SECTION - 12a

STABILITY CRITERIA OF NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS*

One of the effective methods for treating stability problems in the theory of differential equations or of differential difference equations is the so-called direct method or 2nd method of Lyapunov. In this paper we consider the stability properties of the solutions of Neutral functional differential equations and show how most of the results in ordinary differential equations and in functional differential equations can be extended in the natural manner to neutral functional differential equations.

We consider the neutral functional differential equations

\[ \dot{x}(t) = f(t, x(t), x(h_1(t)), \dot{x}(h_2(t))) \]

for \( t > t_0 \) with \( x(t) = x_0(t) \) on \([t_0, t_1]\)

and

\[ \dot{y}(t) = g(t, y(t), y(h_1(t)), \dot{y}(h_2(t))) \]

for \( t > t_0 \) with \( y(t) = y_0(t) \) on \([t_0, t_1]\)

where \( x, y, f \) and \( g \) are \( n \) dimensional vectors, \( h_1 \) is an \( m \) dimensional vector and \( h_2 \) is a \( p \) dimensional vector. Vector functions \( h_1 \) and \( h_2 \) satisfy the condition \( \alpha \leq h_1(t) \leq \beta \)

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for all $t \geq t_0$. Here $\alpha$ may take the value $-\infty$ also. Then the interval $[\alpha, t_0]$ is interpreted as $(-\infty, t_0]$. Equations (1) and (2) arise in a two-body problem of classical electrodynamics.

Equations (1) and (2) may also be of the form

(3) \[ \dot{x}(t) = f(t, x(t), x(h(t)), \dot{x}(h(t))) \quad \text{for } t > t_0 \]

and

(4) \[ \dot{y}(t) = g(t, y(t), y(h(t)), \dot{y}(h(t))) \quad \text{for } t > t_0 \]

in which $h_1(t) = h_2(t) \equiv h(t)$. The term neutral apparently arose because in certain simple cases the equation could equally well be considered as having a retarded argument or an advanced argument. For example if $h(t) = t - \beta$, where $\beta$ is a positive constant, the equations could be solved for $x(t - \beta)$ and $y(t - \beta)$. The continuity and Lipschitz conditions on the given functions $f$, $g$, $h_1$, $h_2$, $\theta_1$ and $\theta_2$ will assure the local existence and uniqueness of a solution of the functional differential equations. If $x$ and $y$ are any two solutions of (1) and (2) respectively with their initial functions $\theta_1(t)$ and $\theta_2(t)$ on $[\alpha, t_0]$, then we define the norm as

(5) \[ \| x - y \| = |x(\alpha) - y(\alpha)| + \int_{\alpha}^{t_0} |\dot{x}(s) - \dot{y}(s)| \, ds \quad (\beta > t_0) \]
We use this norm in our paper. Let $E^{1+n+nm+np}$ denote the euclidean space of $1+n+nm+np$ dimensions. Let $D = D^{1+n+nm}$ be a domain (a connected open set) in $E^{1+n+nm}$, an euclidean space of $1+n+nm$ dimensions, $f$ and $g$ are defined over a domain $D \times E^{np}$. \text{h}^t is $f$ and $g$ are defined over a domain in $E^{1+n+nm+np}$. Let $D^t$ be the projection of $D$ onto the $t$-axis. Let the vectors $h_1(t)$ and $h_2(t)$ are defined over $D^t$. Given an initial function $\xi$ which is an absolutely continuous $n$ vector-valued function on $[\alpha, t_0]$, $\alpha < t_0$, a solution $x(t)$ of (1) satisfies the following conditions

(i) $x(t)$ is defined and continuous on $[\alpha, \beta)$, $\beta > t_0$.

(ii) $x(t) = \xi(t)$ on $[\alpha, t_0]$.

(iii) $\dot{x}(t) = f(t, x(t), x(h_1(t)), x(h_2(t)))$ for almost all $t \in (t_0, \beta)$.

Let $V(t, x, y) \geq 0$ be defined and continuous scalar function on $D^n \times E^n \times E^n$. Suppose that it satisfies Lipschitz's condition in $x$ and $y$ locally. We define its derivative $\dot{V}(1), (2)(t, x, y)$ in the following manner.

(6) $\dot{V}(1), (2)(t, x, y) = \lim_{h \to 0} \frac{1}{h} [V(t+h; x+hf; y+hg) - V(t, x, y)]$

Lemma 1: Let the function $w(t, \tau) \geq 0$ be defined and
continuous for $t \in [t_0, \infty)$ and $r \geq 0$. Let $r(t)$ be the maximal solution of the scalar differential equation

$$\dot{x} = w(t, x) \quad r(t_0) = r_0$$

existing to the right of $t_0$. For $t \in [t_0, \infty)$ and $x$, $y$ defined over $D_1$, let

$$\limsup_{h \to 0^+} \frac{1}{h} \left[ V(t+h, x+h, y+h) - V(t, x(t), y(t)) \right] \leq w(t, v(t, x(t), y(t)))$$

If $x(t)$ and $y(t)$ are any two solutions of (1) and (2) with their initial functions $\theta_1$ and $\theta_2$ defined above such that

$$V(t_0, \theta_1, \theta_2) \leq r_0$$

then

$$V(t, x(t), y(t)) \leq r(t) \quad (t \geq t_0)$$

Proof of Lemma: Let $x(t)$ and $y(t)$ be any two solutions of (1) and (2) respectively with the initial functions $\theta_1$ and $\theta_2$ defined above such that $V(t_0, \theta_1, \theta_2) \leq r_0$. Let $m(t) = V(t, x(t), y(t))$ then $m(t_0) = V(t_0, \theta_1, \theta_2) \leq r_0$

let $r_c(t)$ be any solution of

$$\dot{r} = w(t, r) + \xi, \quad r(t_0) = r_0$$

where $\xi$ is arbitrarily small positive quantity. It is easy to show that

$$m(t) \leq r_c(t) \quad (t \geq t_0)$$

suppose this relation does not hold. Let $t_1$ to the greatest lower bound of all $t$'s $> t_0$ for which the above inequality
does not hold. Since $m(t)$ and $r_C(t)$ are continuous we have

(i) $m(t) \leq r_C(t)$ \hspace{1cm} (t_0 \leq t \leq t_1)

(ii) $m(t_1) = r_C(t)$ \hspace{1cm} (t = t_1)

(iii) $m(t_1 + h) > r_C(t_1 + h)$ \hspace{1cm} (h > 0)

Hence we have
\[
\lim_{h \to 0^+} \frac{1}{h} \left[ m(t_1 + h) - m(t_1) \right] \geq r_C(t_1) = w(t_1, r_C(t_1)) + \varepsilon
\]

since $w(t, r)$ is non-negative, the solution $r_C(t)$ are non decreasing as $t$ increases.

Since $V(t, x, y)$ satisfy the Lipschitz condition in $x$ and $y$ for each $t$, we have for small $h > 0$

\[
(12) m(t_1 + h) - m(t_2) \leq k \left[ ||x(t_1 + h) - x(t_1) - hf|| + ||y(t_1 + h) - y(t_1) - hg|| \right]
\]
\[
+ V[t_1 + h, x(t_1) + hf, y(t_1) + hg] - V(t_1, x(t_1), y(t_1))
\]

which implies that
\[
(14) \lim_{h \to 0^+} \frac{1}{h} \left[ m(t_1 + h) - m(t_1) \right] \leq w(t_1)
\]

which is a contradiction.

Hence the inequality $m(t) \leq r_C(t)$ holds. From this the results follows; The proof of lemma is complete.

Before we state the main theorems of this paper we give the following definitions of stability and boundedness for the neutral functional differential equations of (1) and (2). Let the solutions $x(t)$ and $y(t)$ of (1) and (2) exist
for all $t \geq t_0$, with the initial functions $e_1$ and $e_2$ at $t = t_0$.

(i) Equation (1) or (2) is said to be equinorm bounded with respect to equation (2) or (1) if for each $\delta > 0$ and $t_0 \geq 0$ there exists a positive function $\xi(t_0, \delta)$ continuous in $t_0$ for each $\delta$ such that $||e_1 - e_2|| \leq \delta$ implies $||x(t) - y(t)|| < \xi$ for $t \geq t_0$.

(ii) If $\xi$ in (i) is independent of $t_0$, equation (1) or (2) is said to be uniform norm bounded with respect to equation (2) or (1).

(iii) Equation (1) or (2) is said to be equistable with respect to equation (2) or (1) if for each $\delta > 0$ and $t_0 \geq 0$ there exists $\delta(t_0, \delta) \geq \delta$, continuous in $t_0$ for each $\delta$ such that $||e_1 - e_2|| \leq \delta(t_0, \delta)$ implies $||x(t) - y(t)|| < \delta$ for $t \geq t_0$.

(iv) If $\delta$ in (iii) is independent of $t_0$, equation (1) or (2) is said to be uniformly stable with respect to the equation (2) or (1).

