CHAPTER III

BOUNDS FOR THE EIGENVALUES OF A GRAPH

INTRODUCTION:

The characteristic roots of the adjacency matrix \( A(G) \) of a graph \( G \) are called adjacency eigenvalues of \( G \) and they are denoted by \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p \).

Note that 
\[
\sum_{i=1}^{p} \lambda_i = 0 \quad \text{and} \quad \sum_{i=1}^{p} \lambda_i^2 = 2q
\]

Where \( p \) and \( q \) are the number of vertices and edges of a graph \( G \) respectively.

If \( G \) is regular graph of degree \( r \) then \( \lambda_i = q \) [22]. If \( G \) is bipartite graph then \( \lambda_i = -\lambda_{p+1-i}, 1 \leq i \leq p \) [8,17,57].

The adjacency eigenvalues of complete graph \( K_p \) are \( p-1 \) and \( -1 \) (\( p-1 \) times). The adjacency eigenvalues of complete bipartite graph \( K_{m,n} \) are \( \pm \sqrt{mn} \) and \( 0 \) (\( m+n-2 \) times).

Several bounds on the adjacency eigenvalues of a graph are obtained.

If \( G \) is a graph with \( p \) vertices and \( q \) edges then [65] and [41].
If \( G \) is \( r \)-regular graph then [65]
\[
\lambda_p \geq r - p \quad \text{..... (3)}
\]
If \( G \) is connected graph then [13]
\[
\lambda_2 \leq \sqrt{\frac{q(p-2)}{p}} \quad \text{..... (4)}
\]

The matrix \( L(G) = D(G) - A(G) \) is called Laplacian matrix of \( G \) where \( D(G) \) is a diagonal degree matrix of \( G \). The characteristic roots of \( L(G) \) are called the Laplacian eigenvalues of \( G \) and they are denoted by \( \mu_1 \geq \mu_2 \geq \ldots \geq \mu_p \).

Note that \( \mu_p = 0 \), \( \sum_{i=1}^{p} \mu_i = 2q \) and \( \sum_{i=1}^{p} \mu_i^2 = 2q + \sum_{i=1}^{p} (d_i(G))^2 \).

Where \( d_i(G) \) is the degree of a vertex \( v_i \) in \( G \).

If \( G \) is bipartite regular graph of degree \( i \) then
\[
\mu_i + \mu_{p-i+1} = 2r, \ 1 \leq i \leq p \quad [49].
\]

The characteristic roots of matrix \( B(G) = D(G) + A(G) \) are denoted by \( \theta_1 \geq \theta_2 \geq \ldots \geq \theta_p \).
Note that $\sum_{i=1}^{\rho} \theta_i = 2q$ and $\sum_{i=1}^{\rho} \theta_i^2 = 2q + \sum_{i=1}^{\rho} (d_i(G))^2$

Further $B(G) = RR'$ where $R = R(G)$ is the vertex-edge incidence matrix of $G$.

If $G$ is bipartite graph then Laplacian eigenvalues of $G$ and characteristic roots of $B(G)$ are same.

i.e., $\mu_i = \theta_i, \quad i = 1, 2, ..., p$.

Thus if $G$ is bipartite regular graph of degree $r$ then

$$0_i + 0_{p-i+1} = 2r, \quad 1 \leq i \leq p.$$ 

For more details about graph eigenvalues, one can refer [8, 22, 49, 61, 65].

**BOUNDS FOR THE ADJACENCY EIGENVALUES:**

**THEOREM 1:** Let $G$ and $H$ be the graphs with $p$ vertices. If $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_p$ be the adjacency eigenvalues of $G$ and $\lambda_1' \geq \lambda_2' \geq ... \geq \lambda_p'$ be the adjacency eigenvalues of $H$ then

$$\sum_{i=1}^{p} \lambda_i \lambda_i' \leq 2\sqrt{q_1 q_2}$$

.....(5)

where $q_1$ and $q_2$ be the number of edges of $G$ and $H$ respectively.
PROOF: By Cauchy–Schwarz inequality [10]

\[
\left( \sum_{i=1}^{p} \lambda_i \lambda_i \right)^2 \leq \left( \sum_{i=1}^{p} \lambda_i^2 \right) \left( \sum_{i=1}^{p} \lambda_i^2 \right) = 2q_1 q_2
\]

\[
\sum_{i=1}^{p} \lambda_i \lambda_i' \leq 2\sqrt{q_1 q_2}
\]

