CHAPTER II

INTERRELATIONS BETWEEN THE CHARACTERISTIC POLYNOMIALS OF ADJACENCY MATRIX, LAPLACIAN MATRIX AND INCIDENCE MATRIX

INTRODUCTION:

In the literature of graph spectra, one can see the characteristic polynomial of an adjacency matrix of the compliment $\overline{G}$ of a graph $G$ is expressed in terms of the characteristic polynomial the adjacency matrix of $G$ and also many other graph valued functions such as the Sub-division graph $S(G)$, Line graph $L(G)$, and Total graph $T(G)$ of $G$ whose characteristic polynomials of their adjacency matrices are expressed in terms of the characteristic polynomial of adjacency matrix of $G$. Similar results have been obtained by Kelmans [15] with respect to the Laplacian matrix of the respective graph.

In this chapter, we consider the same problem with respect to the incidence matrix of a graph $G$, and also establish the interrelationship between the characteristic polynomials of adjacency matrix, Laplacian matrix and incidence matrix by using the relations

\[ B = A + D \]
\[ L = D - A \]
\[ L = B - 2A \]
\[ A(L(G)) = R'R - 2I_n \]
\[ A(S(G)) = \begin{bmatrix} 0 & R \\ R & 0 \end{bmatrix} \]

all these relations have well defined and established in the previous chapter. For the sake of clarity, we list the following relations and notations. (Of course, these are mentioned in the previous chapter).

If \( G \) is regular graph of degree \( r \) and of order \( n \), then

\[ B = A + rl_n \]
\[ L = rl_n - A \]
\[ L = 2rl_n - B \]
\[ \psi(G, x) = \det [xl_n - A] \]
\[ \mu(G, x) = \det [xl_n - L] \]
\[ \phi(G, x) = \det [xl_n - B] \]

Although we may not establish the relations among arbitrary class of graphs but we succeeded in establishing the relationships among regular class of graphs.
PROPOSITION 1: Let $G$ be a regular graph of order $n$ with regularity $r$, then

(i) $\phi(G, x) = \Psi(G, x-r)$

(ii) $\mu(G, x) = (-1)^n \phi(G, 2r-x)$

PROOF:

(i) $\phi(G, x) = \det[xI_n - B]$

$\phi(G, x) = \det[xI_n - (A + rI_n)]$

$= \det[(x-r)I_n - A]$

$= \Psi(G, x-r)$

(ii) $\mu(G, x) = \det[xI_n - L]$

$= \det[xI_n - (2rI_n - B)]$

$= \det[(x-2r)I_n + B]$

$= (-1)^n \det[B - (2r-x)I_n]$  

$= (-1)^n \phi(G, 2r-x)$

Ex: Let $G$ be a regular graph of degree 3 and of order 6.

\[
G: \quad A(G) = \begin{bmatrix}
0 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0
\end{bmatrix}
\]
\[ B = D + A = \begin{bmatrix} 3 & 1 & 0 & 0 & 1 & 1 \\ 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 & 1 \\ 0 & 1 & 1 & 3 & 1 & 0 \\ 1 & 0 & 0 & 1 & 3 & 1 \\ 1 & 0 & 1 & 0 & 1 & 3 \end{bmatrix} \]

\[ L(G) = \begin{bmatrix} 3 & -1 & 0 & 0 & -1 & -1 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 & -1 \\ 0 & -1 & -1 & 3 & -1 & 0 \\ -1 & 0 & 0 & -1 & 3 & -1 \\ -1 & 0 & -1 & 0 & -1 & 3 \end{bmatrix} \]

\[ \Psi(G, x) = \det (xI_n - A) \]
\[ = x^6 - 9x^4 - 4x^3 + 12x^2 \]

\[ \Psi(G, x-r) = (x-r)^6 - 9(x-r)^4 - 4(x-r)^3 + 12(x-r)^2 \quad r = 3 \]
\[ = (x-3)^6 - 9(x-2)^4 - 4(x-3)^3 + 12(x-3)^2 \]
\[ = x^6 - 18x^5 + 126x^4 - 436x^3 + 777x^2 - 666x + 216 \]