(v) Equation (1) or (2) is said to be quasiequilibrium norm bounded with respect to equation (2) or (1) if for each $\delta > 0$ and $t_0 \geq 0$ there exists $\xi > 0$ and $T(t_0, \delta, \xi)$ such that $||e_1 - e_2|| \leq \delta$ implies $||x(t) - y(t)|| < \xi$ for $t > t_0 + T(t_0, \delta, \xi)$.

(vi) If $T$ in (v) is independent of $t_0$, equation (1) or (2) is said to be quasiuniform equilibrium norm bounded with respect to equation (2) or (1).
(vii) When (I) and (v) hold simultaneously equation (1) or (2) is said to be equi ultimately - norm - bounded with respect to (2) or (1).

(viii) When (ii) and (vi) hold simultaneously equation (1) or (2) is said to be uniform ultimately - norm - bounded with respect to (2) or (1).

(ix) Equation (1) or (2) is said to be quasi - equi - asymptotically stable with respect to equation (2) or (1) if for each $\epsilon > 0$, $\delta > 0$ and $t_0 \geq 0$ there exists a positive number $T(t_0, \epsilon, \delta)$ such that $||e_1 - e_2|| \leq \delta$ implies $||x(t) - y(t)|| < \epsilon$ for $t > t_0 + T(t_0, \epsilon, \delta)$.

(x) If $T$ in (ix) is independent of $t_0$ equation (1) is said to be quasi - uniform - asymptotically stable with respect to (2) or (1).

(xi) If (iii) and (ix) hold simultaneously equation (1) or (2) is said to be equi - asymptotically stable with respect to (2) or (1).

(xii) If (iv) and (x) hold simultaneously equation (1) or (2) is said to be uniform - asymptotically - stable with respect to equation (2) or (1).

Suppose $g(t, y(t), y(h_1(t)), y(h_2(t))) \equiv 0$ and that $y \in S$ where $S$ is a nonempty set in $\mathbb{R}^n$. 
Let $d(x, S)$ be the distance between a point $x$ and the set $S$ defined by

$$
d(x, S) = \inf \left\{ \| x - y \| ; \ y \in S \right\}
$$

that is our results include the stability and boundedness of a set.

Corresponding to the definitions given above we may reformulate the definitions (ia) to (xiiia) for the scalar differential equation (7) also. That is

(ia) Equation (7) is said to be equibounded if for each $\delta > 0$ and $t_0 \geq 0$ there exists a positive function $\varepsilon(t_0, \delta)$ continuous in $t_0$ for each $\delta$ such that

$$r_0 < \delta \text{ implies } r(t) < \varepsilon(t_0, \delta) \text{ for } t \geq t_0$$

similarly the conditions (iia) to (xiiia) may be reformulated we suppose that

(xiii) There is a function $b(r)$ which is continuous and non-decreasing in $r$, $b(r) > 0$ for $r > 0$ such that

$$b(\| x - y \|) \leq V(t, x, y)$$

(xiv) $b(r) \to \infty$ as $r \to \infty$

Theorem 1: Suppose the assumptions of lemma 1 held together with (xiii) and (xiv). Suppose the scalar differential equation (7) satisfies the conditions (ia), (iia), (iiia) or (iva), then the equations (1) and (2) satisfy the corresponding one of the conditions (i), (ii), (iii) and (iv).
Proof of theorem 1: Suppose the differential equation (1) has the property (la). Then corresponding to \( \delta > 0 \) and \( t_0 \geq \delta \) there exists a \( \varepsilon(t_0, \delta) > 0 \) that is continuous in \( t_0 \) for each \( \delta \) such that \( r_0 < \delta \) implies

(15) \[ r(t) < \varepsilon(t_0, \delta) \quad \text{for} \quad t \geq t_0 \]

Since \( b(r) \to \infty \) as \( r \to \infty \), there exists a \( k = k(t_0, \delta) \) such that

(16) \[ b(k) > \varepsilon(t_0, \delta) \]

Let \( x(t) \) and \( y(t) \) be any two solutions of (1) and (2) respectively, with the initial functions \( \theta_1 \) and \( \theta_2 \) on \([\alpha, t_0] \), \( \alpha < t_0 \), such that \( \psi(t_0, \theta_1, \theta_2) \leq r_0 \leq \delta \).

From the condition (xiii) we have \( \| \theta_1 - \theta_2 \| \leq b^{-1}(\delta) \) where \( b^{-1} \) is the inverse function of \( b \). From the lemma we have \( \psi(t, x(t), y(t)) \leq r(t) \) for \( t \geq t_0 \). Suppose that there exists solutions \( x(t) \) and \( y(t) \) of (1) and (2) respectively for which \( \| \theta_1 - \theta_2 \| \leq b^{-1}(\delta) \) satisfy \( \| x(t_1) - y(t_1) \| = k \) for some \( t = t_1 > t_0 \). Then from (xiii), (15) and (16) we obtain the inequality

\[ b(k) < \psi(t_1, x(t_1), y(t_1)) \leq r(t_1) < \varepsilon(t_0, \delta) < b(k) \]

which is a contradiction.

Hence \( \| \theta_1 - \theta_2 \| \leq \delta \) implies

\( \| x(t) - y(t) \| < k(t_0, \delta) \) for \( t \geq t_0 \)

This proves the condition (1). It is easy to prove the
condition (ii) from the condition (iia). By arguments similar to the above we can easily show that the conditions (iii) and (iv) follow from conditions (iia) and (iva) respectively. Proof of the theorem 1 is complete.

Remarks: Other stability and boundedness properties of the neutral functional differential equations (1) and (2) follow from the corresponding stability and boundedness properties of the scalar differential equation (7).

We extend the above results to perturbed systems

(17) \[ \dot{x}(t) = f(t,x(t),x(h_1(t)),x(h_2(t)))+F(t,x(t),x(h_1(t)),x(h_2(t))) \]

and

(18) \[ \dot{y}(t) = g(t,y(t),y(h_1(t)),y(h_2(t)))+G(t,y(t),y(h_1(t)),y(h_2(t))) \]

For each \( t \in [t_0, \infty) \) and \( x, y \) defined over \( \mathbb{R}^1 \), let the perturbations \( F \) and \( G \) satisfy the condition

(19) \[ \|F(t,x(t),x(h_1(t)),x(h_2(t)))\| + \|G(t,y(t),y(h_1(t)),y(h_2(t)))\| \leq k_2 V(t,x(t), y(t)) \quad (k > 0) \]

If the solutions of (17) and (18) satisfy the conditions (i) to (xii) for all perturbations \( F \) and \( G \) which satisfy the inequality (19), then the equations (1) and (2) are said to satisfy the conditions (i) to (xii), totally or under constantly acting perturbations.

Theorem 2: Let the assumptions of lemma hold except that
the condition (8) is replaced by

\[
\lim_{h \to 0} \sup \frac{1}{h} \left[ V(t+h,x(t)+hf,y(t)+hg)-V(t,x(t),y(t)) \right] \\
+ k \left[ V(t,x(t),y(t)) \right] \leq W(t,V(t,x(t),y(t)))
\]

where \( k = k_1, k_2, k_1 \) being the Lipschitz constant when \( V(t,x,y) \) satisfied the Lipschitz condition. Suppose also that the assumptions (xiii) and (xiv) are satisfied. Then if the scalar differential equation (7) satisfies one of the conditions (iia) and (iv) equations (1) and (2) satisfy totally or under constantly acting perturbations, the corresponding conditions (i) and (ii).

Theorem 3: Let the assumptions of theorem 2 hold. If the scalar differential equation (7) satisfies one of the conditions (iia) and (iv), then the equations (1) and (2) satisfy totally or under constantly acting perturbations, the corresponding one of the conditions (iii) and (iv).

Proof of theorems 2 and 3: Let \( x, y \) be defined over \( D^1 \) since \( V(t,x,y) \) satisfies the Lipschitz condition in \( x \) and \( y \)

\[
V(t+h,x(t)+hf,y(t)+hg) + G(t,y(t),y(h_1(t)),y(h_2(t)))) \\
+ hF(t,x(t),x(h_1(t)),x(h_2(t)))) + y(t) \\
+ hG(t,y(t),y(h_1(t)),y(h_2(t)))) + g(t,y(t),y(h_1(t)),y(h_2(t)))) \\
- V(t,x(t),y(t)) \leq hk[H(t,x(t),x(h_1(t)),x(h_2(t))))] \\
+ g(t,y(t),y(h_1(t)),y(h_2(t)))) \\
+ V(t+h,x(t)+hf,y(t)+hg) - V(t,x(t),y(t))]
\]

\[
y(t)+hg(t,y(t),y(h_1(t)),y(h_2(t)))) - V(t,x(t),y(t)) \
\]
From (19) and (20) we obtain the inequality

\[
\lim_{h \to 0} \sup_{t \in [a, b]} \left[ \frac{1}{h} \left( y(t+h, x(t), x(h_{1}(t)), \dot{x}(h_{2}(t))) + hf(t, x(t), x(h_{1}(t)), \dot{x}(h_{2}(t)))
\right.
\]
\[
+ y(t) + hg(t, y(t), y(h_{1}(t)), \dot{y}(h_{2}(t)))
\]
\[
+ hg(t, y(t), y(h_{1}(t)), \dot{y}(h_{2}(t))) - v(t, x(t), y(t)) \right]
\]
\[
\leq w(t, v(t, x(t), y(t)))
\]

If \( x(t) \) and \( y(t) \) are any two solutions of (17) and (18) with their initial functions \( \vartheta_{1} \) and \( \vartheta_{2} \) on \( [a, t_{0}] \), where \( a < t_{0} \) such that \( v(t_{0}, \vartheta_{1}, \vartheta_{2}) \leq r_{0} \), we can obtain the desired results by applying directly the arguments used in the lemma and theorem 1. Proof of theorems 2 and 3 are complete.
Neutral functional differential equations have been discussed by Bellman and Cooke [45] in the case of constant delays, Driver [45*] in the case of variable delays. The works of other authors can be known from the references given in [45*].