THEOREM 2: If G is a graph with p vertices, q edges and adjacency eigenvalues then

\[
\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p
\]

\[
\lambda_1 \leq \frac{1}{n-1} \left[ \sqrt{2qn(n-1)} + \sum_{i=2}^{n} \lambda_{p-n+i} \right], \quad 2 \leq n \leq p \quad .... (6)
\]

PROOF: Let \( \lambda_1, \lambda_2, \ldots, \lambda_p \) be the adjacency eigenvalues of G. Let \( H = K_s U \bar{K}_{p-s} \). The adjacency eigenvalues of H are \( n-1, 0 \) \( p-n \) times) and \(-1 \) \( n-1 \) times). The number of edges of H are \( n(n-1)/2 \). Using (5) we have

\[
(n-1)\lambda_1 + (0)\lambda_2 + \ldots + (0)\lambda_{p-n+1} - \lambda_{p-n+2}, \ldots, \lambda_p \leq 2 \sqrt{\frac{qn(n-1)}{2}}
\]

\[
(n-1)\lambda_1 \leq \sqrt{2qn(n-1)} + \sum_{i=2}^{n} \lambda_{p-n+i}
\]

\[
\lambda_1 \leq \frac{1}{n-1} \left[ \sqrt{2qn(n-1)} + \sum_{i=2}^{n} \lambda_{p-n+i} \right].
\]

If \( p=n \) then (6) reduces to (1).
Putting $n=2$ in (6) we get following result.

**COROLLARY 3:** If $G$ is a graph with $p$ vertices and $q$ edges then

$$\lambda_1 - \lambda_p \leq 2\sqrt{q} \quad \ldots \quad (7)$$

**COROLLARY 4:** If $G$ is an $r$-regular graph with $p$ vertices then

$$\lambda_p \geq r - \sqrt{2pr}$$

**PROOF:** Let $G$ be an $r$-regular graph with $p$ vertices and $q$ edges. Then $q = \frac{pr}{2}$ and by spectral property $\lambda_1 = p$. Therefore from (7), we get $\lambda_p \geq r - \sqrt{2pr}$.

**COROLLARY 5:** If $G$ is a bipartite graph with $q$ edges then

$$\lambda_1 \leq \sqrt{q} \quad \ldots \quad (8)$$

**PROOF:** Since $G$ is a bipartite graph, $\lambda_1 = -\lambda_p$. Therefore (7) gives $\lambda_1 \leq \sqrt{q}$.

Inequality (8) is improvement over (1) & (2).

**COROLLARY 6:** If $G$ is a $r$-regular bipartite graph with $p$ vertices then

$$\lambda_{p-1} \geq 3r - \sqrt{6pr} \quad \ldots \quad (9)$$
**PROOF:** Putting \( n=3 \) in (6) we get

\[
\lambda_1 \leq \frac{1}{2}\sqrt{12q + \lambda_{p-1} + \lambda_p}
\]

Since \( G \) is \( r \)-regular bipartite graph, \( \lambda_1 = -\lambda_p = r \) and \( q = \frac{pr}{2} \)

\[
2r \leq \sqrt{6pr + \lambda_{p-1}} - r
\]

\[
\therefore \quad \lambda_{p-1} \geq 3r - \sqrt{6pr}.
\]

**THEOREM 7:** Let \( G \) be the graph with \( p \) vertices and \( q \) edges and \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p \) adjacency eigenvalues then,

\[
\sum_{i=1}^{k} \lambda_i \leq \sqrt{\frac{2qk(p-k)}{p}}, \quad 1 \leq k \leq p \quad \text{.... (10)}
\]

**PROOF:** Let \( G \) be the graph with adjacency eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_p \). Let \( H = \bigcup K_n \) where \( p=kn \).

The adjacency eigenvalues of \( H \) are \( n-1 \) \((k\text{-times})\), and \(-1\) \((p-k\text{ times})\). Number of edges of \( H \) are \( \frac{kn(n-1)}{2} \).

