SECTION - 4

STABILITY CRITERIA FOR NONLINEAR DIFFERENTIAL EQUATIONS *

In this paper we shall be concerned with the system of Non-linear differential equations.

\[ \dot{x} = f(t, x) \]

where \( x \) and \( f \) are \( n \) dimensional vectors, \( 0 \leq t < \infty \) and study the \( L^p \) - stability of \((E)\) when \( x^t f(t, x) \) is a concave function. Using the notion of concave and convex functions Mangasarian has studied the stability and instability properties of a non-linear differential system. We extend the results to the perturbed system and study the \( L^p \) - stability of \((E)\) under constantly acting perturbations.

Let the norm of an element \( x \) of Euclidean \( n \) - space be given by \( |x| = \sum_{i=1}^{n} |x_i| \). \( f(t, x) \) is defined on the semi cylinder \( D_\infty = \{ (t, x) : t \geq 0, |x| < M, 0 < M \leq \infty \} \). Let \( D^* = D_\infty \cap \{ (t, 0) ; t \geq 0 \} \).

Assume \( f(t, 0) = 0 \) for all \( t \geq 0 \).

First we give following definitions :-

(i) \( x = 0 \) is stable for \((E)\) if for any \( \epsilon > 0 \) and all \( t_0 \geq 0 \), there exists \( \delta(\epsilon, t_0) > 0 \) that is continuous in \( t_0 \) for each \( \epsilon \) , such that \( |x_0| < \delta \) implies

\[ |x(t, t_0, x_0)| < \epsilon \] for all \( t \geq t_0 \).

(ii) \( x = 0 \) is \( L^p \) - stable for \((E)\) if it is stable and if for all \( t_0 \geq 0 \) there exists \( \delta_0(t_0) > 0 \) such that

*This paper was presented in the 81st Conference of Indian Mathematical Society at Jaipur in December 1965.*
(1) \( |x_0| < \delta_0 \) implies \( \int_0^\infty |x(t, t_0, x_0)|^p \, dt < \infty \)

(iii) \( x = 0 \) is \( L^p \) - stable in the large if \( x = 0 \) is stable, the solution \( x(t, t_0, x_0) \) exists on \([t_0, \infty)\) for every \((t_0, x_0) \in \mathbb{R}^n\) and the integral (1) converges uniformly in some neighbourhood of \((t_0, x_0)\) for every \((t_0, x_0) \in \mathbb{R}^n\).

(iv) \( V(t, x) \) is a scalar function defined on \( \mathbb{R}^n \) whose generalized total derivative with respect to (5) is given by

\[
\dot{V}(E)(t, x) = \lim_{h \to 0} \sup \frac{1}{h} \left[ V(t+h, x+h f(t, x)) - V(t, x) \right]
\]

(v) \( V(t, x) \) is a Lyapunov function on \( \mathbb{R}^n \) for (E) if

(a) \( V(t, 0) = 0 \) for all \( t \geq 0 \)
(b) \( V(t, x) \) is continuous on \( \mathbb{R}^n \)
(c) \( V(t, x) \) is positive definite on \( \mathbb{R}^n \)
(d) \( V(t, x) \) is locally lipschitzian
(e) \( \dot{V}(E)(t, x) \leq 0 \) on \( \mathbb{R}^n \)

(vi) \( L(E) \) is the class of Lyapunov functions

(vii) \( \bar{L}(E) \) is the class of mildly unbounded Lyapunov functions for (E)

\( \alpha(x) \) is called a convex function if for\( \[ a \leq x \leq b \]

\[
(1-\delta) \alpha(x_1) + \delta \alpha(x_2) \leq \alpha\left[ (1-\delta) x_1 + \delta x_2 \right]
\]

where \( x_1 \) and \( x_2 \) are vectors in the concave region of \( \alpha(x) \).

The function \( \alpha(x) \) is convex if the inequality sign
in (3) is reversed. For strictly concave (or convex) functions the equality sign in (3) hold only for \( s = 0 \), 
\( s = 1 \) or \( x_1 = x_2 \).

If \( \alpha(x) \) is twice continuously differentiable then the necessary and sufficient condition for strict convexity of \( \alpha(x) \) is that the symmetric matrix of second partial derivatives \( \frac{\partial^2 \alpha}{\partial x_i \partial x_j} \) be negative definite for all values of \( x \) in the region of definition of \( \alpha \). We make use of the following lemma (Lemma 2 from [36]) which is modified in a suitable form.