Neutral functional differential equations are the differential equations which express $\dot{x}(t)$ as a function of present and past values of $x$ and past values of $\dot{x}$. In this paper we consider the stability and boundedness properties of the solutions of the neutral functional differential equations

\begin{align}
(1) \quad \dot{x}(t) &= f(t, x(t), x(h_1(t)), \dot{x}(h_2(t))) \quad \text{for } t > t_0 \quad \text{with} \\
&\quad x(t) = \xi_1(t) \quad \text{on } [-\alpha, t_0] \\
(2) \quad \dot{y}(t) &= g(t, y(t), y(h_1(t)), \dot{y}(h_2(t))) \quad \text{for } t > t_0 \quad \text{with} \\
&\quad y(t) = \xi_2(t) \quad \text{on } [-\alpha, t_0]
\end{align}

where $x, y, f$ and $g, \xi$ are dimensional vectors, $h_1$ and $h_2$ dimensional and $p$ dimensional vectors respectively are the given functions such that for some $\alpha < t_0$, $\alpha \leq h_{1j}(t) \leq t$ \((j = 1, \ldots, m)\) and $\alpha \leq h_{2k}(t) < t$ \((k = 1, 2, \ldots, p)\).

$\dot{x}(h_{2k}(t))$ represents $\frac{d}{ds} \xi_2(s)$ evaluated at $s = h_{2k}(t)$.

If $\alpha = -\infty$, the interval of the form $[-\alpha, --]$ should be interpreted as $(--, --)$. 

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Suppose \( x(t) \) and \( y(t) \) are any two solutions of (1) and (2) respectively. We use the following norm in our paper:

\[
\| x(t) - y(t) \| = |x(\alpha)| + \int_{\alpha}^{\beta} |\dot{x}(s) - \dot{y}(s)| \, ds
\]

However, in the systems (1) and (2) the retarded arguments will depend upon the dependent variables as well as the independent variables. In such cases (1) and (2) will be of the form

\[
\begin{align*}
\dot{x}(t) &= f(t, x(t), x(t, h_1(t)), \dot{x}(t, h_2(t))) \\
\dot{y}(t) &= g(t, y(t), y(t, h_1(t)), \dot{y}(t, h_2(t)))
\end{align*}
\]

Given an initial function \( \alpha_1(t) \) on \( [\alpha, t_0] \) where \( \alpha < t_0 \), a solution \( x(t) \) of (1) is any function \( x(t_0, \alpha_1) \) satisfying the conditions:

(i) \( x(t_0, \alpha_1) \) is defined and continuous in \( [\alpha, \beta] \), where \( \beta > t_0 \) such that \( (t, x(t), x(g(t))) \in D \) for \( t_0 \leq t < \beta \)

(ii) \( x(t_0, \alpha_1) = \alpha_1 \) on \( [\alpha, t_0] \)

(iii) \( \dot{x}(t) = f(t, x(t), x(h_1(t), \dot{x}(h_2(t))) \) for almost all \( t \in (t_0, \beta) \).

\( D \) in (i) is defined in the previous section. We follow the notations given the previous section. The solution \( y(t_0, \alpha_2) \) of (2) is similarly defined.
More effective and stronger results can be obtained by considering neutral functional differential inequalities of the type

\[ |\dot{x} - f(t, x(t), x(t, h_1(t)), \dot{x}(t, h_2(t)))| \leq w(t, |x(t)|) \]

and

\[ |\dot{y} - g(t, y(t), y(t, h_1(t)), \dot{y}(t, h_2(t)))| \leq w(t, |y(t)|) \]

where \( w(t,r) \) is a scalar function defined and continuous on \( I \times I \), or of the inequalities of the type

\[ |\dot{x} - f(t, x(t), x(t, h_1(t)), \dot{x}(t, h_2(t)))| \leq \epsilon(t) \]

on the interval \( I - s \) where \( I \) denotes the interval \( 0 \leq t_0 \leq t < \infty \) and \( s \) is a countable subset and

\[ |\dot{y} - g(t, y(t), y(t, h_1(t)), \dot{y}(t, h_2(t)))| \leq \epsilon(t) \]

on the interval \( I - s \), \( \epsilon(t) \) being a continuous function of \( I \). We may take \( \epsilon \) as a constant also.

The study of inequalities yield better results of qualitative nature than that of equations and they play a very great role in the qualitative theory of differential equations, partial differential equations, differential difference equations, integro - differential equations and in functional differential equations in general.

In the metric used in (3), if \( x(t) \) and \( y(t) \) are any
two solutions of the same equation (1), we can use the
cauchy-whether existence theory of ordinary differential
equations elaborately presented by Coddington and Leven-
sion [13] to prove the existence of an absolutely continuous
solution \( x(t) \) which satisfies (1) almost every where.

Definition 1: The solution \( x = o \) of equation (1) is
called stable if for every positive \( \epsilon > 0 \) we can find
a positive number \( \delta > 0 \) such that whenever the inequality

\[ \| e_1 \| \leq \delta \]

is satisfied, the relation

\[ \| x(t, t_0, \epsilon) \| < \epsilon \]

holds for all \( t \geq t_0 \)

Definition 2: If whenever definition (1) is satisfied, the
conditions

\[ \lim_{t \to \infty} \| x(t, t_0, \epsilon_1) \| = 0 \]

\[ \| x(t, t_0, \epsilon_1) \| < H_1 \text{ for all } t \geq t_0 \]

\((H_1 = \text{constant})\)

are satisfied for all initial functions \( \epsilon_1(\text{vector}) \) satisfy-
ing the inequality,

\[ \| \epsilon_1 \| \leq H_0 \]

\((H_0 = \text{constant})\)

then the solution \( x = o \) of equation (1) is asymptotically
stable and the region (11) lies in the region of attraction
of the unperturbed motion.

From the formal point, definitions 1 and 2 are the
most natural extension of Lyapunov's definitions of stability and asymptotic stability to equations of the type (1). 

In order to unify our results on stability and boundedness we give the following conditions. We assume hereafter that the solutions $x(t_0, \theta_1, t)$, $y(t_0, \theta_2, t)$ of (1) and (2) exists for all $t \geq t_0$ with the initial functions $\theta_1$ and $\theta_2$ at $t = t_0$. Here $\theta_1$ and $\theta_2$ are $n$-vector functions.

(i) For each $\delta > 0$ and $t_0 \geq 0$ there exists a positive function $\varepsilon(t_0, \delta)$ that is continuous in $t_0$ for each $\delta$ such that

$$
\| \theta_1 \| < \delta \text{ implies } \\
\| x(t_0, \theta_1, t) \| < \varepsilon(t_0, \delta)
$$

for all $t \geq t_0$

(ii) The $\varepsilon$ in (i) is independent of $t_0$

(iii) For each $\alpha > 0$ and $t_0 \geq 0$ there exists a positive function $\beta(t_0, \alpha)$ that is continuous in $t_0$ for each $\alpha$ and such that $\| \theta_1 \| < \beta(t_0, \alpha)$ implies

$$
\| x(t_0, \theta_1, t) \| < \alpha \text{ for all } t \geq t_0
$$

(iv) The $\beta$ in (iii) is independent of $t_0$.

(v) For each $\delta > 0$ and $t_0 \geq 0$ there exists positive
numbers k and T(t₀, s) such that

\[ ||x(t, t₀, e₁)|| < k \]

whenever

\[ ||e₁|| ≤ δ \text{ and } t > t₀ + T(t₀, s) \]

(vi) The T in (v) is independent of t₀.

(vii) For each a > 0, B > 0 and t₀ ≥ 0 there exists a positive number T(t₀, a, B) such that

\[ ||x(t, t₀, e₁)|| < a \]

whenever

\[ ||e₁|| ≤ B \text{ and } t > t₀ + T(t₀, a, B) \]

(viii) The T in (vii) is independent of t₀.

In the definitions above, the solution x(t, t₀, e₁) of (1) may be replaced by the solution y(t, t₀, e₂) of (2) in which e₂ is the initial function.