From (5) we have

\[
(n-1)\lambda_1 + (n-1)\lambda_2 + \ldots + (n-1)\lambda_k - \lambda_{k+1} - \ldots - \lambda_p \leq 2\sqrt{\frac{qkn(n-1)}{2}}
\]
\[ n \sum_{i=1}^{k} \lambda_i - \sum_{i=1}^{k} \lambda_i \leq 2qnk(n-1) \]

\[ \sum_{i=1}^{k} \lambda_i \leq \frac{1}{n} \sqrt{2qnk(n-1)} \]

But \( p= nk \)

\[ \sum_{i=1}^{k} \lambda_i \leq \sqrt{\frac{2q(k(p-k))}{p}} \]

If \( k=1 \) then (10) reduces to (1).

**COROLLARY 8:** If \( G \) is an \( r \)-regular graph with \( p \) vertices and \( q \)-edges then

\[ \lambda_2 \leq 2\sqrt{q-r-r} \quad \ldots \quad (11) \]

**PROOF:** Putting \( k=2 \) in (10) we get

\[ \lambda_1 + \lambda_2 \leq \sqrt{\frac{4q(p-2)}{p}} \]

Since \( G \) is \( r \)-regular graph, \( \lambda_1 = r = \frac{2q}{p} \)

\[ \therefore \quad \lambda_2 \leq 2\sqrt{\frac{q(p-2)}{p}} - r \]

\[ = 2\sqrt{q-\frac{2q}{p}} - r \]
\[ = 2\sqrt{q-r}. \]

Bound (11) is the improvement over (4).

**THEOREM 9:** Let \( G \) be the graph with \( p \) vertices \( q \) edges and \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p \) adjacency eigenvalues then

\[
\sum_{i=1}^{k} [\lambda_i - \lambda_{p-k+i}] \leq 2\sqrt{qk}, \quad 1 \leq k \leq p \quad \text{.... (12)}
\]

**PROOF:** Let \( G \) be the graph with adjacency eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_p \). Let \( H = \bigcup_{k=m,n} K_{k,m,n} \) where \( p = k(m+n) \).

The adjacency eigenvalues of \( H \) are \( \sqrt{mn} \) (k-times), 0 (p-2k times) and \(-\sqrt{mn}\) (k-times) and the number of edges of \( H \) are kmn. Therefore from (5) we have

\[
\sqrt{mn} \lambda_1 + \sqrt{mn} \lambda_2 + \ldots, + \sqrt{mn} \lambda_k + (0) \lambda_{k+1} + \ldots, +(0) \lambda_{p-k} - \sqrt{mn} \lambda_{p-k+1} - \sqrt{mn} \lambda_p \leq 2\sqrt{qkmn}
\]

\[ \therefore \quad \sqrt{mn} \sum_{i=1}^{k} [\lambda_i - \lambda_{p-k+i}] \leq 2\sqrt{qkmn} \]

\[ \therefore \quad \sum_{i=1}^{k} [\lambda_i - \lambda_{p-k+i}] \leq 2\sqrt{qk}. \]

If \( k=1 \) then (12) reduces to (7).
COROLLARY 10: If $G$ is an $r$-regular bipartite graph with $p$ vertices and $q$ edges then
\[ \lambda_2 \leq \sqrt{2q - r} \quad \text{ .... (13)} \]

PROOF: Putting $k=2$ in (12) we get
\[ \lambda_1 + \lambda_2 - \lambda_{p-1} - \lambda_p \leq 2\sqrt{2q} \]
\[ \therefore \quad 2(\lambda_1 + \lambda_2) \leq 2\sqrt{2q} \quad \text{(since $G$ is bipartite hence $\lambda_1 = -\lambda_p$ & $\lambda_2 = -\lambda_{p-1}$)} \]
\[ \therefore \quad r + \lambda_2 \leq \sqrt{2q} \quad \text{since $\lambda_1 = r$} \]
\[ \therefore \quad \lambda_2 \leq \sqrt{2q - r}. \]

Inequality (13) is better than (11).