**Lemma 1**: - Let \( f(t,x) \) be continuous in \( x \) at \( x = 0 \) for 
\( 0 \leq t < \infty \), let \( f(t,0) = 0 \) for \( 0 \leq t < \infty \) and let \( \theta(x) = \lim_{t \to \infty} x' f(t,x) \). If \( x' f(t,x) \) is a strictly concave function of \( x \) for \( 0 \leq t < \infty \) and if \( \theta(x) \leq -c |x|^p \), and 
\( \lim_{x \to 0} \lim_{t \to \infty} f(t,x) = 0 \). Then \( x' f(t,x) \) is negative definite.

The proof is analogous to that of lemma 2 in [36].

We state theorems A and B from [37].

**Theorem A**: - Let \( V \in L(E) \) be such that \( \dot{V}(E) (t,x) \leq -c |x|^p \) in the interval \( 0 \leq t < \infty \), \( |x| < M \) for some \( c > 0 \), \( p > 0 \), then \( x = 0 \) is \( L^p \) - stable for (E).

**Theorem B**: - Let \( V \in \mathcal{L}(E) \) such that \( \dot{V}(F) (t,x) \leq -c |x|^p \) on \( R_0 \) for some \( c > 0 \), \( p > 0 \), then \( x = 0 \) is \( L^p \) - stable in the large.
Theorem 1: Let \( f(t,x) \) be continuous in \( x \) at \( x = 0 \) for \( 0 \leq t < \infty \). Let \( f(t,0) = 0 \) for \( 0 \leq t < \infty \) and let
\[
\varepsilon(x) = \lim_{t \to \infty} x' f(t,x).
\]
If \( x' f(t,x) \) is a concave function of \( x \) for \( 0 \leq t < \infty \) and if \( \varepsilon(x) < -c|x|^p \) for \( x \neq 0 \) in the interval \( 0 \leq t < \infty \) and \( |x| < M \), \( 0 \leq M < \infty \). and
\[
limit_{x \to 0} \lim_{t \to \infty} f(t,x) = 0 \text{ then } x = 0 \text{ is } L^p \text{- stable for the system } (E).
\]

Theorem 2: Let \( f(t,x) \) be continuous in \( x \) at \( x = 0 \) for \( 0 \leq t < \infty \). Let \( f(t,0) = 0 \) for \( 0 \leq t < \infty \) and let
\[
\varepsilon(x) = \lim_{t \to \infty} x' f(t,x).
\]
If \( x' f(t,x) \) is a strictly convex function of \( x \) for \( 0 \leq t < \infty \) and if \( \varepsilon(x) \leq -c|x|^p \) for \( x \neq 0 \), \( 0 \leq t < \infty \), \( |x| < \infty \) and \( \lim_{t \to 0} \lim_{t \to \infty} f(t,x) = 0 \) then \( x = 0 \) is \( L^p \) - stable in the large for \( (E) \).

Proof of theorem 1: Let the Lyapunov function \( V(t,x) \) be equal to \( x' x \). The \( V(t,x) \) satisfies the assumptions in theorem A and \( \dot{V}_{(R)}(t,x) = 2 x' f(t,x) \) which is negative definite by lemma 1 stated above.

Hence it follows from the assumptions in theorem A and from the assumptions in the theorem, that \( y = 0 \) is \( L^p \) - stable for the system \( (E) \).

Proof of theorem 2: Again let \( V(t,x) = x' x \)
\[
\dot{V}_{(R)}(t,x) = 2 x' f(t,x)
\]
Applying lemma 1, theorem B and also from the assumptions in theorem 2 on \( f(t,x) \) and \( x^t f(t,x) \) we can easily deduce the result that \( x = 0 \) is \( L^p \) - stable in the large for the system (E).

Corollary 1: Let \( f(t,0) = 0 \) for \( 0 \leq t < \infty \). Let 
\[
\theta(x) = \lim_{t \to \infty} x^t f(t,x)
\]
and let \( f(t,x) \) be a twice continuously differentiable function of \( x \) for \( 0 \leq t < \infty \). If the matrix 
\[
M_{ij} = \frac{\partial^2 (x^t f(t,x))}{\partial x_i \partial x_j}
\]
is negative definite for \( 0 \leq t < \infty \) and \( |x| < \infty \) and if \( \theta(x) < -c|x|^p \), \( \lim_{t \to \infty} f(t,x) = 0 \) the matrix 
\[
K_{ij} = \frac{\partial^2 (\theta(x))}{\partial x_i \partial x_j}
\]
is negative definite for \( |x| < \infty \), then the solution \( x = 0 \) is \( L^p \) - stable in the large for the system (E).