Let r(t) be the maximal solution of a scalar differential equation.

\[ \dot{x} = w(t, r) \]

where w(t, r) is defined and continuous on I x I where

I : 0 ≤ t < ∞ and I : 0 ≤ r < ∞.

Theorem 1 :- Suppose the functions f(t, x(t), x₁(t), x₂(t)) and g(t, y(t), y₁(t), y₂(t)) of (1) and (2) defined
over a common closed region, satisfy the condition

\[(14) \| f(t, x(t), x(h_1(t)), \dot{x}(h_2(t))) - g(t, y(t), y(h_1(t)), \dot{y}(h_2(t))) \| \leq w(t, \| x(t) - y(t) \|) \]

Then the solutions \( x(t) \) and \( y(t) \) of (1) and (2) with their initial functions \( a_1 \) and \( a_2 \) satisfy the relation

\[(15) \quad \| x(t) - y(t) \| \leq r(t) \quad \text{for} \quad t \geq t_0 \]

whenever \( \| e_1(t) - e_2(t) \| \leq r_0 \)

where \( r_0 = r(t_0) \)

Proof of theorem 1 :- The proof is very simple and is quite analogous to a corresponding one proved in the case of ordinary differential equations.

Theorem 2 :- Suppose \( x_1(t) \) and \( x_2(t) \) are the continuous functions which satisfy the equation (1). Suppose the function \( f(t, x(t), x(h_1(t)), \dot{x}(h_2(t))) \) satisfied the inequality

\[(16) \quad \| f(t, x_1(t), x_1(h_1(t)), \dot{x}(h_2(t))) - f(t, x_2(t), x_2(h_1(t)), \dot{x}_2(h_2(t))) \| \leq w(t, \| x_1(t) - x_2(t) \|) \]

where \( x_1(t) \) and \( x_2(t) \) are the continuous vector functions, then, if \( x_1(t) \) and \( x_2(t) \) are any two solutions of (1) with the same initial function \( a_1(t) \) in the interval \([-\alpha, t_0]\).
\[ \| x_1(t) - x_2(t) \| \leq r(t) \]

where \( r(t) \) is the maximal solution of

\[ \dot{r} = w(t, r) \]  

through \((t_0, 0)\).

The proof of theorem 2 is:

Suppose \( x_1(t) \) and \( x_2(t) \) are any two solutions of (1) with the same initial function \( \theta_1(t) \) in the interval \([a, t_0]\) then using the norm defined in (3) we have

\[ \| x_1(t) - x_2(t) \| = \int_{t_0}^{t} \| f(t, x_1(t), x_1(h_1(t)), x_2(h_2(t))) - f(t, x_2(t), x_2(h_1(t)), x_2(h_2(t))) \| dt \]

\[ \leq \int_{t_0}^{t} w(t, \| x_1(t) - x_2(t) \|) dt \]

If \( r(t) = \| x_1(t) - x_2(t) \| \) then

\[ r(t) \leq \int_{t_0}^{t} w(s, r(s)) ds \]

Hence

\[ \| x_1(t) - x_2(t) \| \leq r(t) \quad \text{for } t \geq t_0 \]

where \( r(t) \) is the maximal solution of \( \dot{r} = w(t, r(t)) \).

The proof of theorem 2 is complete.

Theorem 3:

Suppose the assumptions in theorem 2 hold except that the initial functions of the solutions \( x_1(t) \) and \( x_2(t) \) are \( \theta_1(t) \) and \( \theta_2(t) \) respectively in the
interval $[a, t_0]$.

Also suppose $\| \varphi_1(t) - \varphi_2(t) \|$ is bounded then $\| x_1(t) - x_2(t) \|$ is bounded provided the maximal solution of $\dot{r} = w(t, r)$ is bounded.

Proof of theorem 3: - Proof of theorem 3 is a direct consequence of the proof of theorem 2.

The following lemma is useful to prove the next theorem.

Lemma: - Let $\phi(t), \alpha(t), \beta(t)$ be real valued continuous functions in an interval $I: a \leq t \leq b$. Let the function $w(t, r)$ be continuous and defined on $I \times \mathbb{R}^+$ where $\mathbb{R}^+ = [0, \infty)$. Suppose

$$\phi(t) \leq \alpha(t) + \int_a^b w(s, \alpha(s)) \, ds$$

for $a \leq t \leq b$.

then

$$\phi(t) \leq \alpha(t) + r(t) \text{ on } I$$

where $r(t)$ is the maximal solution of $\dot{r} = w(t, r)$.

Proof of the lemma: - Let

$$R(t) = \phi(t) - \alpha(t)$$

$$R(t) \leq \int_a^b w(s, \alpha(s)) \, ds$$

$$= \int_a^b w(s, R(s) + \alpha(s)) \, ds$$
If \( r(t) \) is the maximal solution of \( \dot{z} = w(t, z + \alpha(t)) \), then \( R(t) \leq r(t) \). This yields the desired result.

Remarks - \( \alpha(t) \) may be taken to be a constant.

Theorem 4: Let the assumptions in theorem 3 hold except that the initial functions of the solutions \( x_1(t) \) and \( x_2(t) \) of (1) are \( \phi_1(t) \) and \( \phi_2(t) \) respectively in the interval \([a, t_0]\). Suppose

\[
|| \phi_1(t_0) - \phi_2(t_0) || \leq \delta
\]

then the solutions \( x_1(t) \) and \( x_2(t) \) satisfy the inequality

\[
|| x_1(t) - x_2(t) || \leq r(t)
\]

where \( r(t) \) is the maximal solution of

\[
\dot{x}(t) = w(t, r(t)) \quad \text{for } t \geq t_0
\]

with \( r(t_0) = \delta \)

Proof of Theorem 4: If \( x_1(t) \) and \( x_2(t) \) are the continuous functions which satisfy the equation (1), then from the norm given in (3) we have

\[
|| x_1(t) - x_2(t) || = || x_1(t_0) - x_2(t_0) ||
\]

\[
+ \int_{t_0}^{t} || f(t, x_1(t), x_1(h_1(t)), x_1(h_2(t))) - f(t, x_2(t), x_2(h_1(t)), x_2(h_2(t))) || \, ds
\]

\[
\leq \delta + \int_{t_0}^{t} w(s, || x_1(s) - x_2(s) ||) \, ds
\]

Applying the above lemma we obtain the result.
The following theorems are the obvious consequences of the above results.

Theorem 5: Let the assumptions of theorem 1 hold. Suppose the only solution \( r(t) \) of \( \dot{r} = w(t, r) \) on \( t_0 \leq t < \infty \) such that \( r(t_0) = \hat{r}(t_0) = 0 \) is the trivial solution. Then there exists at most one and the same solution for both the systems, (1) and (2) on \( t_0 \leq t < \infty \).

Theorem 6: Let the assumptions of theorem 4 hold. Suppose the only solution \( r(t) \) of \( \dot{r} = w(t, r) \) on \( t_0 \leq t < \infty \) such that \( r(t_0) = \hat{r}(t_0) = 0 \) is the trivial solution. Then there exists at most one solution for the system (1) on \( t_0 \leq t < \infty \).

Remarks: Many results are obtained in the ordinary differential equations, differential-difference equations, integral equations, integro-differential equations and functional differential equations in general by using the comparison technique, comparing the solutions of each of these systems with those of the corresponding scalar equations. The properties of the solutions of a functional differential system are studied by comparing them with those of the corresponding ordinary scalar differential equation. It is interesting to put a question whether it is possible to replace the ordinary scalar comparison differential equation by a scalar comparison functional differential equation and then study the properties
of the solutions of the functional differential system, comparing them with those of the corresponding scalar functional differential equation. Laxmi Kantham in [29c] has proved the existence of the maximal solution for scalar functional differential equations.

Theorem 6 in [29c] can be stated in the following manner.

Theorem 6: Let \( m(t) \geq 0 \) be defined and continuous on \( [t_0 - \alpha, \infty) \) and satisfy the inequality

\[
D^- m(t) \leq w(t, m_t) \quad t > t_0
\]

where the functional \( w(t, \theta) \) is defined and continuous on \( I \times C^+_H, C^+_H \) being the set of all non-negative continuous functions belonging to \( C^+_H \). Assume that \( w \) is non-decreasing in \( \theta \) for \((t, \theta)\) and that \( r(t_0, \theta), \theta \in C^+_H \) is the maximal solution of

\[
r(t) = w(t, r_t)
\]

existing for \( t \geq t_0 \) then \( m_{t_0} \leq \theta \) implies

\[
m(t) \leq r(t_0, \theta)(t) \quad t \geq t_0
\]

Various properties, including the stability and boundedness properties of the solutions of neutral functional differential systems can be studied by comparing the solutions of neutral functional differential systems with those of the corresponding scalar functional differential equations.
In this paper we will be concerned with the application of the concept of Lyapunov Functionals to determine the stability of autonomous Neutral Functional Differential systems of the form

\[ \dot{x}(t) = f(x(t), x(t-\alpha_1), x(t-\alpha_2), \ldots, x(t-\alpha_n), x(t), \dot{x}(t-\alpha_1), \ldots, \dot{x}(t-\alpha_n)) \]

where \( \alpha_i \) are positive constants and the derivatives are the right hand derivatives here and in the sequel.

