INTRODUCTION

Once Solomon Lefachetz, has said 'to apply mathematics is to enrich mathematics'. Differential Equations is the part of mathematics very close to applications. In the last few decades the theory of differential equations has grown rapidly under the action of forces working both from within and without - from within as a development and deepening of the concepts and of the topological and analytical methods brought out by Lyapunov, Poincaré, Bendixson, S.Lefschetz, Wazewski etc - from without in the wake of the technological development particularly in communications, servomechanisms, automatic controls and electronics.

This work was stimulated by the recent investigations in the stability theory of differential equations by S.Lefschetz [30], Lasalle [26] [24], R.Bellman [4], Taro Yoshizawa [49] [50] [51], W. Hahn [16], N.N.Krasovskii [25], Massera [37] [38] [39], Antosiewicz [1], Hale [38], Levin and Nohel [41] [42] [35], F.Brauer [10] [11] and R.D. Driver [44].

The most common method of determining the stability of a Nonlinear differential system or a Nonlinear differential difference system is to examine the stability of its linear approximation. A central problem is given the behaviour of the solutions of the unperturbed system, what perturbations are admissible in order that the solutions of the perturbed system
Lyapunov's second method has long been recognised as the most general method for studying the stability problems of differential equations or difference equations. The method was presented by Lyapunov in his classical memoir [25] in 1892. Good sources for the statements and proofs of the mathematical theorems underlying the method can be found in works of W. Hahn [16], Antosiewicz [1], Cesari [14], Krasovskii [22] and Lasalle and Lefschetz [24].

In ordinary differential equations an important tool for the investigation of stability and boundedness is the concept of Lyapunov function. Krasovskii [22] has shown that the ideas of Lyapunov can be extended to differential - difference equations if one replaces the Lyapunov function by a proper Lyapunov functional. Its importance arose mainly due to its very wide applications in the theory of controls and in various branches of technology. The main advantage in using Lyapunov functionals for studying the asymptotic behavior and stability are as follows: (i) the system in question need not be linear, (ii) in cases of perturbed systems, the unperturbed system can be either linear or nonlinear and one does not require an integral representation for solutions.

It is important to note in the Lyapunov method that the exhibit a similar qualitative behaviour.
pertinent conditions on the derivative $\frac{dv}{dt}$ of the Lyapunov function $v(x(t), t)$ along the trajectory $x(t)$ are applied not to the true derivative but to the expression

$$\limsup_{\Delta t \to 0} \frac{v(x(t+\Delta t), t+\Delta t) - v(x,t)}{\Delta t}$$

Whenever a trajectory $x(t)$ can be defined in a natural manner for the system of differential equations and when a condition (D) can be verified along these trajectories, Lyapunov's method admits a rigorous foundation.

Comparing the solutions of a differential system or a functional differential system or an integral system with solutions of a related scalar equation enables us to draw a number of consequences of qualitative nature. This comparison technique and the concept of Lyapunov functions has been widely used to study many problems concerning the behaviour of the solutions. The comparison technique has been used - for differential systems by R. Conti [12], Fred Brauer [11], Laxmikantama [29] - for differential - difference systems by Hale [18] and for integral systems by Nohel [42]. In the comparison technique we take norms of the original equation. Naturally a great deal of information is lost by taking norms. It is therefore not surprising that much deeper knowledge can be obtained in may special cases if this step is not taken.
Boundedness is a type of stability and can itself be investigated by Lyapunov's method. Yoshizawa has made a notable contribution in this direction in his papers [49] and [50].

By extracting the essential elements of the concept of periodicity, Lewis [34*] [34**] introduced the more general concept of synartetic and autosynartetic solutions of Differential Equations and developed a theory very similar to the theory associated with periodic solutions. Stability and boundedness properties of synartetic and autosynartetic Differential Equations is still to be exploited.