Proof of corollary 1 follows from the above arguments.

Now we consider the non-linear perturbed differential systems

\[(PE) \quad \dot{x} = f(t,x) + P(t,x)\]

where \( f(t,x) \) is the perturbed function. \( P(t,x) \) is continuous in \((t,x)\)

(viii) The solution \( x = 0 \) is totally \( L^p \) - stable for the system (E) if the system is stable and if for all \( t_0 \geq 0 \) there exists a \( \delta = \delta(t_0) > 0 \) such that \( |y_0| < \delta \) and
(4) \[ |F(y, t)| < \delta_0 \text{ imply} \]
\[ \int_{t_0}^{\infty} |y(t, t_0, y_0)|^P dt < \infty, \text{ where } y(t, t_0, y_0) \text{ is} \]
the general solution of \((PF)\)

(ix) If \(\delta_0\) is independent of \(t_0\) in the definition \((viii)\)
then \(x = 0\) is uniformly totally \(L^P\) - stable.

(x) If \((x_0, t_0) \in D_\infty\) in the definition \((viii)\) then
\(x = 0\) is totally \(L^P\) - stable in the large.

**Theorem 3:** Let \(f(t, x)\) satisfy the assumptions in theorem 1. Let \(\theta(x) = \lim_{t \to \infty} x' [F(t, x) + F(t, x)]\) where \(F(t, x)\) is the perturbed function which satisfies condition (4) above. Let \(x' F(t, x)\) be negative definite. If \(\theta(x) < -c|x|^P\) in the interval \(0 \leq t < \infty\) and \(|x| < \rho\)
where \(0 \leq M < \infty\), then \(x = 0\) is totally \(L^P\) - stable for the system \((E)\).

**Theorem 4:** Let \(\theta(x) = \lim_{t \to \infty} x' [F(t, x) + F(t, x)]\). \(\theta(x) < -c|x|^P\) in the interval \(0 \leq t < \infty\) and \(|x| < \infty\)
defined in the whole \((x, t)\) space, and if \(x' F(x, t)\) is negative definite, then \(x = 0\) is totally \(L^P\) - stable in the large for the system \((E)\).

**Proof of theorem 3:** As before let \(V(t, y) = x' x\).
It satisfies the conditions of theorem A. Its derivative \(\dot{V}(PF)(t, x) = 2 x' \dot{x} = 2 x' [F(t, x) + F(t, y)] \leq -c|x|^F\).
Applying theorem A we obtain that \( x = 0 \) is \( L^p \) - stable for (PE) and hence totally \( L^p \) - stable in the large for the system (\( E \)).

Proof of theorem 4: The result follows from applying theorem 6 and using the arguments of theorem 3.

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SECTION - 5

STABILITY AND BOUNDEDNESS OF COMPLEX DIFFERENTIAL SYSTEMS

We shall consider the complex differential systems

(1) \[ \frac{dx}{dz} = f(z, x) ; \quad x(z_0) = x_0 \]

(2) \[ \frac{dy}{dz} = g(z, y) ; \quad y(z_0) = y_0 \]

and study their stability and boundedness properties.

\( x, y, f \) and \( g \) are complex \( n \) dimensional vectors. \( f \) and \( g \) are regular analytic in \( z \) for \( z \in D \) where \( D \) denotes the region of the complex plane \( |z| \geq a, \theta_1 \leq \arg z \leq \theta_2 \) where \( a, \theta_1 \) and \( \theta_2 \) are real, positive numbers, and \( z \) is an independent complex variable. Let \( \mathbb{C}^n \) be the \( n \) dimensional complex Euclidean space. \( f(z,x) \) and \( g(z,x) \) are entire in \( x \) on \( \mathbb{C}^n \). \( \mathbb{R}^n \) denotes the \( n \) dimensional real Euclidean space. Let \( [0, \infty) \) be the interval \( 0 \leq t < \infty \). Let the norm of the vector \( x \) in the complex Euclidean space be denoted by \( |x(z)| = \sum_{i=1}^{n} |x_i(z)| \).
\( x(z) \) is said to be a solution of (1) with \( x(z_0) = y_0 \) when \( x(z) \) is a complex vector, regular analytic in \( z \) on \( D \) and satisfies the equation (1) for all \( z \in D \). Similarly we may define a solution \( y(z) \) for the system (2). Let \( |z| = t \) and arg \( z = \theta \). Also let \( |z_0| = t_0 ; z_0, z \in D \).