Many important and useful results in this direction have been obtained by Lasalle for ordinary differential equations and by Hale for functional differential equations. As has been suggested by Lasalle - by employing the compact open topology on the space of the continuous functions it is possible to extend the results to infinite time lag.

The ideas of Lasalle, Hale, Yoshizawa produces some new results for dynamical systems defined by ordinary differential equations which demonstrates the essential nature of a Lyapunov Function and which may be useful in applications. Of greater importance is the possibility that these ideas can be extended to more general classes of dynamical systems. It may be possible to do this for some special cases.

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type of dynamical systems defined by partial differential equations.

Whenever the limit sets of solutions are known to have an invariance property then sharper results can be obtained. The principle of invariance in the theory of stability has drawn the attention of many differential equationists. It had its origin for autonomous and periodic systems \([26], [53]\). Miller in \([39]\) has established an invariance property for almost periodic systems and obtains thereby a similar stability theorem for almost periodic systems. Little attention has been paid to theorems which makes possible estimates of regions of attraction (regions of asymptotic stability) for non-autonomous systems.

2. First we introduce the following notation.

\(\mathbb{R}^n\) denotes the real euclidean space of \(n\) vectors. 
\(|x|\) denotes the norm of the vector \(x\) in \(\mathbb{R}^n\). For \(r \geq 0\), 
\(C = C([-r, 0], \mathbb{R}^n)\) denotes the space of continuous functions with domain \([-r, 0]\) and range in \(\mathbb{R}^n\). The norm in this space will be the uniform one 
\(||\alpha|| = \max_{-r \leq \delta \leq 0} |\alpha(\delta)|\)

for \(\alpha\) in \(C\). Suppose \(x\) is any given function with domain \([-r, \infty)\) and range in \(\mathbb{R}^n\). For any \(t \geq 0\), we will let \(x_t\) denote a translation of the restriction of \(x\)
to the interval \([t - r, t]\). To be more specific -
\(x_t\) is an element of \(C\) defined by \(x_t(e) = x(t + a)\)
\(-r \leq e \leq 0\). That is the graph of \(x_t\) is the graph of \(x\)
on \([t - r, t]\) shifted to the interval \([-r, 0]\).

For a given positive constant \(H\) we define a subset
\(C_H\) of \(C\) as a set \(\alpha\) in \(C\) : \(\|\alpha\| < H\) this is \(C_H\) is the
open ball in \(C\) of radius \(H\).

We consider the following autonomous neutral functional
differential equation

\[(2) \quad \dot{x}(t) = f(x_t, \dot{x}_t) \quad t \geq 0\]
where \(f(\alpha, \dot{\alpha})\) is a function defined for every \(\alpha, \dot{\alpha}\) in
\(C_H\) and \(\dot{x}(t)\) and \(\dot{x}_t\) are the right hand derivatives.

\(x(\alpha)\) is a solution of (2) with initial function \(\alpha\) in
\(C_H\) at \(t = 0\) if there is a \(k > 0\) such that \(x(\alpha)\) is a
function from \([-r, k]\) into \(E^n\) such that \(x_t(\alpha)\) is
in \(C_H\) for \(0 \leq t < k\), \(x_o(\alpha)\) and \(x(\alpha)(t)\) satisfied
(1) for \(0 \leq t < A\). The existence, uniqueness and continuous
dependence upon initial functions of the solutions of more
general neutral functional differential equations have been
discussed in more detail by Driver in \([45\text{a}]\) in a manner
quite analogous to that used for ordinary differential equations.

If \(f(\alpha, \alpha) = \alpha\) then the solution \(x = \alpha\) of (1) is
said to be stable if for every $\epsilon > 0$ there is a $\delta > 0$ such that $||x|| > \delta$ implies $x(\alpha)$ exists for $t \geq 0$ is in $C_H$ and $||x(\alpha)|| < \epsilon$ for all $t \geq 0$. If there is an $H_1 < H$ such that $||x|| < H_1$ implies $x_t(\alpha)$ is in $C_H$ for $t \geq 0$ and $x_t(\alpha) \to 0$ as $t \to \infty$ then the solution $x = 0$ is said to be asymptotically stable.

Let $V(x_t(\alpha))$ be a continuous scalar function on $C_H$. Its derivative $\dot{V}(2)(\alpha)$ along the solutions of (2) is as follows

\begin{equation}
\dot{V}(2)(\alpha) = \lim_{h \to 0} \frac{1}{h} \left[ V(x_h(\alpha)) - V(\alpha) \right]
\end{equation}

when $H = \infty$ then $C_H = C$ and we will have global stability in which case $H_1 = \infty$.

We define the $\omega$-limit set of $\alpha$ as follows. An element $\beta$ of $C$ is in $\mathcal{U}(\alpha)$, the $\omega$-limit set of $\alpha$ if $x(\alpha)$ is defined on $[-T, \infty)$ and there is a sequence of non-negative real numbers $t_n$, $t_n \to \infty$ as $n \to \infty$ such that $||x_{t_n}(\alpha) - \beta|| \to 0$ as $n \to \infty$.

A set $M$ in $C$ is called an invariant set if for any $\alpha$ in $M$ there exists a function $x$ depending on $\alpha$ defined on $(-\infty, \infty)$, $x_t$ in $M$ for $t$ in $(-\infty, \infty)$, $x_0 = \alpha$ such that if $x^*(\theta, x_0)$ is the solution of (1) with initial value $x_0$ at $\theta$ then $x^*(\theta, x_0) = x_t$ for all $t \geq \theta$. 

\[ \text{Shahware} \]
To any element of an invariant set there corresponds a solution which must be defined on \((-\infty, \infty)\). The following lemmas are natural extensions of the corresponding results obtained by Hale for functional differential equations.

**Lemma 1**: If \( x(t) \) is a solution of (1) with initial function \( \alpha \) at 0 defined on \([-r, \infty)\) and \( \| x_t(\alpha) \| \leq M_1 \) for all \( t \) in \([-r, \infty)\) then the family of functions \( x_t(\alpha); \ t \geq 0 \) belongs to a compact subset of \( C \).

**Lemma 2**: If \( \alpha \) in \( C_H \) is such that the solution \( y(\alpha) \) of system (1) with initial function \( \alpha \) at 0 is defined on \([-r, \infty)\) and \( \| x_t(\alpha) \| \leq M_1 < H \) for \( t \) in \([-r, \infty)\) then \( U(\alpha) \) is a nonempty compact, connected, invariant set and

\[ \text{Dist}(x_t(\alpha), U(\alpha)) \to 0 \quad \text{as} \quad t \to \infty. \]

Proofs of these lemmas are analogous to those given by Hale in [20*] for functional differential equations.

**Remark**: \( x_t(\alpha) \) is continuous in \( t, \alpha \) and belongs to a compact subset of \( C \). Therefore the Lipschitz condition of \( f \) could have been replaced by \( \| \alpha \| \leq M \), \( \| f(\alpha) \| \leq L \) for some \( L \).

**Theorem 1**: Let \( V \) be a continuous scalar function on \( C_H \). If \( \mathcal{P} \) designates the region where \( V(\alpha) < p \). Suppose there exists a non-negative constant \( k \) such that

\[ \| \alpha(0) \| \leq k, \quad V(\alpha) \geq 0 \quad \text{and} \quad \hat{V}(\alpha) \leq 0 \quad \text{for all} \quad \alpha \in \mathcal{P}. \]
$\Sigma_p$. If $E$ is the set of all points in $\Sigma_p$ where 
$\dot{V}(2)(\alpha) = 0$ and $M$ is the largest invariant set in $E$, 
then every solution of (2) with initial values in $\Sigma_p$ 
approaches $M$ as $t \to \infty$.

Proof of theorem 1 :- From the assumptions in the theorem 
we have that $V(x_t(\alpha))$ is a non-increasing function of $t$ 
and $V(x_t(\alpha))$ is bounded below, with in $\Sigma_p$. Hence $\alpha$ in 
$\Sigma_p$ implies $x_t(\alpha)$ in $\Sigma_p$ and $\| x(\alpha)(t) \| \leq k$ for $t \geq 0$ 
which implies $\| x_t(\alpha) \| \leq k$ for $t \geq 0$ i.e. $x_t(\alpha)$ is 
bounded and lemma 2 implies $U(\alpha)$ is an invariant set.

But $V(x_t(\alpha))$ has a limit $p_0 < p$ as $t \to \infty$ and $V = p_0$ 
on $U(\alpha)$. Hence $U(\alpha)$ is in $\Sigma_p$ and $\dot{V}(2) = 0$ in $U(\alpha)$.

