SECTION 1

ASYMPTOTIC BEHAVIOR OF DIFFERENTIAL EQUATIONS*

In this paper we study the asymptotic behavior of the solutions of the differential equation

\[ \dot{x} = A(t)x + f(t, x) \]

under certain restrictions on \( A(t) \) and \( f(t, x) \). \( A \) is an \( n \times n \) matrix, \( x \) and \( f \) are column vectors with \( n \) components. First we consider the case when \( A(t) \) is a constant matrix and we extend the results to the case when it is a periodic matrix.

Before we prove the theorems of this paper we define the canonical form of a constant matrix in the following manner.

If \( P \) is a non-singular matrix, the canonical form of a constant matrix \( A \) may be written as

\[
\begin{pmatrix}
J_0 & 0 & 0 & \cdots & 0 \\
0 & J_1 & 0 & \cdots & 0 \\
0 & 0 & J_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & J_q
\end{pmatrix}
\]

where \( J_0 \) is a diagonal matrix with the characteristic roots of \( A \), \( \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_q \) in the diagonal and

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* The results of this paper are generalizations of a corresponding paper on 'Asymptotic Behavior of Differential Systems' which was first presented in the 30th Conference of Indian Mathematical Society at Karnataka University, Dharwar in Dec. 1964 and later published in 'the Proceedings of the National Academy of Sciences, India Vol. XXXV (1965) pages 234-241.'
Theorem 1: If

(2) a) The canonical form of $A$ has submatrices $J_k$, $x \geq 0$

b) $r + 1$ is the maximum number of rows in any matrix

$$J_k, \quad k \geq 0, \quad r \geq o$$

c) $|f(t,x)| \leq |R(t)||f_1(t,x)|$ and $f(t,o) = o$

d) $f_1(t,x)$ be continuous for small $|x|$ and $t \geq o$

e) given any $\epsilon > 0$, there exists $s$ and $t_\epsilon$ so that

$$|f_1(t,x)| \leq \epsilon |x| \text{ for } |x| \leq s \text{ and } t \geq t_\epsilon$$

f) $\int_1^\infty t^\theta |R(t)| \, dt < \infty \quad r \geq o$

g) $\dot{y} = Ay$ has a solution of the form

$$e^{\alpha_j t} \cdot t^k \cdot c + O(e^{\alpha_j t} \cdot t^{k-1})$$

where $c$ is a constant vector, then

$$\lim_{t \to \infty} \mathbf{e}(t) e^{-\alpha_j t} \cdot t^{-k} = c$$

where $\mathbf{e}(t)$ is a solution of (1)
The norm of a vector $x$ is defined by

$$|x| = \sum_{i=1}^{n} |x_i|$$

$x_1, x_2, \ldots, x_n$ being the components of the vector, and

the norm of a matrix $A$ is defined by

$$|A| = \sum_{i,j=1}^{n} |a_{ij}|$$

we give similar definitions for the norms of the other vectors and matrices that are considered in this paper.

**Proof of Theorem 1:** Suppose $R \alpha_j = b$ where $R \alpha_j$ denotes the real part of the characteristic root. Let

$$e^A(t-s) = Y_1(t,s) + Y_2(t,s)$$

where $Y_1(t,s)$ contains terms of the form $e^{\omega_k(t-s)} t^p s^m$ when $R \alpha_k \leq b$ and $Y_2(t,s)$ contains such terms when $R \alpha_k > b$ where $0 \leq p + m \leq r$. Here $k$ is different from $j$. We can find constants $k_1$ and $k_2$ such that

1. $|Y_1(t,s)| \leq k_1 e^{b(t-s)} t^k s^{r-k+1}$  \hspace{1cm} $t \geq s \geq 1$
2. $|Y_2(t,s)| \leq k_2 e^{b(t-s)} t^k s^{r-k}$  \hspace{1cm} $s \geq t \geq 1$

Now we consider the integral equation

$$(4) \quad \phi(t,c) = e^{At} c + \int_{t}^{\infty} Y_1(t,s)f(s,\phi(s,c)) \, ds + \int_{t}^{\infty} Y_2(t,s)f(s,\phi(s,c)) \, ds$$

Using the method of successive approximations to solve equation

$(4)$ let $\phi_0(t,c) = 0$ and
\( (6) \quad e_{m+1}(t,c) = e^{\alpha t} \cdot tk \cdot c + \int_{t_0}^{t} \gamma_1(t,s)f(s,e_m(s,c)) \, ds \)
\[ - \int_{t}^{\infty} \gamma_2(t,s)f(s,e_m(s,c)) \, ds \]

Also let
\( (6) \quad |e^{\alpha t} \cdot tk \cdot c| \leq k_0 \cdot e^{bt} \cdot tk \)
where \( k_0 \) is a constant.