Either by examining linear approximations or by Lyapunov's method we can find the asymptotic stability of a Nonlinear differential system. But neither of these procedures gives the extent of asymptotic stability or the size of the region of asymptotic stability. Lasalle has made a great contribution to this problem. The set of all initial values with the property that \( x(t,x^0) \to 0 \) as \( t \to \infty \) is called the region of asymptotic stability. It is never completely satisfactory to know only that the system is asymptotically stable without some idea of the size of the region of asymptotic stability. The following theorems of Lasalle gives us some idea of the region of asymptotic stability.
Theorem A:- Let \( \Sigma \) be bounded (compact) set with the property that every solution of \( \dot{x} = X(x) \) which begins in \( \Sigma \) remains for all future time in \( \Sigma \). Suppose there is a scalar function \( V(x) \) which has continuous first partials in \( \Sigma \) and is such that \( \dot{V}(x) \leq 0 \) in \( \Sigma \). Let \( E \) be the set of all points in \( \Sigma \) where \( \dot{V}(x) = 0 \). Let \( M \) be the largest invariant set in \( E \). Then every solution starting in \( \Sigma \) approaches \( M \) as \( t \rightarrow \infty \).

Theorem B:- Let \( \Sigma \) denote the closed region defined by \( V(x) \leq b \) and suppose \( V(x) \) has continuous first partials in \( \Sigma \). If in addition, \( \Sigma \) is bounded and \( \dot{V}(x) \leq 0 \) in \( \Sigma \) then every solution starting in \( \Sigma \) approaches \( M \) as \( t \rightarrow \infty \) where \( M \) is, as in theorem A, is the largest invariant set in \( E \) which is the set of all points in \( \Sigma \) where \( \dot{V}(x) = 0 \).

There are many physical systems where the desired state of the system is not stable but yet the system always tends to return sufficiently close to the desired state that the performance is acceptable and from a practical point of view is 'stable'.

In the fundamental system

\[
(F) \quad \dot{x} = X(x,t) \quad t \geq 0
\]

The equilibrium is at the origin. \( X(0,t) = 0 \) for all \( t \geq 0 \).

The perturbed system is

\[
(FP) \quad \dot{y} = X(y,t) + p(y,t)
\]

We are given a number \( \delta \) and two sets \( Q \) and \( Q_0 \). \( Q \) is closed and bounded containing the origin and \( Q_0 \) is a subset of \( Q \).
Let $y(t, x^0, t_0)$ be a solution of (FP) satisfying 
$y(t_0, x^0, t_0) = x_0$. Let $P$ be the set of all perturbations $p$ satisfying 

$$
\| p(y, t) \| \leq \delta \text{ for all } t \geq 0
$$

and for all $y$. If for each $p$ in $P$ each $x^0$ in $Q_0$ and each 
$t_0 \geq 0$, $y(t, x^0, t_0)$ is in $Q$ for all $t \geq 0$, then the origin 
is said to be practically stable. Practical stability is closely related to ultimate boundedness and Yoshizawa has developed extensively Lyapunov methods for determining ultimate boundedness.

![Diagram](image)

Figure 1

stability and asymptotic stability may not assure practical stability. Hence one needs to know the size of the region of asymptotic stability and then based on estimates of conditions under which the system will actually operate required on its performance etc., one can judge whether or not the system is sufficiently stable to function properly and may be able to see how to improve stability.
Problems in Nonlinear Oscillations is the field of Differential Equations in which there has been the greatest recent progress. Modern Mathematics and topology in particular have given us new methods. If a periodic differential system

\[ \dot{x} = x(x,t) \]

has a unique periodic solution and if all solutions of \((E_1)\) approaches this periodic solution as \(t \to \infty\) then system \((E_1)\) is said to have a steady state oscillation. Equation \((E_1)\) is said to be extremely stable if for any pair of solutions \(x(t)\) and \(\bar{x}(t)\) of \((E_1)\) we have \(x(t) - \bar{x}(t) \to 0\) as \(t \to \infty\). Lyapunov's method has been used to study the problems in oscillations and extreme stabilities. Lasalle has shown that if the system \((E_1)\) is extremely stable and if it has a bounded solution which is unique then all solutions of \((E_1)\) approaches this unique periodic solution as \(t \to \infty\), which means that the system \((E_1)\) has a steady state oscillation. Yoshizawa has shown how extreme stability can be decided by the construction of suitable Lyapunov functions. By generalizing the concept of extreme stability, Yoshizawa has obtained a Lyapunov functional for a functional differential system which guarantees the existence of periodic and almost periodic solutions of the functional differential system. He has obtained results [52] which correspond to steady state oscillations in ordinary differential equations.
A Lyapunov functional which is decreasing along the solutions of the functional differential equation and which satisfies an inequality

$$\| \alpha - \beta \| \leq v(t, \alpha, \beta) \leq x \| \alpha - \beta \|$$

has been used by Yoshizawa to study this problem [32].