Before we prove our results, we give the following definitions of stability and Boundedness.

(i) Given any \( \epsilon > 0 \) and \( |z_0| \geq 0 \), there exists \( \delta(|z_0|, \epsilon) > 0 \) continuous in \( t_0 \) for each \( \epsilon \) such that \( |x(z_0) - y(z_0)| < \delta \) implies \( |x(z) - y(z)| < \epsilon \) for all \( |z| \geq |z_0| \).

(ii) \( \delta \) in (i) is independent of \( |z_0| \).

(iii) Given any \( \delta > 0 \) and \( |z_0| \geq 0 \), there exists \( \epsilon(|z_0|, \delta) > 0 \) continuous in \( |z_0| \) for each \( \delta \) such that \( |x(z_0) - y(z_0)| \leq \delta \) implies \( |x(z) - y(z)| < \epsilon \) for all \( |z| \geq |z_0| \).

(iv) \( \epsilon \) in (iii) is independent of \( |z_0| \).

(v) Given any \( \epsilon > 0, \delta > 0 \) and \( |z_0| \geq 0 \), there exists \( T(|z_0|, \epsilon, \delta) > 0 \) such that \( |x(z_0) - y(z_0)| \leq \delta \) implies \( |x(z) - y(z)| < \epsilon \) for all \( |z| \geq |z_0| + T(|z_0|, \epsilon, \delta) \).

(vi) \( T \) in (v) is independent of \( |z_0| \).

(vii) Given any \( \delta > 0 \) and \( |z_0| \geq 0 \), there exists positive numbers \( B \) and \( T(|z_0|, \delta) \) such that \( |x(z_0) - y(z_0)| \leq \delta \) implies \( |x(z) - y(z)| < \epsilon \) for all \( |z| \geq |z_0| + T(|z_0|, \delta) \).
(viii) $T$ in (vii) is independent of $|z_0|$

(ix) (iii) and (vii) hold simultaneously

(x) (iv) and (viii) hold simultaneously

(xi) (i) and (v) hold simultaneously

(xii) (ii) and (vi) hold simultaneously

(xiii) Given any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ and $\alpha > 0$ such that $|z_0| \geq \alpha$, $|x(z_0) - y(z_0)| < \delta$ implies $|x(z) - y(z)| < \varepsilon \exp \left[-\alpha(|z| - |z_0|)\right]$ for $|z| \geq |z_0|$

(xiv) Condition (i) holds and for all $|z_0| \geq \alpha$ there exists a $\delta = \delta(|z_0|) > 0$ such that $|x(z_0) - y(z_0)| < \delta(|z_0|)$ imply

\[
\int_{|z_0|}^{\infty} |x(z) - y(z)|^p \mathrm{d}(|z|) < \infty
\]

We state the following lemma A and lemma B which are analogous to lemmas 1 and 2 in [29(b)].

Lemma A: Let the function $U(t, r)$ be continuous and defined on $I \times R^+$ where $R^+ = [0, \infty)$. Let the functions $f(z, x)$ and $g(z, y)$ of (1) and (2) satisfy the condition.

\[
|x(z) - y(z) + h[f(z, x) - g(z, y)]| \leq |x(z) - y(z)| + h W(t, |x-y|)
\]

where $h$ is sufficiently small, real, positive number. Let $r(t)$ be the maximal solution of

\[
f = W(t, r) \quad ; \quad r(t_0) = r_0
\]

then, if $x(z)$ and $y(z)$ are any two solutions of (1) and (2)
such that \(|x(z_0) - y(z_0)| \leq r_0\), satisfies \(|x(z) - y(z)| \leq r(t)\) for \(|z| \leq |z_0|\).

Lemma B: In addition to other assumptions in lemma A, let

\[ |x(z) - y(z) + h[f(x,z) - g(y,z)]| \leq |x(z) - y(z)| (1-h\alpha) + h W(t, |x-y|e^{\alpha|z|}) e^{-\alpha|z|} \]

where \(\alpha > \alpha_0\) is a real number.

(7) If \(|x(z_0) - y(z_0)| \leq r_0\) then

(8) \(|x(z) - y(z)| e^{\alpha|z|} \leq r(t)\) for all \(|z| \geq |z_0|\)

Proof of Lemma A: Let \(x(z)\) and \(y(z)\) be any two solutions of (1) and (2) respectively such that

\[ |x(z_0) - y(z_0)| \leq r_0. \]

Let \(z = t e^{i\theta}\).