\( (7) \quad |e_1(t,c) - e_0(t,c)| \leq k_0 \cdot e^{bt} \cdot tk \)

\( (8) \quad |e_2(t,c) - e_1(t,c)| \leq \int_{t_0}^{t} |\gamma_1(t,s)| |f(s,e_1(s,c))| \, ds \)
\[ + \int_{t}^{\infty} |\gamma_2(t,s)| |f(s,e_1(s,c))| \, ds \]
\[ t \geq t_0 \]

Using (2 c), (2 d), (2 e), (3) and (7), we obtain
\[ |e_2(t,c) - e_1(t,c)| \leq \varepsilon \cdot k_0 \cdot k_1 \cdot e^{bt} \cdot tk -1 \int_{t_0}^{t} s^{r+1} |R(s)| \, ds \]
\[ + \varepsilon \cdot k_0 \cdot k_2 \cdot e^{bt} \cdot tk \int_{t}^{\infty} s^r |R(s)| \, ds \]
provided \( b \) takes negative values. Since \( t \geq s \) in the first integral we get
\[ |e_2(t,c) - e_1(t,c)| \]
\[ \leq \varepsilon \cdot k_0 \cdot k_1 \cdot e^{bt} \cdot tk \int_{t_0}^{t} s^r |A(s)| \, ds \]
\[ + \varepsilon \cdot k_0 \cdot k_2 \cdot e^{bt} \cdot tk \int_{t}^{\infty} s^r |R(s)| \, ds \]
\[ \leq \varepsilon \cdot k_0 (k_1 + k_2) \cdot e^{bt} \cdot tk \int_{t_0}^{\infty} s^r |R(s)| \, ds \]
\[ \leq \frac{k_0 \cdot e^{bt} \cdot tk}{2} \]

where
\[ \varepsilon (k_1 + k_2) \int_{t_0}^{\infty} s^r |R(s)| \, ds < \frac{1}{\varepsilon} \]
when \( t_0 \) is sufficiently large and by induction we get
(9) \(|e_m(t, c) - e_{m-1}(t, c)| \leq k_0 \cdot e^{bt} \cdot t^k \)

thus the successive approximations converge uniformly to \(e(t)\) which satisfies

(10) \(|e(t)| \leq 2 k_0 \cdot e^{bt} \cdot t^k\), and

(11) \(e(t) = e^{\alpha t} \cdot t^k \cdot c + \int_{t_0}^{t} \int_{t_0}^{s} Y_1(t, s) f(s, \theta(s)) \, ds\)

\(= \int_{t_0}^{t} \int_{t_0}^{s} Y_2(t, s) f(s, \theta(s)) \, ds\)

that is \(|e(t) - e^{\alpha t} \cdot t^k \cdot c| \leq \int_{t_0}^{t} |Y_1(t, s)| |f(s, \theta(s))| \, ds\)

+ \int_{t_0}^{t} |Y_2(t, s)| |f(s, \theta(s))| \, ds

From the above assumptions we obtain

(12) \(|e(t) - e^{\alpha t} \cdot t^k \cdot c| \leq 2 \epsilon k_0 \cdot k_1 e^{bt} t^k |R(s)| \int_{t_0}^{t} s^{r+1} |R(s)| \, ds\)

+ 2 \epsilon k_0 \cdot k_2 e^{bt} t^k |R(s)| \int_{t_0}^{t} s^r |R(s)| \, ds

In the first integral of the above inequality \(t \geq s\)

Thus \(|e(t) - e^{\alpha t} \cdot t^k \cdot c|\)

\(\leq 2 \epsilon k_0 \cdot k_1 e^{bt} t^{k-\frac{1}{2}} \int_{t_0}^{t} s^{r+\frac{1}{2}} |R(s)| \, ds\)

+ 2 \epsilon k_0 (k_1 + k_2) e^{bt} t^k \int_{t_0}^{t} s^r |R(s)| \, ds

that is
Hence the result follows as $t \to \infty$. Here $\Theta(t)$ satisfies

(4) and also (1). The proof of theorem 1 is completed.