Therefore, $U(\alpha)$ invariant implies $U(\alpha)$ is in $M$ and 
lemma 2 implies $x_t(\alpha) \to M$ as $t \to \infty$.

This completes the proof of theorem 1.

Corollary 1 :- Let the assumptions of theorem 1 hold and 
$\dot{V}(2)(\alpha) < 0$ for all $\alpha \neq \alpha$ in $\Sigma_p$ then every solution 
of (1) with initial values in $\Sigma_p$ approaches $\alpha$ as $t \to \infty$.

Theorem 2 :- Suppose $f(\alpha, \alpha)$ and there exists a function 
$a(s)$ that is continuous and increasing for $0 \leq s < t$ with 
a(0) = 0. Also suppose there is a continuous scalar function 
$V(\alpha)$, $V(\alpha) = 0$ defined on $C_\alpha$ such that
(5) \[ a(||x(o)||) \leq V(o) \]

(6) \[ \dot{V}(o) \leq 0 \]

for all \( o \) in \( C_H \). Then the solution \( x = o \) of (1) is stable.

Also if the initial value \( o \) satisfies \( V(o) < p_o \), \( p_o = \lim_{r \to H} a(r) \) and the only invariant set in \( \dot{V}(o) = e \) is \( o \) then the solution \( x = o \) of (1) is asymptotically stable and every solution of (1) approaches \( o \) as \( t \to \infty \).

Proof of theorem 2: We can find a continuous non-decreasing function \( b(r) \) for \( r \leq o \) sufficiently small such that \( b(o) = 0 \) and \( V(o) \leq b(||x||) \) for \( ||o|| \) sufficiently small.

For any \( \varepsilon, o < \varepsilon < H \), choose \( s < \varepsilon \) small enough such that \( b(s) < a(o) \).

If \( o \) is in \( C_\delta \) then \( V(x_t(o)) \) non-increasing implies.

(7) \[ a(||x(o)(t)||) \leq V(x_t(o)) \leq b(o) < a(o) \] for \( t \geq o \).

Therefore \( ||x(o)(t)|| < \varepsilon \) for \( t \geq o \) and hence

\[ ||x_t(o)|| < \varepsilon \] for \( t \geq o \).

From this it follows that \( x = o \) is stable.

Since \( a \) is increasing, the set \( \Sigma_p \) of \( o \) for which \( V(o) < p \) satisfies the conditions of theorem 1 if \( p < p_o \).
By corollary 1 every solution of (1) approaches zero as \( t \to \infty \). This completes the proof of theorem 2.

Corollary 2: Suppose \( f(0,0) = 0, V(\alpha) \) satisfied the condition (5). Suppose there exists a continuous, non-negative and non-decreasing function \( C(r) \) on \([-o, o]\) such that

\[
(8) \quad \dot{V}_2(\alpha) \leq -C(||\alpha(0)||)
\]

then the solution \( x = 0 \) of (1) is stable and if \( C(r) > 0 \) for \( r \neq 0 \) then \( x = 0 \) is asymptotically stable.

Proof of Corollary 2: The stability follows from theorem 2. If \( C(r) > 0 \) for \( r \neq 0 \) then the largest invariant set in the set where \( \dot{V}_2 = 0 \) are such solutions of (1) for which \( ||x(t)|| = 0 \) for \( -\infty < t < \infty \). That is the solution \( x = 0 \) is asymptotically stable.

Theorem 3: Let \( C = C \) and \( V \) be a continuous scalar function on \( C \). If \( V(\alpha) \geq 0, \dot{V}_2(\alpha) \leq 0 \) for all \( \alpha \) in \( C \) and \( B \) is the set of \( \alpha \) in \( C \) for which \( \dot{V}_2 = 0 \) \( M \) is the largest invariant set in \( B \) then all solutions of (2) bounded for \( t \geq 0 \) approach \( M \) as \( t \to \infty \).

If, in addition, there exists a continuous non-negative function \( u(r) \) for \( 0 \leq r < \infty \), \( u(r) \to \infty \) as \( r \to \infty \) such that
for all \( \alpha \) in \( C \), then all solutions of (1) are bounded for \( t \geq 0 \).

Proof of theorem 3: - The first part of the theorem follows from the proof of theorem 1.

For any \( \alpha_0 \) in \( C \) there is a constant \( m \) such that \( V(\alpha) > V(\alpha_0) \) for \( || \alpha(0) || \geq m \).

Since \( V(x_t(\alpha)) \) is a non-increasing function of \( t \), it follows that \( || x(\alpha)(t) || < m \) for \( t \geq 0 \) which implies \( || x_t(\alpha) || < m \) for \( t \geq 0 \). Hence the result follows.

Corollary 3: - If \( f(\alpha, o) = o \) all the conditions of theorem 3 are satisfied and \( V(o) = 0, \dot{V}(2) < 0 \) for \( o \neq o \) the above results can be extended to non-autonomous systems. It is also possible to obtain results quite similar to those of Miller, for almost periodic neutral functional differential equations and study an invariance property of such equations.

The following is an extension of Getaev's instability theorem. As before let \( V \) be a continuous scalar function on \( C \). Let \( G \) be any set in \( E^n \). We shall say that \( V \) is a Lyapunov function on \( G \) for equation (1) if \( V(\alpha) \geq 0 \) and

\[
\dot{V}(\alpha) \leq -w(\alpha) \quad \text{for} \ t > 0 \ \text{and all} \ \alpha \ \text{in} \ \mathcal{J} \ \text{where} \ w \ \text{is continuous on} \ E^n \ \text{to} \ E^1 \ \text{and}
\]
\[ V = \frac{\partial V}{\partial \xi} + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i \]

\( \overline{G} \) is the closure of \( G \).

**Theorem 4s**: Let \( p \in C \) be an equilibrium point of (1) contained in the closure of an open set \( U \) and let \( N \) be a neighbourhood of \( p \). Suppose that

(i) \( V \) is a Lyapunov function on \( G = U \cap N \)

(ii) \( M \cap G \) is either the empty set or \( p \)

(iii) \( V(\alpha) < \beta \) on \( G \) when \( \alpha \neq p \) and \( V(p) = \beta \) and \( V(\alpha) = \delta \) on that part of the boundary of \( G \) inside \( N \).

(iv) If \( N_0 \) is a bounded neighbourhood of \( p \) properly contained in \( N \) then each trajectory starting at a point of \( G_0 = G \cap N_0 \) other than \( p \) leaves \( N_0 \) in finite time.

Proof of theorem 4s: The conditions in the theorem imply that each trajectory starting inside \( G_0 \) at a point other than \( p \) must either leave \( G_0 \), approach its boundary or approach \( p \). Conditions (i) and (iv) imply that it cannot reach or approach that part of the boundary of \( G_0 \) inside \( N_0 \) nor can it approach \( p \) as \( t \to \infty \).

Now condition (ii) states that there are no points of \( M \) on that part of the boundary of \( N_0 \) inside \( G \). Hence each such trajectory must leave \( N_0 \) in finite time.
p is either in the interior or on the boundary of \( Z \), each neighbourhood of \( p \) contains such trajectories, and \( p \) is therefore unstable.

§3. We extend the notion of an invariant set to almost periodic neutral functional differential equations and consider the interaction of this notion of invariance with the theory of Lyapunov functions. The notion of an invariant set has been widely exploited in the theory of autonomous ordinary differential equations. R.K. Miller has made a generalization of invariant sets to almost periodic systems. The present section will depend heavily on the work of Miller.

We consider the systems of \( n \) equations of the form

\begin{equation}
\dot{x}(t) = f(t, x_t, \dot{x}_t)
\end{equation}

where the prime denotes the right hand derivative here and in the sequel.

The existence, uniqueness and continuous dependence of solutions of Neutral Functional Differential Equations have been discussed by many authors - Driver [45*], Bellman and Cooke [4(b)].

Definition 1: A function \( x(t_0; x) \) is said a solution of (12) with initial function \( \alpha \) at \( t = t_0 \) if there exists a number \( A > 0 \) such that
for each $t$, $t_0 \leq t \leq t_0 + A$

Definition 2: A continuous function $f(t, \alpha, \delta)$ of the variables $t \in I$, $\alpha \in C([-r, 0] \to \Omega)$, $\delta \in C([-r, 0] \to \Omega)$ is said to be almost periodic in $t$ uniformly with respect to $\alpha \in C_H$, $\delta \in C_H$, $H > a$ if for any $a > a$ it is possible to find a $p(a)$ such that in any interval of length $p(a)$ there is a $\delta$ such that the inequality

$$||f(t + \delta, \alpha, \delta) - f(t, \alpha, \delta)|| \leq \beta$$

is satisfied for all $t \in R$, $\alpha \in C_H$, $\delta \in C_H$. ($R$ denotes the real line).