\[ \phi (G, x) = \det (xI_n - B) \]
\[ = x^6 - 18x^5 + 126x^4 - 436x^3 + 777x^2 - 666x + 216 \]
\[ \therefore \phi (G, x) = \Psi (G, x-r) \]

\[ \mu (G, x) = x^6 - 18x^5 + 126x^4 - 428x^3 + 705x^2 - 450x \]

\[ \phi (G, 2r-x) = (2r-x)^6 - 18(2r-x)^5 + 126(2r-x)^4 - 436(2r-x)^3 + 777(2r-x)^2 - 666(2r-x) + 216 \]
\[ = (6-x)^6 - 18(6-x)^5 + 126(6-x)^4 - 436(6-x)^3 + 777(6-x)^2 - 666(6-x) + 216 \]
\[ = x^6 - 18x^5 + 126x^4 - 428x^3 + 705x^2 - 450x \]
\[ \therefore \mu (G, x) = (-1)^n \phi (G, 2r-x) \.

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**COROLLARY 1.1**: For any graph $G$ of order $n$ with regularity $r$,
\[ \mu(G, x) = (-1)^n \varphi(G, r-x) \]

**PROOF**: by (ii) \[ \mu(G, x) = (-1)^n \varphi(G, 2r-x) \]
\[ = (-1)^n \varphi(G, 2r-x-r) \]
Prop. by (i)
\[ = (-1)^n \varphi(G, r-x) \]

**LEMMA 2**: Let $G$ be a graph of order $n$, size $m$. \[ \det [x_l^m - R'R] = x^{m-n} \varphi(G, x) \]

**PROOF**: Consider two partitioned matrices
\[ A = \begin{bmatrix} x_l^n - R \\ 0 & I_m \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} I_n \\ R^t \end{bmatrix} \]

Clearly, \[ AB = \begin{bmatrix} x_l^n - RR^t & 0 \\ R^t & x_l_m \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} x_l^n & 0 \\ x R^t & x_l_m - R'R \end{bmatrix} \]

By an elementary result concerning the determinant of a partitioned matrix, that is, \[ \det \begin{bmatrix} M & N \\ P & Q \end{bmatrix} = \det M \det \begin{bmatrix} Q - PM^{-1}N \end{bmatrix} \]

we have \[ \det (AB) = \det [x_l^n - RR^t] \det [x_l_m - R^t(x_l^n - RR^t)^tO] \]
\[ = \det [x_l^n - RR^t] \det [x_l_m] \]
\[ = x^n \det [xI_n - B] \]
\[ = x^n \phi (G, x) \]

and

\[ \det (BA) = \det [xI_n] \det \left( [xI_m - R'R] - xR'(xI_n)^{-1}O \right) \]
\[ = x^n \det [xI_m - R'R] \]
\[ = x^n \phi (G, x) \quad \ldots (2) \]

Since \( \det (AB) = \det (BA) \) and from (1) and (2) we have,

\[ \det [xI_m - R'R] = x^{n-n} \phi (G, x) \]

**PROPOSITIONS 3:** For any regular graph \( G \) of order \( n \), size \( m \)
with regularity \( r \),

\[ \phi (L(G), x) = (x - 2r + 4)^{n-n} \phi (G, x - 2r + 4). \]

**PROOF:** Clearly, \( B (L(G)) = A (L(G)) + D (L(G)) \)
\[ = R'R - 2I_m + (2r - 2)I_m \]
\[ = R'R + (2r - 4)I_m \]
\[ \therefore \phi (L(G), x) = \det [xI_m - B (L(G))] \]
\[ = \det [xI_m - (R'R + (2r - 4))I_m] \]
\[ = \det [(x - 2r + 4)I_m - R'R] \]
\[ = (x - 2r + 4)^{n-n} \phi (G, x - 2r + 4) \]
COROLLARY 3.1 (H. Sachs [59 & 22]): If $G$ is a regular graph of degree $r$ with $n$ vertices and $m = \frac{1}{2}nr$ edges, then

$$
\Psi(L(G), x) = (x + 2)^{m-n}\Psi(G, x + 2 - r)
$$

PROOF: By proposition 1 (i),

$$
\phi (L(G), x) = \Psi(L(G), x - (2r - 2)) = \Psi(L(G), x - 2r + 2)
$$

.... (1)

