequence of $R$-modules, where $R$ is a commutative integral domain. Let $r \neq 0 \in R$. Since $R$ is a commutative integral domain, $rR$ is projective and hence $p.dim(R/rR) \leq 1$. Hence $R/rR$ is $\epsilon$-projective. Now the theorem follows from an equivalent condition of Warfield [33, Proposition 2].

4.1.9 Proposition: Over a Prufer domain a short exact sequence of $R$-modules is pure if and only if it is 1-c-pure.

Proof: Only if: Follows by Proposition 4.1.7.

If: Follows by Proposition 4.1.8 and using the fact that over a Prufer domain purity and $RD$-purity are equivalent.

4.1.10 Proposition: A commutative integral domain $R$ is a Prufer domain if and only if every 1-c-pure submodule of an $R$-module $M$ is pure.

Proof: Only if: Follows from Proposition 4.1.8 and [33, Proposition 2].

If: Let $M$ be a torsion-free $R$-module. Let $M = F/K$ for some free $R$-module $F$ and $K$ a submodule of $F$. Then the canonical short exact sequence $0 \rightarrow K \rightarrow F \rightarrow F/K \rightarrow 0$ is $\epsilon$-pure by Proposition 2.1.8.
Hence it is 1-c-pure. By hypothesis, $K$ is pure in $F$ and hence $F/K$ is flat. By [20, Corollary 2.31], $R$ is a Prüfer domain.

4.1.11 Proposition: Suppose an $R$-module $M$ satisfies the following condition

(*) Every $R$-homomorphism from a right ideal $I$ of $R$ of $p.dim < (n - 1)$ into $M$, can be extended to $R$.

Then $n$-c-pure submodules of $M$ also satisfy the condition (*).

Proof: We first note that an $R$-module $M$ satisfies the condition (*) if and only if $\text{Ext}(R/I, M) = 0$ for every right ideal $I$ of $R$ with $p.dim \leq (n - 1)$. Let $A$ be an $n$-c-pure submodule of $M$. Then, the canonical short exact sequence $\epsilon: 0 \rightarrow A \rightarrow M \rightarrow M/A \rightarrow 0$ is $n$-c-pure. Let $I$ be a right ideal of $R$ with $p.dim \leq (n - 1)$. Then $p.dim(R/I) \leq n$. We have the following exact sequence, $0 \rightarrow \text{Hom}(R/I, A) \rightarrow \text{Hom}(R/I, M) \rightarrow \text{Hom}(R/I, M/A) \rightarrow \text{Ext}(R/I, A) \rightarrow \text{Ext}(R/I, M)$. Since $M$ satisfies (*), $\text{Ext}(R/I, M) = 0$ by the above observation. Since $\epsilon$ is $n$-c-pure, $f$ is surjective and hence $\text{Ext}(R/I, A) = 0$. Therefore $A$ satisfies (*).

4.1.12 Proposition: Let $R$ be a commutative integral domain. If every short exact sequence of $R$-modules is 1-c-pure, then $R$ is a field.

Proof: Let $I$ be a principal ideal of $R$. Since $R$ is an integral domain, $I$ is projective. Hence $p.dim R/I \leq 1$. By hypothesis, the canonical short
exact sequence \( \varepsilon : 0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0 \) is 1-c-pure. Hence \( R/I \) is \( \varepsilon \)-projective and so \( I \) is a direct summand of \( R \). Hence \( R \) is von Neumann regular ring. As, \( R \) is a commutative integral domain, it is a field.
4.2 n-ABSOLUTELY c-PURE MODULES

4.2.1 Definition: An $R$-module $M$ is said to be $n$-absolutely $c$-pure if every short exact sequence $0 \to M \to A \to B \to 0$ of $R$-modules is $n$-c-pure.

We note that, $M$ is $n$-absolutely $c$-pure if and only if it is $n$-c-pure submodule of its injective hull $E(M)$.

4.2.2 Proposition: An $R$-module $A$ is $n$-absolutely $c$-pure if and only if $\text{Ext}_R(N, A) = 0$ for every cyclic $R$-module $N$ of $p.\text{dim} \leq n$.