Let \(m(t) = |x(z) - y(z)|\). When \(\theta\) is fixed

\[ m(t + h) = |x(z + h e^{i\theta}) - y(z + h e^{i\theta})| \leq |x(z) - y(z)| + h[f(z,x) - g(z,y)] + |\beta h| + |\delta h| \]

where \(\beta\) and \(\delta\) tends to zero as \(h \to 0\). Using the inequality (4) we obtain

(9) \(\limsup_{h \to 0} \frac{1}{h} [m(t + h) - m(t)] \leq W(t, m(t))\)

Let \(r(t_0, \epsilon)\) be a solution of \(\tau = W(t, r) + \epsilon\) ; \(r(t_0) = r_0\)

where \(\epsilon\) is an arbitrarily small positive quantity.

(10) Now we can prove that \(m(t) \leq r(t_0, \epsilon)\) for \(t \geq t_0\)

suppose this relation does not hold; then let \([t_1, t_2]\)

be an interval where \(m(t) > r(t, \epsilon)\).

At \(t_1\) we have \(m(t_1) = r(t_1, \epsilon)\) Hence it follows that
\[ \lim_{h \to 0} \sup \frac{1}{h} [m(t_1 + h) - m(t_1)] \geq r(t_1, \varepsilon) \]
\[ = w(t_1, r(t_1, \varepsilon)) + \varepsilon \]
which contradicts the inequality (9). Hence (10) holds and
\[ \lim_{\varepsilon \to 0} r(t, \varepsilon) = r(t). \]
Hence we obtain the result. The proof of lemma A is complete.

Proof of lemma B is analogous to that of lemma A

Corresponding to the definitions (i) to (xiv) we formulate definitions (ia) to (xiva) for the scalar differential equation (6). For example we define (ia) in the following manner.

(ia) Given any \( \varepsilon > 0 \) and \( t_0 \geq 0 \) there exists a \( s(t_0, \varepsilon) > 0 \) that is continuous in \( t_0 \) for each \( \varepsilon \) such that \( r(t_0) < s(t_0, \varepsilon) \) implies \( r(t) < \varepsilon \) for all \( t \geq t_0 \). Similarly we may formulate the definitions (1ia) to (xiva).

Theorem 1: Let the assumptions of lemma A be satisfied. If the scalar differential equation (5) satisfies the conditions (ia) to (xiva), then the complex differential systems (1) and (2) satisfy the corresponding conditions (i) to (xiv)

Proof of theorem 1: Suppose the scalar differential equation (5) satisfies the condition (ia). If \( x(z) \) and \( y(z) \) are the solutions of (1) and (2) respectively such that \( |x(z_0) - y(z_0)| \leq r(t_0) \) then, it follows from lemma A, that \( |x(z) - y(z)| \leq r(t) \) for \( |z| \geq |z_0| \).
From this we can easily obtain that the complex differential systems (1) and (2) satisfy the condition (1). Similar arguments hold for other cases.

Theorem 2: Let the assumptions of lemma B be satisfied.
If the scalar differential equation (5) satisfies the conditions (la), (ilb), (iiia) and (iva) then the complex differential systems (1) and (2) satisfy the conditions (xi), (xii), (ix) and (x) respectively.

Proof of theorem 2: Suppose the differential equation (5) satisfies the condition (la). Let \( x(s) \) and \( y(z) \) be any solutions of (1) and (2) such that \( |x(z_0) - y(z_0)| \leq r(t_0) \) then it follows from lemma B that \( |x(s) - y(z)| e^{\alpha|z|} \leq r(t) \) for all \( |z| \geq |z_0| \). From this we obtain that the systems (1) and (2) satisfy the condition (xi). Similar arguments hold to prove the other cases. Proof of theorem 2 is completed.

Now we extend the above results to the complex differential systems.

(12) \[ \dot{x}(z) = f(x,z) + F(x,z) \]
(13) \[ \dot{y}(z) = g(y,z) + G(y,z) \]
where \( F \) and \( G \) are the vectors which correspond to the perturbed functions in the real systems.

Suppose the functions \( F \) and \( G \) satisfy the inequality.
If the solutions of (12) and (13) satisfy the conditions (1) to (xiv) with \( F \) and \( G \) satisfying the inequality (14), then the systems (1) and (2) are said to satisfy the conditions weakly.