By proposition 3,

$$
\phi (L(G), x) = (x - 2r + 4)^{m-n}\phi(G, x - 2r + 4)
$$

= $(x - 2r + 4)^{m-n}\Psi(G, x - 2r + 4 - r)$ Prop. by (i)

= $(x - 2r + 4)^{m-n}\Psi(G, x - 3r + 4)$ .... (2)

∴ by (1) and (2), we have

$$
\Psi(L(G), x - 2r + 2) = (x - 2r + 4)^{m-n}\Psi(G, x - 3r + 4)
$$

By taking $y = x - 2r + 2$, we have

$$
\psi(L(G), y) = (y + 2)\psi(G, y + 2 - r)
$$

COROLLARY 3.2: (Kelmans [43]): If $G$ is a regular graph of degree $r$ with $n$ vertices and $m = \frac{1}{2}nr$ edges, then

$$
\mu(L(G), x) = (x - 2r)^{m-n}\mu(G, x).
$$
**PROOF:** By proposition 1 (ii).

\[ \mu (L(G), x) = (-1)^m \phi (L(G), 2(2r-2)-x) \]

\[ = (-1)^m \phi (L(G), 4r-4-x) \]

\[ = (-1)^m \phi (L(G), y) \quad y = 4r-4-x \]

\[ = (-1)^m (y-2r+4)^m-n \phi (G, y-2r+4) \quad \text{by prop. 3.} \]

\[ = (-1)^m (4r-4-x-2r+4)^m-n \phi (G, 4r-4-x-2r+4) \quad \text{putting value of } y. \]

\[ = (-1)^m (2r-x)^m-n \phi (G, 2r-x) \]

\[ = (-1)^m (2r-x)^m-n (-1)^n \mu (G, x) \quad \text{Prop. 1 (ii)} \]

\[ = (-1)^m (x-2r)^m-n (-1)^m-n \mu (G, x) \]

\[ = (-1)^m (x-2r)^m-n \mu (G, x) \]

\[ = (x-2r) \mu (G, x). \]

which is as required.

**PROPOSITION 4 (Cve [22]):** If G is a regular graph of degree r with n vertices and \( m = \frac{1}{2} nr \) edges, then

\[ \psi (G, x) = (-1)^n \frac{x-n+r+1}{x+r+1} \psi (G, -x-1) \]
Let $G$ be a regular graph of degree 4 and order 6.

\[ V(G, x) = x^6 - 3x^4 + 3x^2 - 1 \]
\[ V(G, x) = x^6 - 12x^4 + 16x^3 \]
\[ V(G, -1 + x) = (-1 + x)^6 - 12(-1 + x)^4 - 16(-1 + x)^3 \]
\[ V(G, x) = (-1)^n x^{n-6} + 1 \]
\[ x^{n-6} + 1 + 4 + 1 \]
\[ x^{n-5} - 12(-1 + x)^4 - 16(-1 + x)^3 \]
\[ x^6 - 3x^4 + 3x^2 - 1. \]

**PROPOSITION 5:** If $G$ is a regular graph of degree $r$ with $n$ vertices and $m = \frac{1}{2}nr$ edges, then

\[ \Phi(G, x) = (-1)^n \frac{x-n+2r+2}{x-n+2r+2} \Phi(G, -x + n - 1) \]

**PROOF:** By the definition of the matrix $B$ of $\overline{G}$, we have

\[ B(\overline{G}) = A(\overline{G}) + (n-r-1)I_n \]