Proof: Let $A$ be an $R$-module. We have the natural short exact sequence $0 \to A \to E(A) \to E(A)/A \to 0$ where $E(A)$ is the injective hull of $A$. Let $N$ be a cyclic $R$-module of $p.\text{dim} \leq n$. Then we have the induced exact sequence $0 \to \text{Hom}(N, A) \to \text{Hom}(N, E(A)) \to \text{Ext}(N, A) \to \text{Ext}(N, E(A)) = 0$. Here, $f$ is surjective if and only if $\text{Ext}(N, A) = 0$. Hence the result.

4.2.3 Proposition: An $R$-module $A$ is $n$-absolutely $c$-pure if and only if every homomorphism from a right ideal $I$ of $R$ with $p.\text{dim} \leq n - 1$ to $A$, can be extended to $R$.

Proof: Let $A$ be an $R$-module and $I$ be a right ideal of $R$ with $p.\text{dim} \leq n - 1$. Then $R/I$ is of $p.\text{dim} \leq n$ and we have an exact sequence $0 \to \text{Hom}(R/I, A) \to \text{Hom}(R, A) \to \text{Hom}(I, A) \to \text{Ext}(R/I, A)$.
\[ \rightarrow \text{Ext}(R, A) = 0. \] In this sequence, \( f \) is surjective if and only if \( \text{Ext}(R/I, A) = 0. \) Hence the result.

4.2.4 Proposition: Let \( 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \) be a short exact sequence of \( R \)-modules, if \( A \) and \( C \) are \( n \)-absolutely \( c \)-pure, then so is \( B \).

4.2.5 Proposition: If \( B \) is an \( n \)-absolutely \( c \)-pure \( R \)-module, then every \( n \)-\( c \)-pure submodule of \( B \) is also \( n \)-absolutely \( c \)-pure.

Proof: Let \( A \) be an \( n \)-\( c \)-pure submodule of \( B \). Then the canonical short exact sequence \( 0 \rightarrow A \rightarrow B \rightarrow B/A \rightarrow 0 \) is \( n \)-\( c \)-pure. For a cyclic \( R \)-module \( N \) of \( p.\text{dim} \leq n \), we have an exact sequence \( 0 \rightarrow \text{Hom}(N, A) \rightarrow \text{Hom}(N, B) \rightarrow \text{Ext}(N, A) \rightarrow \text{Ext}(N, B) = 0. \) Since \( f \) is surjective, \( \text{Ext}(N, A) = 0. \) By Proposition 4.2.2, \( A \) is \( n \)-absolutely \( c \)-pure.

4.2.6 Proposition: In a right hereditary ring \( R \), an \( R \)-module \( A \) is injective if and only if it is \( 1 \)-absolutely \( c \)-pure.

Proof: Only if: Obvious.

If: Let \( I \) be a right ideal of \( R \). By hypothesis, \( R \) is right hereditary, \( I \) is projective and hence \( p.\text{dim}(R/I) \leq 1. \) Since by hypothesis, \( A \) is \( 1 \)-absolutely \( c \)-pure and hence \( \text{Ext}_R(R/I, A) = 0, \) (See Proposition 4.2.2).

So \( \text{Hom}(R, A) \rightarrow \text{Hom}(I, A) \) is surjective. Hence \( A \) is injective.

4.2.7 Proposition: If \( R \) is a commutative integral domain, then every
1-absolutely c-pure $R$-module is divisible.

**Proof:** Let $D$ be any 1-absolutely c-pure $R$-module. Let $d \in D$ and $r \neq 0 \in R$. Since $R$ is a commutative integral domain, $rR$ is projective and hence $p.dim(R/rR) \leq 1$. Then by hypothesis, $Ext(R/rR, D) = 0$. Then $Hom(i, D) = i^* : Hom(R, D) \longrightarrow Hom(rR, D)$ is surjective where $i : rR \longrightarrow R$ is the natural inclusion. Define a map $f : rR \longrightarrow D$ by $f(rr') = dr'$. Since $R$ is a commutative integral domain, $f$ is a well-defined homomorphism. There exists $g \in Hom(R, D)$ such that $gi^* = f$. Now $d_0 = g(1) \in D$ and $d_0 r = g(1)r = g(r) = f(r) = d$. So $d = d_0 r$. Hence the result.

**4.2.8 Corollary:** In a commutative integral domain, a torsion-free $R$-module is 1-absolutely c-pure if and only if it is injective.