Theorem 3 := Suppose \( f(x, z) \) and \( g(z, y) \) satisfy the condition

\[
|x(z) - y(z) + h|f(z, x) - g(z, y)|| \leq |x(z) - y(z)| + h \left| f(z, x) - g(z, y) \right|, 
\]

(15) where \( h, \alpha \) and \( W(t, r) \) are the same as in lemmas A and B.

Suppose the inequality (14) holds. If equation (5) satisfies the conditions (ia) to (xiv) then equations (1) and (2) satisfy the corresponding conditions (1) to (xiv), weakly.

Proof of theorem 3 :-

\[
|x - y + h|f(z, x) + f(z, x) - g(z, y) - G(z, y)|| \leq |x(z) - y(z) + h|f(z, x) - g(z, y)| + h \left| f(z, x) - g(z, y) \right|, 
\]

(16) From the conditions (14) and (15), the inequality (16) reduces to

\[
|x - y + h|f(z, x) + f(z, x) - g(z, y) - G(z, y)|| \leq |x(z) - y(z)| + h \left| f(z, x) - g(z, y) \right|.
\]

Applying lemma A we can deduce the result.

Theorem 4 := Suppose \( W(t, r) \) satisfies the conditions in lemma A for \( 0 < t < a \) and for \( r \geq 0 \). Suppose the inequality (4) in lemma A be satisfied for \( |z| \leq a \). Also let the only solution of \( \dot{r} = W(t, r) \) for \( 0 < |z| < a \), is the trivial solution. Then there is at most one solution for both the
systems (1) and (2) on \( 0 \leq |z| < a \) when the argument \( \theta \) of \( z \) is fixed.

Now we consider the equation

\[
\dot{x} = A(z) x
\]

(17)

\[
\dot{y} = A(z) y + f(z,y)
\]

(18)

where \( A(z) \) is an \( n \times n \) matrix which is regular analytic in \( z \). In the case of real differential equations, many important results have been obtained by Coddington and Levenson [13].

Here we can extend the above results for the equations (17) and (18).

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SECTION - 6

STABILITY OF NONLINEAR DIFFERENTIAL EQUATIONS USING FIXED POINT THEOREMS

Although Lyapunov Functions and its extension have been widely used in the 'stability theory of Non-linear Differential Equations', another important technique to study various problems regarding the behaviour of the solutions of non-linear Differential Equations is the application of fixed point theorems. Hukumura and Bellman [8] have used fixed point theorems - particularly Schauder's fixed point theorem to study this problem. Tychonoff's fixed point theorem has been used by Stokes in his doctoral dissertation to study the problem of
stability and boundedness of Non-linear Differential Equations.

By Tychonoff's fixed point theorem we could reduce n-dimensional vector differential Equations to first order differential equations and could derive many results in the theory of stability and Boundedness of the equations, in a simplified manner.

Let $E$ be the real vector space of all continuous functions from the non-negative reals into $\mathbb{R}^n$, the n-dimensional vector space over the real field. The pseudo-norms $\{p_n\}_{n=1}^{\infty}$ where $x \in E$, defined by

$$p_n (x) = \sup_{0 \leq t \leq 1} |x(t)|$$

where $|x(t)|$ is any vector norm. A system of neighbourhoods $[V_n]_{n=1}^{\infty}$ where $V_n = \{x \in E, p_n (x) \leq 1\}$. With this topology, $E$ becomes a complete, locally convex linear space.

Fixed point theorem :- Let $E$ be a complete locally convex linear space. Let $T : E \to E$ be continuous and compact and let $A$ be closed, convex, bounded subset of $E$. If $T(A) \subseteq A$ then there exists a fixed point of $T$ in $A$.

We make use of this theorem to prove our results in this paper.

We shall consider the equations
(1) \[ \dot{x} = f(t, x) \]

(2) \[ \dot{x} = A(t) x + f(t, x) \] where the fundamental matrix solution \( X(t) \) of \( \dot{x} = A(t)x \) satisfies the conditions

\[ |X(t)| \leq K, \ |X(t) X^{-1}(s)| \leq K, \ K > 0 \] and

\[ |X(t) X^{-1}(s)| \leq K e^{\alpha(t-s)}; \ K > 0; \ \alpha > 0; \ t_0 \geq 0 \]

where \( A(t) \) is a continuous function of \( t \) for \( t \geq t_0 \geq 0 \)