The set of frequencies of an almost periodic function $f(t)$ is the set of real numbers $m_j$ for which

$$\lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T} f(t)e^{-im_j t} dt = 0$$

the module of an almost periodic function $f(t)$ is the set of all numbers which are finite sums with integer coefficients of the set $\{m_j\}$. If $\{m_j\}$, $\{n_j\}$ are the set of frequencies of the almost periodic functions $f(t)$, $g(t)$ respectively then the module of $f(t)$, $g(t)$ is the set of all numbers which are finite sums with integer coefficients of the combined set $\{m_j, n_j\}$. If $x(t)$ is any function and $A$ any subset
of $\mathbb{R}^n$, $x(t) \to A$ as $t \to \infty$ if for each $\varepsilon > 0$ there is a $t(\varepsilon) > 0$ such that $d(x(t), A) < \varepsilon$ for all $t \geq t(\varepsilon)$ where $d(x, A)$ denote the distance from the point $x$ to the set $A$. Here $\mathbb{R}^n$ is the real $n$-dimensional Euclidean space.

If $x(t)$ is an element of $C^\infty$, the positive limit set of $x(t)$ denoted by $L_+(x(t))$ consists of all points $z$ for which there is a sequence $t_m \to \infty$ as $m \to \infty$ with $x(t_m) \to z$. Similarly, the positive limit set $N^+(x(t))$ consists of all functions $\alpha \in C^\infty$ such that there is a sequence $t_m \to \infty$ as $m \to \infty$ with $\| \alpha - x_{t_m} \| \to 0$.

If $f(t, \alpha, \tilde{\alpha})$ is almost periodic in $t$ let $\mathcal{V}(f)$ be the set of functions $G(f) = \{f(t+h, \alpha, \tilde{\alpha}), -\infty < t < \infty\}$.

If $f(t, \alpha, \tilde{\alpha})$ is uniformly almost periodic in $t$ and if $U$ is a compact subset of $C^\infty$ then $|f(t, \alpha, \tilde{\alpha})|$ is bounded on $\mathbb{R}^1 \times U$, where $\mathbb{R}^1$ is a set of all real numbers.

If $f(t, \alpha, \tilde{\alpha})$ is uniformly almost periodic in $t$ and if $h_m$ is any real sequence, then there is a subsequence $h_{mk}$ and a uniformly almost periodic function $f^*(t, \alpha, \tilde{\alpha})$ such that

$$f(t + h_{mk}, \alpha, \tilde{\alpha}) \to f^*(t, \alpha, \tilde{\alpha}) \quad \text{as} \quad k \to \infty,$$

uniformly for all $t \in \mathbb{R}^1$ and $\alpha$ and $\tilde{\alpha}$ on compact subsets $C^\infty$. $\mathbb{R}^1$ is set of all real number on a real line.
The proofs of these statements are analogous to corresponding ones given for delay differential equations [39*].

Let \( \mathcal{G}(f) \) be the uniform closure in the sense of (14) of \( \mathcal{G}(f) \).

Definition 3: If \( U \) is any subset of \( C_H \), then \( U \) is called quasi-invariant with respect to the almost periodic system \( \dot{x} = f(t, x_t, \dot{x}_t) \) if and only if for each element \( \tau \in U \) there corresponds an almost periodic function \( f^*(t, \alpha, \dot{\alpha}) \in \mathcal{W}(f) \) and a solution \( y(t) \) of \( \dot{y}(t) = f^*(t, y_t, \dot{y}_t) \) such that \( y_0 = \Theta \) and \( y_t \) exists and remains in a compact subset of \( U \) for \( -\infty < t < \infty \). Here and in the sequel the existence of the right-hand derivatives of \( x_t \) and \( y_t \) are assumed and assured.

The positive limit set \( N^*(x_t) \) is a nonempty, quasi-invariant set with respect to (12)

(15) For each compact subset \( D^* \subseteq C_H \); \( |f(t, \alpha, \dot{\alpha})| \) is bounded on \( \mathbb{R}^1 \times C([-r, 0] \rightarrow D^*) \times C([-r, 0] \rightarrow \mathbb{R}^1) \). If \( V(t, x_t) \) is a scalar function on \( I \times C_H \) we define \( \dot{V}(12) (t, x_t) \) by the following relation

\[
\dot{V}(12)(t, x_t) = \lim_{h \to 0^+} \frac{1}{h} [V(t+h, x_{t+h}) - V(t, x_t)]
\]

For \( -r \leq h \leq 0 \) we define the pseudo norm \( \| \alpha \|_{[-r, 0]} \)
on $C_H$ as $\|\alpha\|_{[h,0]} = \sup_{h \leq u \leq 0} |\alpha(u)|$

Theorem 1: Let the function $f(t, \alpha, \dot{\alpha})$ satisfies the condition that for each compact subset $\overline{D^*} \subseteq C_H$, $|f(t, \alpha, \dot{\alpha})|$ is bounded on $R^1 \times C([-r, 0]) \rightarrow D^*$.

Let $V(t, \alpha)$ be a continuous function on $I \times C_H$.

If $U_m$ designates the region $U_m = (t, \alpha) \ ; \ V(t, \alpha) < m$ suppose there exists a non-negative constant $k$ such that $\|\alpha\|_{[h,0]} \leq k$ and $V(t, \alpha) \geq 0$ and $\hat{V}(12)(t, \alpha) \leq 0$ for all $(t, \alpha) \in U_m$.

Define $\Sigma = \chi \ ; \ |\chi| \leq k \ ; \ C_H$ then all solutions which start in the set $U_m$ exist and are bounded for all future time.

Proof of theorem 1: Let $x(t)$ be a solution of $(12)$ with initial values $(t_0, x_{t_0}) \in U_m$. Since $V(t, x_t) < m$ we see that $V(t, x_t) < m$ and $x(t) \in \Sigma$ for as long as $x(t)$ exists. But $\Sigma$ is compact. It follows $x(t)$ exists and remains in $\Sigma$ for all $t \geq t_0$. Hence the theorem follows.

Theorem 2: Suppose $V(t, x_t)$ is a continuous scalar function on $R^1 \times C_H$ with $V(t, x_t) \leq 0$ for each solution $x(t)$ of $(12)$. Suppose $f(t, \alpha, \dot{\alpha})$ and $V(t, \alpha)$ are uniformly almost periodic in $t$ and satisfied the condition(15).
Let $U$ be the subset of $C_H$ consisting of all trajectories $y_t$ such that corresponding to $y_t$ there exists $f^*(t, \theta, \dot{\theta}) \in \overline{\mathcal{U}}(f)$ and $V^* \in \overline{\mathcal{U}}(V)$ such that

1. $\dot{y}(t) = f^*(t, y_t, \dot{y}_t)$ for $-\infty < t < \infty$
2. $y(t)$ has range in a compact set $D^*_y \subseteq C_H$ and
3. $\dot{V}^*(t, y_t) = 0$ for $-\infty < t < \infty$. If $x(t)$ is a solution of (12) which exists for all large $t$ and has range in a compact set $D^* \subseteq C_H$ then $x_t \to U$ as $t \to \infty$.

Proof of theorem 2:- We will prove now that the positive limit set $N^+(x_t)$ is contained in $U$.

Since the range of $x(t)$ is in the compact set $D^*$, we obtain from (15) that the set $x = x_t$; $t$ sufficiently large is equicontinuous and uniformly bounded. Since $x$ has compact closure in $C_H$, $|V(t, x_t)|$ is bounded. But $\dot{V}(t, x_t) \leq 0$ which implies that $V(t, x_t)$ is bounded, but $\dot{V}(t, x_t) \geq 0$ which implies that $V(t, x_t) = V_0$ as $t \to \infty$.

For any fixed $\theta \in N^+(x_t)$ we can find a sequence $t_n \to \infty$ as $n \to \infty$ such that $x_{tn} \to \theta$, $V(t + t_n, x_t) \to V^*(t, x_t)$ and $f(t + t_n, x_t, \dot{x}) \to f^*(t, \theta, \dot{\theta})$. The corresponding path $y_t$ satisfies (i) and (ii).

Since $V^*(t, y_t) \equiv V_0$, constant (iii) is also satisfied. Therefore $N^+(x_t) \subseteq U$. The proof of theorem 2 is complete.
Remark 1: There are also some special classes of non-autonomous systems where the limit sets of solutions have an invariance property. The simplest of these are neutral functional periodic differential systems of the form:

\[ \dot{x} = f(t, x_t, \dot{x}_t) \]

\[ f(t+T, x_T, \dot{x}_T) = f(t, x_t, \dot{x}_t) \]

for all \( t \) and \( x \).

Remark 2: A system of the form

\[ \dot{x} = f(x_t, \dot{x}_t) + g(t, x_t, \dot{x}_t) + h(t, x_t, \dot{x}_t) \]

is said to be asymptotically autonomous if (i) \( g(t, x_t, \dot{x}_t) \to 0 \) as \( t \to \infty \) uniformly for \( x_t \) and \( \dot{x}_t \) in an arbitrary compact set of \( C([-\infty, 0], \mathbb{R}^n) \)

(ii) \( \int_{-\infty}^{\infty} |h(t, x(t), \dot{x}(t))| \, dt < \infty \) for all \( x \) bounded and continuous on \( C^1 \).