BOUNDS FOR LAPLACIAN EIGENVALUES:

THEOREM 11: Let $G$ and $H$ be the graphs with $p$ vertices. If $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p$ are the adjacency eigenvalues of $G$ and $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_p$ be the Laplacian eigenvalues of $H$ then
\[ \sum_{i=1}^{p} \lambda_i \mu_i \leq \sqrt{2q_1 \left[ 2q_2 + \sum_{i=1}^{n} (d_i(H))^2 \right]} \quad \text{ .... (14)} \]

Where $q_1$ and $q_2$ be the number of edges of $G$ and $H$ respectively.
**PROOF:** By Cauchy – Schwarz inequality [10]

\[
\left( \sum_{i=1}^{p} \lambda_i \mu_i \right)^2 \leq \sum_{i=1}^{p} \lambda_i^2 \sum_{i=1}^{p} \mu_i^2
\]

\[= 2q \left[ 2q_2 + \sum_{i=1}^{p} (d_i(H))^2 \right] \]

\[\therefore \sum_{i=1}^{p} \lambda_i \mu_i \leq \sqrt{2q_2 \left[ 2q_2 + \sum_{i=1}^{p} (d_i(H))^2 \right]} \]

**THEOREM 12:** If G is a graph with p vertices, q edges and \( \mu_1 \geq \mu_2 \geq ... \geq \mu_p \) be its Laplacian eigenvalues then

\[
\mu_1 \leq \sqrt{n-1 \left[ \frac{2q + \sum_{i=1}^{p} (d_i(H))^2}{n-1} \right]} + \frac{1}{n-1} \sum_{i=2}^{n} \mu_{p-n+i}, \quad 2 \leq n \leq p \quad .... (15)
\]

**PROOF:** Let \( \mu_1, \mu_2, ..., \mu_p \) be the Laplacian eigenvalues of G. Let

\[H = K_n \cup K_{p-n} \quad \text{The adjacency eigenvalues of } H \text{ are } n-1, 0(p-n \text{ times}) \text{ and } -1(\text{n-1 times}). \]

The number of edges of H are \( \frac{n(n-1)}{2} \). Using (14) we have

\[n-1 \mu_1 + (0)\mu_2 + ... + (0)\mu_{p-n+1} - \mu_{p-n+2} - ... - \mu_p \leq \sqrt{\frac{2n(n-1)}{2} \left[ 2q + \sum_{i=1}^{p} (d_i(G))^2 \right]} \]

\[\therefore \mu_1 \leq \sqrt{\frac{n}{n-1} \left[ \frac{2q + \sum_{i=1}^{p} (d_i(G))^2}{n-1} \right]} + \frac{1}{n-1} \sum_{i=2}^{n} \mu_{p-n+i} \]

putting \( n=2 \) in (15) and noting that \( \mu_p = 0 \) we get following result.
COROLLARY 13: If G be a graph with p vertices and q edges

\[ \mu_i \leq \sqrt{2q + \sum_{i=1}^{p} (d_i(G))^2} \] \hspace{1cm} \text{.... (16)}

COROLLARY 14: If G be a graph with p vertices and q edges

\[ \mu_i \leq \sqrt{\frac{p-1}{p} \left[ 2q + \sum_{i=1}^{p} (d_i(G))^2 \right] + \frac{2q}{p}} \] \hspace{1cm} \text{.... (17)}

PROOF: Put \( n=p \) in (15)

\[ \mu_i \leq \sqrt{\frac{p-1}{p} \left[ 2q + \sum_{i=1}^{p} (d_i(G))^2 \right] + \frac{1}{p-1} \sum_{i=2}^{p} \mu_i} \]

\[ = \sqrt{\frac{p-1}{p} \left[ 2q + \sum_{i=1}^{p} (d_i(G))^2 \right] + \frac{1}{p-1} (2q - \mu_i)} \]

\[ \therefore \mu_i \leq \sqrt{\frac{p-1}{p} \left[ 2q + \sum_{i=1}^{p} (d_i(G))^2 \right] + \frac{2q}{p-1}} \]

\[ \therefore \mu_i \leq \sqrt{\frac{p-1}{p} \left[ 2q + \sum_{i=1}^{p} (d_i(G))^2 \right] + \frac{2q}{p}}. \]

THEOREM 15: Let G be the graph with p vertices, q edges. Let \( \mu_1 \geq \mu_2 \geq \ldots \geq \mu_p \) be its Laplacian eigenvalues then
PROOF: Let \( \mu_1, \mu_2, \ldots, \mu_p \) be the Laplacian eigenvalues of \( G \). Let \( H = \bigcup_k K_n \) where \( p = kn \).

The adjacency eigenvalues of \( H \) are \( n-1 \) (\( k \) times) and \(-1(\ p-k \) times) and number of edges of \( H \) are \( \frac{kn(n-1)}{2} \).