Now we consider the equation

\[(14) \dot{x} = A(t) x + f(t,x)\]

where $A(t)$ is a periodic matrix with period $\omega$.

**Theorem 2**: If

\[(15)\]

a) The canonical form of the constant matrix associated with $A(t)$ has submatrices $J_k$, $k \geq 0$

b) $\alpha_j$ is a characteristic exponent of $A(t)$

c) $\dot{y} = A(t) y$ has a solution of the form

$$e^{\alpha_j t} t^k p(t) + O(e^{\alpha_j t} t^{k+1})$$

where $p(t)$ is a periodic vector with the period $\omega$ and if the other assumptions are the same as in theorem 1, then a solution $\Theta(t)$ of (14) is such that

\[(16) \text{limit}_{t \to \infty} e(t) e^{-\alpha_j t} t^{-k} \to p(t)\]

**Proof of theorem 2**: The fundamental solution matrix $\Phi(t)$ of $\dot{y} = A(t) y$ is $P(t) e^{tB}$ where $P(t)$ is a periodic matrix with period $\omega$ and $B$ is a constant non-singular matrix.

Let $x = P(t)z$. Then equation (14) for $z$ becomes

$$\dot{z} = Bz + p^{-1} f(t,pz)$$
or

(17) \( \dot{x} = Bz + F(t, z) \) where \( F(t, z) = p^{-1} f(t, pz) \)

Equation (17) is similar to (1) and satisfies the conditions of theorem 1.

Hence applying theorem 1 we can easily obtain the result of theorem 2. Proof of theorem 2 is complete.

Remarks:

Similar results can be obtained by taking different cases of the matrix \( A(t) \) in (1). That is when

a) \( A(t) \) is a continuous matrix of complex functions on a real \( t \) interval \( I \),

b) \( A(t) \) is an almost constant matrix, that is

\[ \lim_{t \to \infty} A(t) = A^* \]

where \( A^* \) is a constant matrix,

c) \( A(t) \) is an almost periodic matrix etc.

The above results can be extended to \( n^{th} \) order differential equation.

(18) \( \ln x = x^{(n)} + \left[ a_n(t) x^{(n-1)} + f_n(t, x^{(n-1)}) \right] + \left[ a_{n-1}(t) x^{(n-2)} + f_{n-1}(t, x^{(n-2)}) \right] + \cdots \)

\[ + \left[ a_1(t) x + f_1(t, x) \right] = 0 \]

where \( a_k(t) \) are constants or periodic functions,

\[ x^{(n)} = \frac{d^n x}{dt^n} \]
(18) Can be transformed into a form of the equation (1), that is a vector matrix equation
\[ \dot{y} = A(t) \, y + f(t, y) \]
where
\[
A(t) = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 1 \\
a_1 & a_2 & a_3 & a_4 & \ldots & a_n
\end{bmatrix}
\]
and
\[ f(t, y) = \begin{bmatrix}
f_1(t, x) \\
f_2(t, x(1)) \\
\vdots \\
f_n(t, x(n-1))
\end{bmatrix} \]
with certain restrictions on \( a_k(t) \), \( f_k(t, x(k-1)) \), \( r_k(t) \) and \( f_{k1}(t, x(k-1)) \) quite similar to those of \( A(t) \), \( f(t, x) \), \( R(t) \) and \( f_1(t, x) \) respectively, in the theorems 1 and 2. The results of those theorems can easily be extended to the \( n \)th order differential equation.
Remarks: The solutions of the equations in the different cases considered above behave like the corresponding one for the case when \( f(t,x) \neq 0 \). Condition (2a) in theorem 1 implies that the theorem is true even when \( A \) has repeated eigenvalues.

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SECTION 2

ASYMPTOTIC STABILITY OF A NON-LINEAR DIFFERENTIAL SYSTEM

Considerable research has been done on the problem of studying the stability behavior of the solutions of a differential system

\[
\frac{dx}{dt} = Ax + f(t,x)
\]

under various conditions upon the non-linear function \( f(t,x) \).

In this paper we study the behavior of the solutions under a powerful restriction upon the non-linear term. We make use of the following lemma 1 in the proof of our Theorem.

Lemma 1: Let \( \theta, a_1, a_2 \) be real valued continuous or piecewise continuous functions in the interval \( I : a \leq t \leq b \).