Weiss and Infante did excellent work [38] and has obtained some results towards the establishment of a comprehensive qualitative theory of stability of a differential system of nth order

$$\frac{dx}{dt} = f(x, t)$$

defined over a finite interval of time \( I = [t_0, t_0 + T] \). The theory parallels the classical liapunov theory of stability and the theorems are analogous to some of the well known theorems in the classical case. Definitions of finite time stability and instability are as follows:

**Definition 1:** A system \((E_2)\) is said to be stable with respect to \((\alpha, \beta, t_0, T, \| \cdot \|)\), \(\alpha \leq \beta\) if for every trajectory \(x(t), \| x(t_0) \| < \alpha\) implies \(\| x(t) \| < \beta\) for all \(t \in I\).

**Definition 2:** A system \((E_2)\) is unstable with respect to \((\alpha, \beta, t_0, T, \| \cdot \|)\), \(\alpha \leq \beta\) if there exists a trajectory \(x(t)\) where \(\| x(t_0) \| < \alpha\) and a value of time \(t_1 \in (t_0, t_0 + T)\) such that

$$\| x(t_1) \| = \beta$$

**Definition 3:** The system \((E_2)\) is quasi-contractively
stable with respect to \((\alpha, \beta, t_0, T, \| \cdot \|)\) if for every trajectory \(x(t)\) where \(\| x(t_0) \| < \alpha\) there exists \(t_1 \in (t_0, t_0+T)\) such that \(\| x(t) \| < \beta\) for all \(t \in (t_1, t_0+T)\) and \(\beta < \alpha\).

**Definition 4:** The system \((E_2)\) is contractively stable with respect to \((\alpha, \beta, \delta, t_0, T, \| \cdot \|)\) if (i) it is stable with respect to \((\alpha, \delta, t_0, T, \| \cdot \|)\) and (ii) it is quasi-contractively stable with respect to \((\alpha, \beta, t_0, T, \| \cdot \|)\).

The basic difference between the classical theory and the finite time stability theory presented in [48] stems from the necessarily different concepts of stability which are employed in the later case.

The most obvious difference between the definition of finite time stability and the usual stability definition is that, in the former case there are fixed prespecified bounds on the trajectories. Definitions (1) and (2) are the finite time analogues of Lyapunov stability and instability respectively while definitions (3) and (4) are the finite time analogues of quasi- asymptotic stability and asymptotic stability respectively. The stability, instability, quasi-contractive stability and contractive stability of the system \((E_2)\) depends upon certain conditions which a function \(V(x,t)\) and its derivative \(\dot{V}(x,t)\) satisfies. But there is no requirement of definiteness on either \(V(x,t)\) or \(\dot{V}(x,t)\) in the theorems provided by Weiss and Infante [48]. The instability theorem
of Weiss and Infente is analogous to the classical instability theorem of Chetaev [24].

Functional differential equations describes physical systems whose future behavior depends not only upon the present state of the system but also upon some portion of its past history. They are natural generalizations of differential equations with time delay and have been called equations with a past history. As has been mentioned before, Krasovskii was the first to construct Lyapunov functionals for functional differential equations. With the help of Lyapunov functionals many of the classical theorems of Lyapunov can be extended to functional differential equations. Differential-difference equations have been called as differential equations with retarded arguments. A. N. Myshkis, was perhaps the first to give a fine presentation of this subject in his classical paper [54]. Equations with an advancing argument are also possible.

\[ M \ddot{x}(t) + K x(t + \theta) = 0 \quad (\theta > 0) \]

These equations do not have the immediate physical meaning ascribed to the case of 'retarded' argument. The forces acting on a point and causing changes in its motion cannot be determined by its position at a later time. Hence they have very little practical importance. In the case of an advanced argument, the solution of a naturally formulated initial problem ceases to depend continuously on initial conditions.
Krasovskii has done an invaluable work on the stability of delay differential equations (chapters 6 and 7 in [23]).

\[ \frac{dx_i}{dt} = x_i(x_i(t + \theta), \ldots, x_n(t + \theta), t) \quad (i = 1, 2, \ldots, n) \]

where \(-h \leq \theta \leq 0\) and \(h > 0\).