(3) \[ \dot{x} = A(t)x + f(t, x) \] where \( |X(t)| \leq K e^{\alpha(t-t_0)} \)

\[ |X(t) X^{-1}(s)| \leq K e^{\alpha(t-s)}; \ K > 0; \ \alpha > 0; \ t_0 \geq 0 \]

where \( A(t) \) is a continuous function of \( t \) for \( t \geq t_0 \geq 0 \)

(4) Assume \( |f(t, x)| \leq G(t, |x|) \) for \( t \geq t_0 \); \( x \in \mathbb{R} \subset \mathbb{R}^n \)

where \( G(t, r) \) is piecewise continuous on \( \mathbb{R}^2 \), positive for \( t, r \geq 0 \) and non-decreasing in \( r \) for fixed \( t \) and \( D \) is some subset of \( \mathbb{R}^n \). With the equation (1) above, we shall associate the integral operator

(5) \[ T_b(x)(t) = b + \int_{t_0}^{t} f(s, x(s)) \ ds \]

with the equations (2) and (3) we shall associate the integral operator

(6) \[ T_b(x)(t) = X(t)b + \int_{t_0}^{t} X(t) X^{-1}(s) f(s, x(s)) \ ds \]

where \( b \) is a vector in \( \mathbb{R}^n \).

Fixed points of these integral operators which are compact in the topology of the function space \( E \), correspond to solutions of the associated equations. Let \( B \) be a bounded subset of \( \mathbb{R}^n \), and let \( A \) be a subset of \( E \) defined by a positive real valued function \( g(t) \), continuous for \( t \geq t_0 \geq 0 \) that is
With $g$ defined above, $A$ is a subset of $F$, satisfying the conditions of the fixed point theorem. From the integral operator in (5), we get

$$|T_b(x)(t)| \leq |b| + \int_{t_0}^{t} |f(s, x(s))| \, ds \leq |b| + \int_{t_0}^{t} |G(s, |x(s)|)| \, ds \quad \text{(from (4))}$$

Hence for the operator in (5) $T_b(A) \subseteq A$ if $g$ satisfies

$$|b| + \int_{t_0}^{t} G(s, g(s)) \, ds \leq g(t) \quad \text{for } b \in B, \, t \geq t_0 \geq 0$$

Similarly for the integral operator in (6), if the fundamental matrix solution $X(t)$ satisfies the inequalities in (2), we get $T_b(A) \subseteq A$ if $g$ satisfies

$$K|b| + K \int_{t_0}^{t} G(s, g(s)) \, ds \leq g(t) \quad \text{for } b \in B, \, t \geq t_0 \geq 0$$

In the same way, if $X(t)$ satisfies the inequalities in (3) we have $T_b(A) \subseteq A$ if $g$ satisfies

$$K|b| e^{-\alpha t} = K \int_{t_0}^{t} e^{-\alpha (t-s)} G(s, g(s)) \, ds \leq g(t)$$

for $b \in B, \, t \geq t_0 \geq 0$.

Now $g(t)$ satisfies (9) if it satisfies the differential inequality.

$$g(t) \geq g(t_0) \geq |b| \quad \text{for } b \in B, \, t \geq t_0 \geq 0$$
g(t) which satisfies (10), also satisfies the differential inequality,

\[ g(t) \leq K \cdot g(s, g(s)) \; ; \; g(t_0) \leq K |b| \text{ for } b \in B \; ; \; t \leq t_0 \leq 0 \]

and \( g \) which satisfies (11) also satisfies the differential inequality

\[ \dot{g}(t) \leq -\alpha g(t) + K \cdot g(t, g(t)) \; ; \; g(t_0) \leq K |b| \; ; \text{ for } b \in B \; ; \; t \geq t_0 \]

Hence a solution is meant a differentiable function \( g \) satisfying (12), (13) or (14) where \( g \) is continuous.

Hence for the existence of a solution to the equation (1), (2) or (3) we must choose a subset \( B \subset \mathbb{R}^n \) and a \( g \) satisfying (12), (13) or (14) and also the condition \( |x(t)| \leq g(t) \) implying \( x(t) \in D \).