Thus,

\[ \Phi(\overline{G}, x) = \det [xI_n - B(\overline{G})] \]
\[
= \det \left[ x I_n - A(\overline{G}) - (n - r - 1) I_n \right]
\]
\[
= \det \left[ (x - n + r + 1) I_n - A(\overline{G}) \right]
\]
\[
= \psi(\overline{G}, x - n + r + 1).
\]
\[
= \psi(\overline{G}, y) \quad \text{where } y = x - n + r + 1.
\]
\[
=(-1)^n \frac{y - n + r + 1}{y + r + 1} \psi(G, -y - 1) \quad \text{by proposition (4)}
\]
\[
=(-1)^n \frac{y - n + r + 1}{y + r + 1} \psi(G, z) \quad \text{where } z = -y - 1.
\]
\[
=(-1)^n \frac{y - n + r + 1}{y + r + 1} \phi(G, z + r) \quad \text{by proposition 1 (i)}
\]
\[
=(-1)^n \frac{y - n + r + 1}{y + r + 1} \phi(G, -y - 1 + r) \text{ by replacing } Z \text{ by } -y - 1.
\]
\[
=(-1)^n \frac{x - n + r + 1 - n + r + 1}{x - n + r + 1 + r + 1} \phi(G, -x + n - r - 1 - 1 + r)
\]
\[
\quad \text{by replacing } y \text{ by } x - n + r + 1
\]
\[
=(-1)^n \frac{x - 2n + 2r + 2}{x - n + 2r + 2} \phi(G, -x + 2n - 2)
\]
\[
=(-1)^n \frac{x - 2n + 2r + 2}{x - n + 2r + 2} \phi(G, -x + n - 2)
\]
which is exactly as required.
Ex: Let

\[ \phi(G, x) = x^7 - 28x^6 + 322x^5 - 1974x^4 + 6965x^3 - 14126x^2 + 15225x - 6728 \]

\[ \phi(G, x) = x^7 - 28x^6 + 322x^5 - 1974x^4 + 6965x^3 - 14126x^2 + 15225x - 6728 \]

\[ \phi(\overline{G}, x) = \frac{x - 14 + 8 + 2}{x - 7 + 8 + 2} \left[ \phi(G, -x + n - 2) \right] \]

\[ = x^7 - 14x^6 + 77x^5 - 210x^4 + 294x^3 - 196x^2 + 49x - 4. \]

**PROPOSITION 6:** If \( G \) is a regular graph of degree \( r \) with \( n \) vertices and \( m = \frac{1}{2} n \) edges, then

\[ \phi(S(G), x) = (x - 2)^{n-m} \phi(G, (x - r)(x - 2)) \]

**PROOF:** We know that

\[ B(S(G)) = A(S(G)) + D(S(G)) \]

\[ \begin{bmatrix} 0 & R \\ R^t & 0 \end{bmatrix} + \begin{bmatrix} rI_n & 0 \\ 0 & 2I_m \end{bmatrix} = \begin{bmatrix} rI_n & R \\ R^t & 2I_m \end{bmatrix} \]

\[ \therefore \phi(S(G), x) = \det [xI_{n+m} - B(S(G))] \]
PROPOSITION 7: If $G$ is a regular graph of degree $r$ with $n$ vertices and $m = \frac{1}{2}nr$ edges, then

$$\phi(T(G), x) = \Psi(T(G), x - 2r)$$

PROOF: By the definition of $B(T(G))$,

We have

$$B(T(G)) = A(T(G)) + D(T(G)) = \left[\begin{array}{cc} A(G) & R \\ R^t & L(G) \end{array}\right] + \left[\begin{array}{cc} 2rI_n & 0 \\ 0 & 2rI_m \end{array}\right]$$

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\[
\begin{bmatrix}
A(G) + 2rI_n & R \\
R^t & L(G) + 2I_m
\end{bmatrix}
\]

Thus,

\[
\phi(T(G), x) = \det [xI_{n+m} - B(T(G))]
\]

\[
= \det \begin{bmatrix}
(x-2r)I_n & -A(G) \\
-R^t & (x-2r)I_m - L(G)
\end{bmatrix}
\]

\[
= \det [(x-2r)I_{n+m} - T(G)]
\]

\[
= \Psi(T(G), x-2r)
\]