This then extends the results of Yoshizawa [53] who has obtained some results in the study of invariance principle for *asymptotically autonomous* ordinary differential systems which includes the results of Markus and Opial as special cases - by showing that the positive limit sets of solutions of \( \dot{x} = F(x) + g(t, x) + h(t, x) \) are invariant sets of \( \dot{y} = F(y) \).
The references for the work of Markus and Opial are given in [53].

It is only with in the last one decade that the geometric theory of differential equations has been successfully carried over to functional differential equations. Krasovskii [23] has demonstrated the effectiveness of a geometric approach in extending the classical Lyapunov theory, including the converse theorems, to functional differential equations. It is this extension that has had so far the greatest success in studying stability properties of the solutions of autonomous functional differential systems.

\[ \ddot{x} = f(x_t) \]

Hale had obtained excellent results in this direction in his paper [20*] with which he has excelled all the other differential equationists all over the globe.
SECTION 12d

ANOTHER METHOD OF STUDYING THE STABILITY OF THE EQUATIONS OF NEUTRAL FUNCTIONAL DIFFERENTIAL SYSTEMS

B.S. Razumihin has studied the problem of stability of differential equations with retarded arguments \([45^{**}]\). We follow his method and study the stability of neutral differential difference equations of the form

\[(1) \quad \frac{dx(t)}{dt} = f(t, x(t - \theta), \dot{x}(t - \theta))\]

where \(x\) and \(f\) are \(n\) dimensional vectors in a real Euclidean \(n\) space \(\mathbb{R}^n\). It is quite interesting and important to note that many of the stability and boundedness properties of \((1)\) could be derived by studying the system with anticipation which corresponds to this system \((1)\).

First we introduce a new independent variable \(\alpha\) by putting \(t = t^* - \alpha\) where \(t^* > t_0\) is a known parameter. Then we set \(\frac{dx}{dt} = - \frac{dx}{d\alpha}\). System \((1)\) can now be written as

\[(2) \quad \frac{dx(t^* - \alpha)}{d\alpha} = f(t^* - \alpha, x(t^* - \alpha), \dot{x}(t^* - \alpha))\]

where the dots denote the right hand derivatives. Suppose

\[(3) \quad x(t^* - \alpha) = y(\alpha)\]
\[f(t^* - \alpha) = -g(\alpha)\]

then \((2)\) can be written as

\[(4) \quad \frac{dy}{d\alpha} = g(\alpha, y(\alpha + \epsilon), \dot{y}(\alpha + \epsilon))\]
The derivative of $y(t + \theta)$ is assumed to exist. We call (4) as the system with anticipation which corresponds to the system (1). Suppose $V(t, x)$ is a continuous scalar Lyapunov function with continuous partial derivatives with respect to its arguments. Its derivative with respect to system (1) is

\begin{equation}
\frac{dV}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \cdot f(t, x(t-\theta), \dot{x}(t-\theta)) = W(t, x(t-\theta), \dot{x}(t-\theta))
\end{equation}

is a continuous function defined on the solutions of system (1). The dot in (5) represents scalar product. Thus the Lyapunov function $V(t, x)$ has as many values as there are integral curves at the point $x(t)$.

Suppose

\begin{equation}
W(t^*-\alpha, x(t^*-\alpha-\theta), \dot{x}(t^*-\alpha-\theta)) = U(\alpha, y(\alpha+\theta), \dot{y}(\alpha+\theta))
\end{equation}

defined on the solutions of the system with anticipation (4).

We now extend the technique of Lyapunov functions to a system of the form (1)

Theorem 1: If for the differential equation (1) there exists a Lyapunov function, $V(t, x)$ whose derivative $\frac{dV}{dt}$ relative to this system is such that the functional $U(\alpha, y(\alpha), \dot{y}(\alpha))$ is negative or $\leq 0$ along every continuous
solution of the anticipated system (4) satisfying the conditions

\[ y(0) = x \]

(7) \[ V(t - a, y(a)) \leq V(t, x) \text{ for } a \leq t \]

(8) \[ 0 \leq a \leq t - t_0 \]

then the differential difference equation (1) is stable.

Proof of Theorem 1: Let \( \varepsilon \) be an arbitrarily given positive number \( < \delta = \inf V(t, x) \) for \( t \geq t_0 \) and for \( x(t) \) belonging to the boundary of the region \( ||x|| \leq \delta \), in which the conditions of the existence and uniqueness theorems for (1) are satisfied.

Here \( ||x|| = \sum_{i=1}^{n} |x_i| \)

let \( c = \inf V(t, x) \) on the sphere \( ||x|| = \varepsilon \) and for \( t \geq t_0 \) and let \( \delta(\varepsilon) = ||x|| \) on the set \( t \geq t_0 \), \( V(t, x(t)) = c \). Suppose \( x(t) \) is the solution of (1) corresponding to an arbitrary system of initial functions \( \delta(s) \) which satisfy the condition

(9) \[ ||\theta|| \leq \delta_1(\varepsilon) \]

where \( \delta_1(\varepsilon) < \delta(\varepsilon) \) is small enough, so that the corresponding solution does not leave the region \( ||x|| \leq \delta(\varepsilon) \) in the interval \( t_0 \leq t \leq t_0 + h \). We use the method of
Contradiction. Suppose the assertion of the theorem is not true. Let there exists at least one system of initial functions such that \( \| \theta \| < \delta_1 (C) \) and the corresponding solution does leave the sphere \( \| x \| = \xi \), that there exists an instant \( t_1 > t_0 + h \) such that

\[
||x(t)|| \begin{cases} < \xi & \text{for } t < t_1 \\ > \xi & \text{for } t_1 + \beta > t > t_1 \end{cases}
\]

where \( \beta \) is a non zero positive number or is \( \infty \).

Since \( \xi = \inf V(t, x) \) on \( \| x(t) \| = \xi \) there exists an instant \( t_2, t_0 + h \leq t_2 \leq t_1 \) such that along this solution

\[
V(t, x(t)) \begin{cases} < \xi & \text{for } t < t_2 \\ > \xi & \text{for } t_2 + \delta > t > t_2 \end{cases}
\]

where \( \delta \) is a non zero, positive number or is \( \infty \).

Condition (11) implies that

\[
\frac{dv}{dt} \bigg|_{t = t_2} > 0
\]

However under a change of the independent variable, the integral curve we have been considering is an integral
curve of system (4) satisfying conditions (7) and (8).

Hence the inequality

\[ \frac{dv}{dt} \bigg|_{t = t_2} > 0 \]

is impossible along this curve. The proof of theorem 1 is complete.

Theorem 2: If the conditions of theorem 1 are satisfied for (1) and if the function admits of an infinitesimal upper bound while the functional \( U(o, y(o), y(o)) \) is negative definite under the conditions (7) and (8), then the system (1) is asymptotically stable.

Proof of this theorem is analogous to the proof of theorem 2 in [45**].

A simpler problem is the problem of integration on the set of solutions of system (3) satisfying the conditions (7) and (8) on a segment of length \( T \) where \( T \geq a \geq 0 \).

Let \( \theta(s) \) be an initial vector function in the interval \( t - T - h \leq s \leq t - T, \quad T > h, \ T > t_0 + T \) and satisfying the condition \( V(s, \theta(s)) \leq V(t, x(t)) \) where \( \theta \) and \( x \) are vector functions with \( n \) components.

The corresponding solution on the interval \( (t - T, t) \) can be found by the method. Let \( E_0(t, T) \) be the set of initial functions satisfying the condition \( V(s, \theta(s)) \leq V(t, x(t)) \).
and let $E(t, T)$ be the set of the corresponding solutions. Let $E^*(t, T) \subseteq E(t, T)$ be the subset of solution $x^*(a)$ which satisfy

$$V(a, x^*(a)) \leq V(t, x(t))$$

$$t - T \leq a \leq t$$

$$x^*(t) = x(t)$$

**Theorem 3** - If for the system (1) there exists a positive number $T > h$ and a positive definite function $V(t, x(t))$ whose derivative $\frac{dV}{dt} = W(t, x(t - h), \dot{x}(t - h))$ relative to this system is negative or $\leq 0$, functional on the set of functions $E^1(t, T)$ for every $t > t_0 + T$, then the system (1) is stable.

**Remark 1** - The above technique can be employed to study the total stability or stability under persistent disturbances of (1) by considering the system

$$(12) \quad \frac{dx}{dt} = f(t, x(t - h), \dot{x}(t - h)) + F(t, x(t - h), \dot{x}(t - h))$$

where $F$ is a perturbed function. We can also extend I.G. Malkins theorem for ordinary differential equations that uniform asymptotic stability implies total stability.

**Remark 2** - It is natural to ask whether it might be more convenient to use a vector Lyapunov function rather than a
scalar function. Richard Bellman has dealt with advantages of using vector Lyapunov functions to study the stability properties of ordinary differential equations [see 5*]. Vector Lyapunov functions can also be used to study the stability properties of neutral functional differential equations.