Using (14) we have

\[
(n-1)\mu_1 + (n-1)\mu_2 + \ldots + (n-1)\mu_k - \mu_{k+1} - \ldots - \mu_p \leq \sqrt{\frac{2kn(n-1)}{2} \left[ 2q + \sum_{i=1}^{p} (d_i(G))^2 \right]}
\]

\[
\therefore \sum_{i=1}^{k} \mu_i - \sum_{i=1}^{p} \mu_i \leq \sqrt{kn(n-1) \left[ 2q + \sum_{i=1}^{p} (d_i(G))^2 \right]}
\]

\[
\therefore \sum_{i=1}^{k} \mu_i \leq \sqrt{\frac{k(n-1)}{n} \left[ 2q + \sum_{i=1}^{p} (d_i(G))^2 \right] + \frac{2q}{n}} \quad \text{since} \sum_{i=1}^{p} \mu_i = 2q
\]

\[
= \sqrt{\frac{k(p-k)}{p} \left[ 2q + \sum_{i=1}^{p} (d_i(G))^2 \right] + \frac{2qk}{p}} \quad \text{since} \ n = \frac{p}{k}
\]

If \( k=1 \) then (18) reduces to (17).
COROLLARY 16: If $G$ is bipartite regular graph of degree $r$ with $p$ vertices then
\[ \mu_2 \leq \sqrt{2r(p-2)(r+1)} \] .... (19)

PROOF: Putting $k=2$ in (18) we get
\[ \mu_1 + \mu_2 \leq \sqrt{\frac{2(p-2)}{p} \left[ 2q + \sum_{i=1}^{k} \left( d_i(G) \right)^2 \right] + \frac{4q}{p}}. \]

Since $G$ is bipartite regular graph of degree $r$, $d_i(G)=r$, $2q=pr$ and $\mu_1 = 2r$.
\[ \therefore \quad \mu_2 \leq \sqrt{2r(p-2)(r+1)}. \]

THEOREM 17: Let $G$ be the graph with $p$ vertices and $q$ edges. Let its Laplacian eigenvalues be $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_p$ then
\[ \sum_{i=1}^{k} [\mu_i - \mu_{p-k+i}] \leq \sqrt{2k \left[ 2q + \sum_{i=1}^{p} \left( d_i(G) \right)^2 \right] + \frac{2qk}{p}} \quad 1 \leq k \leq p \] .... (20)

PROOF: Let $\mu_1, \mu_2, \ldots, \mu_p$ be the Laplacian eigenvalues of $G$. Let $H = \bigcup_k K_{m,n}$ where $p = k(m+n)$. The adjacency eigenvalues of $H$ are $\pm \sqrt{mn}$ (k times), $0(p-2k$ times). The number of edges of $H$ are $kmn$. 

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Using (14) we get
\[ \sqrt{mn} \mu_1 + \sqrt{mn} \mu_2 + \ldots + \sqrt{mn} \mu_k + (0) \mu_{k+1} + \ldots + (0) \mu_{p-k} \]
\[ -\sqrt{mn} \mu_{p-k+1} - \ldots - \sqrt{mn} \mu_p \leq \sqrt{2kmn} \left[ 2q + \sum_{i=1}^{p}(d_i(G))^2 \right] \]
\[ \therefore \sqrt{mn} \sum_{i=1}^{k} [\mu_i - \mu_{p-k+i}] \leq \sqrt{2kmn} \left[ 2q + \sum_{i=1}^{p}(d_i(G))^2 \right] \]
\[ \therefore \sum_{i=1}^{k} [\mu_i - \mu_{p-k+i}] \leq 2k \left[ 2q + \sum_{i=1}^{p}(d_i(G))^2 \right] \]

If \( k=1 \) then (20) reduces to (16).