R.D. Driver has extended his results to cases in which the delay \(h\) is a function of \(t\). That is the equations takes the form:

\[ \frac{dx_i}{dt} = x_i(x_i(t - h(t)), \ldots, x_n(t - h(t)), t) \]

where the right member of (DE2) is a functional and not merely a function. Almost all the results in the stability theory of ordinary differential equations in which Lyapunov methods have been used are extended to delay differential systems in which case Krasovskii has used both Lyapunov functions as well as Lyapunov functionals.

Filippov has made an authoritative study on the behaviour of the solutions of differential equations with discontinuous right side. He has shown that some of the results of ordinary differential equations with continuous right side can be extended to differential equations with discontinuous right side.

Fixed point theorems have also been widely used to study the stability properties of differential equations [15] functional differential equations and integro-differential equations [3].
The study of finite difference approximations to partial differential equations is well advanced. This study has been actively pursued due to the increasing use and significance of digital computers. One question that has not properly been discussed is that of finding discrete approximations that manifestly exhibit the boundedness, positivity and so on of the original continuous operator. Approximations of this type are very useful computationally because of their stability properties.


Although Massera has proved some converse theorems of Lyapunov, the study of converse theorems of Lyapunov is still incomplete. These converse theorems enable us to understand how stable systems behave under perturbations and to identify those properties of stable systems unchanged by small perturbations.

The one question which remains a challenge is - how can one effectively use modern computing machines to apply Lyapunov's second method to the study of stability of the real systems?
Another important aspect that has attracted the differential equationists all over the globe is the extension of the geometric theory of ordinary differential equations to functional differential equations. Krasovskii [29] has demonstrated the effectiveness of a geometric approach in extending the classical Lyapunov theory, including the converse theorems to functional differential equations. Hale's extension in [29] of this approach to autonomous functional differential equations of the form:

\[ \dot{x} = f(x_t) \]

has had so far the greatest success in studying the stability properties of the solutions of systems (DE3) and it is possible that this may lead to a similar theory for special classes of systems defined by partial differential equations.

In the delay - differential equations:

\[ \frac{dx_i}{dt} = f_i(x_1(t+\theta), \ldots, x_n(t+\theta), t) \]

\[ (i = 1, 2, \ldots, n) \]

where the right - hand members \( f_i(x_1(\theta), \ldots, x_n(\theta), t) \) are functionals defined for piecewise - continuous functions \( x_i(\theta) \) of the argument \( \theta \), which is restricted to the interval \(-h \leq \theta \leq 0\) where \( h \) is a positive constant.

Let the norm of the vector \( x \) be defined as

\[ \| x \|_{(h)} = \sup (|x_i(\theta)|) \text{ for } -h \leq \theta \leq 0 \]

\[ (i = 1, 2, \ldots, n) \]

suppose \( f_i \) is defined for
(2) $||x||^{(h)} < H, \quad t \geq 0$

where $H$ is a fixed constant.

Let there be a certain moment of time $t = t_0 \geq 0$ and the piecewise-continuous functions

(3) $x_0(\theta_0) = x_{10}(\theta_0)$

\[ i = 1, 2, \ldots, n; \quad -n \leq \theta_0 \leq 0 \]

We call the function $x(t) = x_1(t)$ a solution if they are continuous for $t \geq t_0$ and satisfy the condition

(4) $\lim_{t \to 0} \frac{x_1(t + \tau) - x_1(t)}{\tau} = f_1(x_1(t+\tau), \ldots, x_n(t+\tau), t)$

provided that for $t + \alpha \leq t_0$ we have $x_1(t+\alpha) = x_{10}(\theta_0)$

(5) $\theta_0 = t + \omega - t_0$

Just as for ordinary differential equations, the existence, uniqueness and continuation of the solution $x(t)$ of (1) for all values of $t \geq t_0$ for which the curve $x(x_0(\theta_0), t_0; t)$ lies in the region (3).

If $t \geq t_0, \quad \omega > 0$, then

(6) $x(x_0(\theta_0), t_0, t+\omega) = x(x_0(\theta_0), t, t+\omega)$

where

(7) $x_0^*(\theta_0) = x(x_0(\theta_0); t_0; t+\rho)$

(see figure 2)
**Definition 1**: The solution $x = 0$ of equation (DE4) is called stable if for every positive number $\epsilon > 0$ we can find a positive number $\delta > 0$ such that whenever the inequality

$$(8) \quad \|x_0(\theta_0)\|^{(h)} \leq \delta$$

is satisfied, the relation

$$(9) \quad \|x(x_0(\theta_0), t_0, t)\| < \epsilon$$

holds for $t \geq t_0$.