Let \( a_2(t) > 0 \) on \( I \) and suppose for \( t \in I \) that

\[
\theta(t) \leq a_1(t) + \int_a^t a_2(s) \theta(s) \, ds \quad \text{then on } I
\]

\[
\theta(t) \leq a_1(t) + \int_a^t a_2(s) a_1(s) \exp \left( \int_a^t a_2(u) \, du \right) \, ds
\]

* This paper is published in 'The proceedings of the National Academy of Sciences, India Vol. XXV (1966) pages 61-63.
Proof of Lemma 1:-

Let \( R(t) = \int_a^t a_2(s) \theta(s) \, ds \), \( R'(t) = a_2(t) \theta(t) \)

then

(2) \( R'(t) - a_2(t) R(t) \leq a_2(t) R(t) \leq a_2(t) a_1(t) \)

(3) Now we obtain \( R'(t) - a_2(t) R(t) \leq a_2(t) a_1(t) \)

The solution of (3) may be written as

\[
R(t) = \exp \left( -\int_a^t a_2(s) \, ds \right) \int_a^t a_2(s) \, a_1(s) \exp \left( -\int_s^t a_2(u) \, du \right) \, ds
\]

that is

\[
R(t) \leq \int_a^t a_2(s) \, a_1(s) \exp \left( \int_s^t a_2(u) \, du \right) \, ds
\]

Hence the result follows.

Lemma 2:-

In Lemma 1 if \( \theta(t) \geq a_1(t) - \int_a^t a_2(s) \theta(s) \, ds \)

then \( \theta(t) \geq a_1(t) - \int_a^t a_2(s) \, a_1(s) \exp \left( \int_s^t a_2(u) \, du \right) \, ds \)

Proof of Lemma 2:- The proof is quite similar to Lemma 1.

Theorem :- If

(4) (a) for \( |x| \leq \delta \) with \( \delta > 0 \)

\[
\frac{|f(t; x)|}{|x|} \leq M. e^{\alpha t} \cdot |x|^{\beta}
\]

where in \( \alpha \) and \( \beta \) are positive constants for all \( t \geq 0 \)

(b) A is a constant matrix with the characteristic 
roots all have negative real parts so that 

\[ |Y(t)| \leq k. e^{\alpha t} \] where \( Y(t) \) is the principal 
matrix solution of the linear system \( \frac{dx}{dt} = Ax \)
K and $a$ are positive constants then, if $\alpha < M a$, the trivial solution of (1) is asymptotically stable.

Proof of the theorem: The solution $\theta$ of (1) with $|\theta(0)|$ small for $t \geq 0$ can be written as

$$(8) \quad \theta(t) = e^{tA} \theta(0) + \int_0^t e^{(t-s)A} f(x, \theta(s)) \, ds$$

using (4b) we obtain

$$(6) \quad |\theta(t)| = k |\theta(0)| e^{-at} + k \int_0^t e^{-(t-s)A} |f(s, \theta(s))| \, ds$$

which can be written as

$$(7) \quad e^{at} |\theta(t)| \leq k |\theta(0)| + k \int_0^t |f(s, \theta(s))| e^{as} \, ds$$

so long as $|\theta(t)| \leq \delta$, using the hypothesis 4(a) we obtain

$$(8) \quad e^{at} |\theta(t)| \leq k |\theta(0)| + k \int_0^t e^{as} \beta |\theta(s)| e^{as} \, ds$$

Applying Lemma 1 we get

$$(9) \quad e^{at} |\theta(t)| \leq k |\theta(0)| \exp \left[ k m \int_0^t |\theta(s)|^\beta e^{as} \, ds \right]$$

(10) Hence $\phi(t) \leq |\theta(0)| \exp \left[ k m \int_0^t |\theta(s)|^\beta \, ds -(\alpha - \kappa) t \right]$ where $\phi(t) = \frac{|\theta(t)| e^{at}}{K}$.

Let $|\theta(0)| < \frac{(\alpha - \kappa)}{K m}$ so that $\phi(t) < \frac{(\alpha - \kappa)}{K^2 m}$ on an interval $0 \leq t \leq b$, $b > 0$

The exponent on the right hand side of (10) is negative. Hence the result easily follows.

Remarks: The theorem holds good when the constant matrix $A$ is replaced by a periodic matrix.