**COROLLARY 18:** If \( G \) is bipartite regular graph of degree \( r \) with \( p \) vertices then
\[ \mu_2 \leq \sqrt{pr(r+1)} \]
\[ \ldots \text{(21)} \]

**PROOF:** Putting \( k=2 \) in (20) we get
\[ \mu_1 + \mu_2 - \mu_{p-1} - \mu_p \leq \sqrt{4 \left[ 2q + \sum_{i=1}^{p}(d_i(G))^2 \right]} \]
\[ \ldots \text{(22)} \]

Since \( G \) is bipartite regular graph of degree \( r \)
\[ \mu_i + \mu_{p-i+1} = 2r, \quad i = 1, 2, ..., p \]
\[ \therefore \mu_i = 2r \]
\[ \therefore \quad (22) \text{ becomes } 2r + \mu_2 - (2r - \mu_2) \leq \sqrt{4 \left[ 2q + \sum_{i=1}^{p} (d_i(G))^2 \right]} \]

\[ \therefore \quad \mu_2 \leq \sqrt{pr(r+1)}. \]

**BOUNDS FOR THE EIGENVALUES OF B:**

**THEOREM 19:** Let G and H be the graphs with p vertices. If \( \lambda_1, \lambda_2, \ldots, \lambda_p \) be the adjacency eigenvalues of G and \( \theta_1, \theta_2, \ldots, \theta_p \) be the eigenvalues of B of H then

\[ \sum_{i=1}^{p} \lambda_i \theta_i \leq \sqrt{2q_1 \left[ 2q_2 + \sum_{i=1}^{p} (d_i(H))^2 \right]} \quad \text{ .... (23)} \]

where \( q_1 \) and \( q_2 \) be the number of edges of G and H respectively.

**PROOF:** Proof is similar as Theorem 11, taking

\[ \sum_{i=1}^{p} \theta_i^2 = 2q_2 + \sum_{i=1}^{p} (d_i(H))^2. \]

**THEOREM 20:** If G is a graph with p vertices, q edges and \( \theta_1 \geq \theta_2 \geq \ldots \geq \theta_p \) be the eigenvalues of B of G then

\[ \theta_1 \leq \sqrt{\frac{n}{n-1} \left[ 2q + \sum_{i=1}^{p} (d_i(G))^2 \right] + \frac{1}{n-2} \sum_{i=p-n+1}^{n} \theta_{p-n+i}, \quad 2 \leq n \leq p \quad \text{ .... (24)} \]
**PROOF:** Similar to Theorem 12.

**COROLLARY 21:** If $G$ is bipartite graph with $p$ vertices and $q$ edges then

$$\theta_i \leq \sqrt{2 \left[ 2q + \sum_{i=1}^{p} (d_i(G))^2 \right]} \quad \text{.... (25)}$$

**PROOF:** If $G$ is bipartite then $\theta_p = 0$.

Putting $n=2$ in (24) we get the result.

Putting $n=p$ and noting $\sum_{i=1}^{p} \theta_i = 2q$ in (24) we get following.

**COROLLARY 22:** If $G$ is a graph with $p$ vertices and $q$ edges then

$$\theta_1 \leq \sqrt{\frac{p-1}{p} \left[ 2q + \sum_{i=1}^{p} (d_i(G))^2 \right] + \frac{2q}{p}} \quad \text{.... (26).}$$

**THEOREM 23:** Let $G$ be the graph with $p$ vertices and $q$ edges.

Let $\theta_1 \geq \theta_2 \geq ... \geq \theta_p$ be the eigenvalues of $B$ of $G$ then

$$\sum_{i=1}^{k} \theta_i \leq \sqrt{\frac{k(p-k)}{p} \left[ 2q + \sum_{i=1}^{p} (d_i(G))^2 \right] + \frac{2qk}{p}}, \quad 1 \leq k \leq p \quad \text{.... (27)}$$

**PROOF:** Proof is similar as Theorem 15.
**COROLLARY 24:** If G is bipartite regular graph then
\[ \theta_2 \leq \sqrt{2r(p-2)(r+1)} \] ...

**PROOF:** Putting \( k=2 \) in (27) and noting that \( d_i(G) = r \), \( 2q = pr \) and \( \theta_1 = 2r \) we get the required result.

**THEOREM 25:** Let G be the graph with p vertices and q edges.
Let \( \theta_1 \geq \theta_2 \geq \ldots \geq \theta_p \) be the eigenvalues of B of G then
\[ \sum_{i=1}^{k} [\theta_i - \theta_{p-k+1}] \leq \sqrt{2k[2q + \sum_{i=1}^{p} (d_i(G))^2]} \quad 1 \leq k \leq p \] ...

**PROOF:** Similar as Theorem (17).

**COROLLARY 26:** If G is bipartite regular graph of degree r with p-vertices then
\[ \theta_2 \leq \sqrt{pr(r+1)} \] ...

**PROOF:** Similar as corollary 18.