**Definition 2**: If when definition 1 is satisfied, the conditions

$$(10) \quad \text{limit } \|x(x_0(\theta_0), t_0, t)\| = 0 \text{ for } t \to \infty$$

$$(11) \quad \|x(x_0(\theta_0), t_0, t)\| < H_1 \text{ for all } t \geq t_0 \quad (H_1 = \text{constant})$$

is satisfied for all initial curves $x_0(\theta_0)$ satisfying the inequality.
then the solution \( x = 0 \) of (12) is asymptotically stable and the region (12) lies in the region of attraction of the unperturbed motion.

From the formal point of view definitions 1 and 2 are the most natural extensions of Lyapunov's definitions of stability and asymptotic stability to equations with time delay. A statement of the stability problem for equations with time delay involves the question: what set of initial curves is sufficiently broad for the problem under consideration? In every case a stability criterion would be considered correct whenever it guaranteed stability for a definite class of initial curves known to include all initial curves arising from possible or admissible perturbations of the given system. The chief practical problem is obviously a statement of the stability problem in a manner or form that is sufficiently wide to include small perturbations of the equation and of the initial curvves that are admissible in the structure of the system. In stating the stability problem given by definitions 1 and 2 it is convenient to study as an element \( x(x_0(t_0), t_0, t) \) of the trajectory not the entire vector \( x_1(x_0(t_0), t_0, t) \) but a vector segment of this trajectory \( x_1(x_0(t_0), t_0, t + \epsilon) \) for \(-h \leq \epsilon \leq h\) (see figure 2)
New frontiers of mathematics are the domains of biology and medicine. One of the great advantages of mathematical approach to problem solving lies in the fact that analytic technique developed to treat specific questions in one scientific area can readily be applied to any area in which analogous functional relations hold. It follows that there is a constant interplay between the general and the particular. General methods come out of the investigation of particular problems and particular problems are solved by means of powerful general methods. Let us briefly outline a simplified model of a chemotherapeutic process.

![Graph](image_url)

**Figure 3**

![Diagram](image_url)

**Figure 4**

Heart and 2 Organs in parallel
Fuller details will be found in the references given in (5**). The basic idea is to consider the body to be an "input-output" system and to apply some of the well-known concepts of Mathematical Economics. We take the heart to be a source of pumping energy and consider a number of organs in parallel as shown in figure 4.

The consequence of injecting a chemical into the bloodstream are idealized in the following fashion. The chemical is carried by the circulatory system into organ I and organ II where chemical interactions effect the concentration of the original chemical in the blood and the chemical constitution of the organs. In addition by means of diffusion some of the original chemical passes over into organs. The outputs of organs I and II are mixed by the heart and the composite is fed as a new input into the organs. This process is repeated until a steady-state sets in.

The various parts of this process can be analysed in some detail and in some complexity. This is not our purpose here. We wish only to point out that the non-negligible time required for circulation of the blood means that the equations that describe the process are not conventional differential equations. Ordinarily in dealing with dynamic processes we are accustomed to systems of Differential Equations of the form
\[
\frac{du_i}{dt} = g_i(u_1, u_2, u_3 \ldots u_n, t)
\]

\[
u_1(0) = c_1
\]

\[i = 1, 2, \ldots N.
\]

Equations of this type can be solved numerically with the aid of a digital computer, very accurately and very rapidly for values of \(N\) of the order of 100 and even 500. In chemotherapy we meet equations of the form

\[
\frac{du_i}{dt} = g_i(u_1(t), u_2(t) \ldots u_n(t), u_1(t-\alpha), \ldots u_n(t-\alpha)
\]

where \(\alpha\) is a positive constant and the initial conditions are

\[u_i(t) = h_i(t) \quad 0 \leq t \leq \alpha; \quad i = 1, 2, \ldots N.
\]

These are Differential - Difference equations with a theory which is much richer than the theory of ordinary differential equations and correspondingly more difficult. Actually these equations are themselves simplified versions of more complicated ones, but they suffice to illustrate the points we wish to make here. For detailed analysis for these classes of equations see [**] and its references.

In the following pages the author has made some attempt to study 'the asymptotic behavior and stability problems of Differential Equations, difference equations, integral equations and differential - difference equations.'