**4.2.9 Corollary:** In a Dedekind domain $R$, an $R$-module is 1-absolutely c-pure if and only if it is injective.
4.3 n-c-FLAT MODULES

4.3.1 Definition: An R-module $M$ is n-c-flat if every short exact sequence $0 \rightarrow A \rightarrow B \rightarrow M \rightarrow 0$ of R-modules is n-c-pure.

4.3.2 Proposition: Let $R$ be a commutative integral domain. If an $R$-module is 1-c-flat then it is torsion-free.

Proof: Let $M$ be an 1-c-flat $R$-module and $M = F/K$ for some free module $F$ and a submodule $K$ of $F$. Then, $K$ is 1-c-pure in $F$. Let $\bar{x} \in M$ and $r \neq 0 \in R$ such that $\bar{x}r = \bar{0}$. This implies, $xr \in K$. By Proposition 4.2.7, 1-c-pure implies $RD$-pure and so $xr \in Kr$. Hence $xr = kr$ for some $k \in K$. This implies $(x - k)r = 0$. Since $F$ is torsion-free, $x - k = 0$. Then $x = k$ and hence $x \in K$. This implies $\bar{x} = \bar{0}$.

4.3.3 Proposition: Suppose $R$ is a commutative integral domain and $M$ is an $R$-module. The following conditions are equivalent.

i) $M$ is torsion-free.

ii) $M$ is 1-c-flat.

iii) $M$ is c-flat.

Proof: i) $\Rightarrow$ iii) Follows by Proposition 2.1.8.

iii) $\Rightarrow$ ii) is obvious.

ii) $\Rightarrow$ i) Follows by Proposition 4.3.2.

4.3.4 Proposition: For a ring $R$ the following conditions are equivalent.
i) $R$ is semi-simple.

ii) $R$ is right hereditary and every cyclic $R$-module is 1-c-flat.

**Proof:** i) $\implies$ ii) is obvious.

ii) $\implies$ i) Let $R$ be a right hereditary ring. Then every right ideal is projective and hence every cyclic $R$-module is of $p.dim \leq 1$. Since every cyclic $R$-module is 1-c-flat, every right ideal is a direct summand of $R$. Hence, $R$ is semi-simple.
4.4. $n$-c-REGULAR MODULES

4.4.1 Definition: An $R$-module $M$ is $n$-c-regular if every short exact sequence $0 \rightarrow A \rightarrow M \rightarrow B \rightarrow 0$ of $R$-modules is $n$-c-pure.

We state the following proposition whose proof is straightforward.

4.4.2 Proposition: If $R$ is $n$-c-regular as a right $R$-module, then every right ideal of $R$ with $p.dim \leq n - 1$ is a direct summand of $R$.

4.4.3 Proposition: If $R$ is $n$-c-regular as a right $R$-module, then every $R$-module is $n$-c-regular.

Proof: We need only prove that every short exact sequence of $R$-modules is $n$-c-pure. Let $I$ be any right ideal of $R$ with $p.dim \leq n - 1$. Since $R$ is $n$-c-pure, the canonical short exact sequence $0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0$ splits. Let $A$ be any $R$-module. Then we have the exact sequence $0 \rightarrow \text{Hom}(R/I, A) \rightarrow \text{Hom}(R, A) \xrightarrow{f} \text{Hom}(I, A) \rightarrow \text{Ext}(R/I, A) \rightarrow \text{Ext}(R, A) = 0$. Since $f$ is surjective, $\text{Ext}(R/I, A) = 0$. Hence $\text{Ext}(N, A) = 0$ for any $R$-module $A$ and any cyclic $R$-module $N$ with $p.dim \leq n$. Consider any short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ of $R$-modules. Let $N$ be any cyclic $R$-module of $p.dim \leq n$. We have an exact sequence $0 \rightarrow \text{Hom}(N, A) \rightarrow \text{Hom}(N, B) \xrightarrow{f} \text{Hom}(N, C) \rightarrow \text{Ext}(N, A) = 0$. Hence $f$ is surjective. So every short exact sequence of $R$-modules is $n$-c-pure.
4.4.4 Proposition: If $R$ is a right hereditary and $R$ is 1-c-regular then $R$ is semi-simple.

Proof: If $R$ is 1-c-regular, by Proposition 4.4.3, every cyclic $R$-module is 1-c-regular and hence 1-c-flat. Hence the proof follows from Proposition 4.3.4.