SECTION 8
STABILITY AND BOUNDEDNESS OF INTEGRAL EQUATIONS

Many problems in ordinary differential equations have been studied by comparing the solutions of the systems with those of related scalar differential equations. This comparison enable us to draw a number of consequences of a qualitative nature. This comparison technique has been extended to integral equations by Nohel. In this paper we make use of this idea and study the behavior of solutions of the complex integral equations

\begin{align*}
(1) \quad x(z) &= h_1(z) + \int z q(z - z_1) f(z_1, y(z_1)) \, dz_1 \\
(2) \quad y(z) &= h_2(z) + \int z q(z - z_1) g(z_1, y(z_1)) \, dz_1 \\
& \text{we can extend the results to more general systems.} \\
(3) \quad x(z) &= h_1(z) + \int z F(z, z_1, x(z_1)) \, dz_1 \quad |z| \geq 0 \\
(4) \quad y(z) &= h_2(z) + \int z G(z, z_1, y(z_1)) \, dz_1 \quad |z| \geq 0
\end{align*}

where \( h_1, h_2, f, g \) and \( F \) and \( G \) are \( n \)-dimensional vectors with \( n \) complex components and \( q \) is a given \( n \times n \) matrix defined on \( 0 \leq |z| < |z_0|, |x| < \infty, |y| < \infty \) for some \( |z_0| > 0 \). Let the norm of a vector \( x(z) \) be denoted by \( |x(z)| \) and let \( |x(z)| = \sum_{i=1}^{n} |x_i(z)| \) and let the norm of the matrix \( |q| = \sum_{i,j} |q_{ij}| \), and \( |y(z)| = \sum_{i=1}^{n} |y_i(z)| \).
We assume the existence of \( n \) solutions of (1) and (2).

Let \( D \) denote the region of the complex plane, \( |z| \leq a \), \( \alpha \leq \arg z \leq \beta \) where \( a, \alpha \) and \( \beta \) are real numbers. Let \( \mathbb{C}^n \) denote the \( n \)-dimensional complex Euclidean Space. \( h_1, h_2 \) and \( q \) are regular analytic in \( z \) on \( D \). \( f(z,x) \) and \( g(z,x) \) are regular analytic in \( z \) on \( D \) and entire in \( x \) on \( \mathbb{C}^n \).

For any \( c, 0 < c < t_0(=|z_0|) \)

\begin{align*}
|q| & \in L[\bar{c}, c]. \text{ Let there exists non-negative functions } H, q, W \text{ such that for some } |z_0| > c \\
|h_1(z) - h_2(z)| & \leq H(t) \quad (6) \\
|q(z)| & \leq q(t) \quad (7) \\
|f(z,x) - g(z,y)| & \leq W(t, |x - y|) \quad (8)
\end{align*}

and \( |z_0| = t_0 \). The scalar comparison integral equation associated with the systems (1) and (2) is

\begin{align*}
\mathbf{r}(t) = H(t) + \int_0^t W(t, r(t_1)) \, dt_1 \quad (9)
\end{align*}

We shall have the following theorem which is analogous to theorem 2.1 in [43].

**Theorem 1:** Let \( H \) be continuous on \( 0 \leq t < t_0 \); Let \( \mathbf{q} \in L \) on every finite subinterval of \( [0, t_0] \). Let \( W \) be continuous in \( (t, r) \) for \( 0 \leq t < t_0 \). Let \( W(t, r) \) be non-negative, non-decreasing in \( r \) for each fixed \( t \). For some \( b \leq t_0 \) let \( \mathbf{r}(t) \) be the maximal solution of (9) on \( 0 \leq t < b \). If \( x(z) \) and \( y(z) \) are the solutions of (1) and (2) respectively, then
\( x(z) \) and \( y(z) \) satisfy the condition \( |x(z) - y(z)| \leq r(t) \)
\( (0 \leq t < b) \)
\( z \in D \)

The proof of theorem 1 is analogous to that of theorem 2.1 in [42].

Let \( H(t) = k \) and \( Q(t) = k \) where \( k > o \) is a constant.

Then (9) is equivalent to the scalar differential equation.

\( r' = kW(t, r) \quad r(0) = k \)

Let \( r(t) \) be the maximal solution of (10). We state the following conditions of stability and boundedness for (10).

(i) For each \( \xi > 0 \) and \( t_0 \geq 0 \) there exists a positive function \( \delta = \delta(t_0, \xi) \) that is continuous in \( t_0 \) for each \( \xi \) such that \( r(t) < \xi \) for all \( t \geq t_0 \) whenever \( r(t_0) \leq \xi(t_0, \xi) \)

(ii) \( \delta \) in (i) is independent of \( t_0 \)

(iii) For each \( \alpha > 0 \) and \( t_0 \geq 0 \) there exists a positive function \( \beta(t_0, \alpha) \) that is continuous in \( t_0 \) for each \( \alpha \) such that \( r(t) < \beta \) for all \( t \geq t_0 \) whenever \( r(t_0) \leq \beta \)

(iv) \( \beta \) in (iii) is independent of \( t_0 \)

(v) For each \( \xi > 0, \xi \geq 0 \) and \( t_0 \geq 0 \) there is a positive number \( T = T(t_0, \xi, \delta) \) such that \( r(t) < \xi \) for all \( t \geq t_0 + T \) whenever \( r(t_0) \leq \xi \)

(vi) \( T \) in (v) is independent of \( t_0 \)

(vii) For each \( \delta \geq 0, \delta \geq 0 \) and \( t_0 \geq 0 \) there is a positive number \( B \) and a positive number \( T = T(t_0, \delta) \) such that \( r(t) < B \) for \( t \geq t_0 + T \) whenever \( r(t_0) \leq \delta \).
(viii) In (vii) is dependent of \( t_0 \).

Every stability and boundedness property of (10) may be translated, via theorem 1, into a similar result for solutions of (1) and (2).

Many properties regarding the behaviour of the solutions of the integral equations (1) and (2) can be studied by comparing them with those of the associated scalar integral equation (9). These results can be extended to the more general systems (3) and (4).