SECTION 10

$\mathbb{L}^p$ - STABILITY OF DIFFERENTIAL - DIFFERENCE EQUATIONS

In this paper we study $\mathbb{L}^p$ - stability of Differential - Difference Equation.

(1)  
\[ \dot{x}(t) = f(x(t), t) \]

$\mathbb{L}^p$ - Stability has been first studied by Strauss in his doctoral dissertation, taking ordinary differential equations. Now we extend his results to differential - difference equations.

$\mathbb{R}^n$ is the space of $n$ vectors, $|x|$, where $x \in \mathbb{R}^n$, is any vector norm. For a given $h > 0$, $\mathcal{C}$ is the space of continuous functions - mapping the interval $[-h, 0]$ into $\mathbb{R}^n$. For $\Theta \in \mathcal{C}$, $|\Theta| = \sup_{-h \leq \alpha \leq 0} |\Phi(\alpha)|$. $\mathcal{C}_H$ denotes the set of $\Theta \in \mathcal{C}$ for which $|\Theta| \leq H$. For any continuous function $y(u)$ defined on $-h \leq u \leq A$, and any fixed $t_0 \leq t \leq A$, $y_t = y(t + \alpha)$, $-h \leq \alpha \leq 0$ i.e. $y_t \in \mathcal{C}$, the function $y(u)$ is defined in the interval $t-h \leq u \leq t$. A function $x_t(t_0, \Theta)$ is said to be a solution of (1) with initial function $\Theta$ at $t = t_0$ if there exists a number $A > 0$ such that

(i)  
For each $t$, $t_0 \leq t \leq t_0 + A$  $x_t(t_0, \Theta)$ is defined and $\in \mathcal{C}_H$

(ii)  
$x_{t_0}(t_0, \Theta) = \Theta$

(iii)  
$\dot{x}_t(t) = f(x(t), t), t_0 \leq t \leq t_0 + A$
Before we state our theorems, we give the following definitions

(i) The solution \( x_t = 0 \) of (1) said to be stable if for every \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that
\[
|\varepsilon|^{(h)} \leq \delta \quad \text{implies} \quad |x_t(t_0, \varepsilon)|^{(h)} < \varepsilon \quad \text{for} \quad t \geq t_0 \geq 0
\]

(ii) The solution \( x_t = 0 \) is \( L^F \)-stable for (1) if it is stable and if for all \( t_0 \geq 0 \) there exists \( \delta_0 = \delta_0(t_0) > 0 \) such that \( |\varepsilon|^{(h)} < \delta_0 \) implies
\[
\int_{t_0}^{\infty} \|x_t(t_0, \varepsilon)|^{(h)} |^F dt < \infty
\]

(iii) The functional \( V(t, \varepsilon) \) defined for the continuous \( \varepsilon(a), -\infty \leq a \leq 0 \) for (A) : \( |\varepsilon|^{(h)} < H, \quad t \geq 0 \) is called positive definite in the region (A) if there exists a continuous function \( w(r) \) that satisfies the conditions.
\[
V(t, \varepsilon) \geq w(|\varepsilon|^{(h)}); \quad w(r) > 0 \quad \text{for} \quad r \neq 0
\]

(iv) The functional \( V(t, \varepsilon) \) is said to have an infinitely small upper bound in the space (A) if there exists a continuous function \( w_1(r) \) satisfying the conditions
\[
V(t, \varepsilon) \leq w_1(|\varepsilon|^{(h)}); \quad w_1(0) = 0. \quad V(t, \varepsilon) \text{ is a continuous functional in } t, \varepsilon \text{ for } t \geq 0, \ \varepsilon \in C_H
\]

(v) The derivative of \( V(t, \varepsilon) \) along the solution of (1) will be denoted by \( \dot{V}_1(t, \varepsilon) \) and is defined as
\[
\dot{V}_1(t, \varepsilon) = \lim_{h \to 0^+} \frac{1}{h} \left[ V(t+h, x_{t+h}(t_0, \varepsilon)) - V(t, x_t(t_0, \varepsilon)) \right]
\]
where $a$ is the solution of (1) at time $t_0$.

(vi) The derivative $\dot{V}(1)(t,0)$ is said to be negative definite in the region $(A)$ if we can find a continuous function $w_2(r)$ satisfying the conditions

$$\dot{V}(1)(t,0) \leq -w_2|\theta|^{(h)} ; \quad w_2(r) > 0, \quad (t \geq t_0, e < r < H)$$

We now state a theorem which is quite similar to Krasovskii's theorem (Theorem 30.1 in [23]) on delay-differential equations.

Theorem 1: If to the differential-difference equation (1) there corresponds a Lyapunov functional $V(t,0)$ which is positive definite in the region $(A)$ and has an infinitely small upper bound and if its derivative $\dot{V}(1)(t,0)$ is negative definite along the solutions of the equation (1), then the null solution $x_t = 0$ of (1) is asymptotically stable.

Proof of theorem 1: The proof is similar to that of Krasovskii's theorem (Theorem 30.1 in [23]).

Theorem 2: Let the Lyapunov functional satisfies the above assumptions. Also let

$$(2) \quad \dot{V}(1)(t,x_t(t_0,0)) \leq -c \|x_t(t_0,0)\|^{(h)}$$

where $x_t(t_0,0)$ is the solution of (1), for some $c > 0$, $p > 0$, then $x_t = 0$ is $L^p$-stable for (1).

Proof of theorem 2: From theorem 1, $x_t = 0$ is asymptotically stable. Let for $t \geq t_0$. 
Let $t$ be fixed in $[t_0, \infty)$. Since $\beta(t_0) = V(t_0, \theta)$ we have $\beta(t) \leq V(t_0, \theta)$ for $t \geq t_0$. Hence from (3)

$$0 \leq V(t, x_t(t_0, \theta)) \leq -c \int_{t_0}^{t} \|x_t(t_0, \theta)\| \, dt + V(t_0, \theta)$$

for all $t \geq t_0$, so that

$$c \int_{t_0}^{t} \|x_t(t_0, \theta)\| \, dt \leq \frac{1}{c} V(t_0, \theta).$$

Hence the theorem is proved.

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