To unify our results on stability and boundedness of (1) we state the following conditions

1. Given any \( \varepsilon > 0 \), and \( t_0 \geq 0 \), there exists a \( s(t_0, \varepsilon) > 0 \) that is continuous in \( t_0 \) for each \( \varepsilon \), such that \( |x(t_0)| \leq \varepsilon \) implies \( |x(t)| < \varepsilon \) for all \( t \geq t_0 \)

2. \( s \) in (1) is independent of \( t_0 \)

3. Given any \( \alpha > 0 \) and \( t_0 \geq 0 \) there exists a \( \beta(t_0, \alpha) > 0 \) that is continuous in \( t_0 \) for each \( \alpha \), such that \( |x(t_0)| \leq \alpha \) implies \( |x(t)| < \beta(t_0, \alpha) \) for all \( t \geq t_0 \)
(iv) \( \beta \) in (iii) is independent of \( t_0 \)

(v) For each \( \xi > 0 \) and \( t_0 \geq 0 \) there exists positive numbers \( B \) and \( T(t_0, \xi) \) such that \( |x(t_0)| \leq \xi \) implies \( |x(t)| < B \) for \( t \geq t_0 + T(t_0, \xi) \)

(vi) \( T \) in (v) is independent of \( t_0 \)

(vii) \( (i) \) and (v) hold simultaneously

(viii) \( (ii) \) and (vi) hold simultaneously

(ix) For each \( \epsilon > 0, \xi > 0 \) and \( t_0 \geq 0 \) there exists a positive number \( T(t_0, \epsilon, \xi) \) such that \( |\tau(t)| < \epsilon \) provided \( |x(t_0)| \leq \xi \) and \( t \geq t_0 + T(t_0, \epsilon, \xi) \)

(x) \( T \) in (ix) is independent of \( t_0 \)

(xi) Conditions (iii) and (x) hold simultaneously

(xii) Conditions (iv) and (x) hold simultaneously

Corresponding to the above conditions we may formulate conditions (ia) to (xiiia) for the equation \( \tau = G(t, r) \) in the following way

(ia) Given any \( \epsilon > 0 \) and \( t_0 \geq 0 \) there exists a \( \delta(t_0, \epsilon) > 0 \) that is continuous in \( t_0 \) for each \( \epsilon \) such that \( r(t_0) \leq \delta(t_0, \epsilon) \)

implies \( r(t) < \epsilon \) for all \( t \geq t_0 \). Similarly we may formulate the definitions (iia) to (xiiia)

Theorem 1: If \( \tau = G(t, r) \) satisfies the conditions (ia) to (xiiia) then equation (1) satisfies corresponding conditions (i) to (xii) respectively.
Proof of theorem 1: Let $r(t)$ be a solution of $\dot{r} = G(t, r)$ with $r(t_0) \geq |x(t_0)|$ such that $r(t)$ satisfies the inequality (12). So the operator in (5) maps $A$ into $A$ where $A$ is a subset of $E$, which is defined in (7) and satisfies all the conditions of the fixed point theorem. So by the fixed point theorem, $T_b$ has a fixed point in $A$ where $b \in \mathbb{R}^n$ and the solution passes through $b$ at $t = t_0$. If the solution $r(t)$ of $\dot{r} = G(t, r)$ satisfies the condition (1a), then the solution $x(t)$ of (1) satisfies the condition (1).

Similar arguments hold to prove the other cases.

The above results can be extended to the perturbed differential system

(15) \[ \dot{x} = f(t, x) + F(t, x) \] where $F(t, x)$ is the perturbed function. The corresponding inequality is

(16) \[ \dot{g}(t) \geq G(t, g(t)) + G_1(t, g(t)) \] where $G_1(t, r)$ is piecewise continuous on $\mathbb{R}^2$, positive for $t \geq t_0 \geq 0$, $r \geq 0$ and non-decreasing in $r$ for fixed $t$ and satisfies the condition.

(17) \[ |F(t, x)| \leq G_1(t, |x|) \] for $t \geq t_0 \geq 0$; $x \in D \subset \mathbb{R}^n$

Corresponding to the conditions (i) to (xii) we may reformulate the conditions (i*) to (xii*) for the perturbed differential system (15). Similarly corresponding to the
conditions (ia) to (xiiia) the conditions (ia*) to (xiiia*) may be formulated for the related first order equation

\begin{equation}
\dot{x} = G(t,x) + G_1(t,x).
\end{equation}

If the equation (15) satisfies the conditions (i*) to (xii*) along with the condition (17), then system (1) is said to satisfy the conditions (i) to (xii) totally or under constantly acting perturbations.

**Theorem 2**: If the equation \( \dot{x} = G(t,x) + G_1(t,x) \) satisfies the conditions (ia*) to (xiiia*) then the differential system (1) satisfies the corresponding conditions (i) to (xii) totally or under constantly acting perturbations.

**Proof of Theorem 2**: Following the arguments given in the proof of theorem 1, we can obtain the result.

**Remarks**: Our results partly generalizes the results of Stokkes.

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