STABILITY OF DIFFERENTIAL - DIFFERENCE EQUATIONS

In the stability theory of non-linear differential equations, equations of first variations or linear variational equations associated with the given non-linear differential equations, are very important. The stability or instability of the non-linear differential equations depends upon the stability or instability of the associated variational equations. Lyapunov in his classical memoir [35] has obtained some results on orbital stability and on asymptotic orbital stability of nonlinear differential equations by considering the stability and asymptotic stability of the variational equations. Similar discussion is valid for differential difference equations also.

We consider first a scalar differential difference equation of the form

\[ a \ddot{x}(t) = f(x(t), x(t-w), t) \]  

where \( a \) and \( w \) are constants.

Let \( a(t) \) be a particular solution of (1).

Let \( y = x - z \) then \( a \dot{y}(t) = a \dot{x}(t) - a \dot{z}(t) \)

\[ = f(x(t), x(t-w), t) - f(z(t), z(t-w), t) \]

By the theorem of the mean

\[ a \dot{y}(t) = f_1(z(t), z(t-w), t)y(t) \]

+ \( f_2(z(t), z(t-w), t)y(t-w) + F(y(t), y(t-w), t) \)

\[ + \frac{F_2}{a} \]
Where $f_1$ and $f_2$ are the partial derivatives of $f$ with respect to $x(t)$ and $x(t-w)$ respectively.

The equation of first variation of (1) with respect to $s(t)$ is

$$ (3) \quad a\dot{y}(t) = f_1(t, z(t-w), t)y(t) + f_2(t, z(t-w), t)y(t-w) $$

If $f$ is independent of $t$ then equation (1) has the form

$$ (4) \quad a\ddot{x}(t) = f(x(t), x(t-w)). $$

If $f(c, c) = 0$ for a constant $c$, then $z(t) = c$ is a constant solution of (4) for $t > w$. Then the variational equation becomes

$$ (5) \quad a\dot{y}(t) = f_2(c, c)y(t) + f_2(c, c)y(t-w) = k_1y(t) + k_2y(t-w) $$

we make use of the following theorem to prove the asymptotic stability of equation (2)

Theorem: Suppose

$$ (6)(a) \quad \text{Every continuous solution of} \quad a\ddot{x}(t) + b_1x(t) + b_2x(t-w) = 0 $$

where $b_1$, $b_2$ are constants, approach zero as $t \to \infty$

(b) $f(u, v)$ is a continuous function of $u$ and $v$, in the

neighbourhood of the origin $|u| + |v| \leq c_1$ and

$$ \lim_{|u|+|v| \to 0} \frac{f(u, v)}{|u|+|v|} = c_2 \max_{0 \leq t \leq w} |g(t)| $$

small depending on $c_1$, $a$, $b_1$ and $b_2$, then any solution of

the Non-linear scalar differential - difference equation

$$ a\ddot{x}(t) + b_1x(t) + b_2x(t-w) = f(x(t), x(t-w)) $$

with the initial
condition \( t > w \), \( u(t) = g(t) \quad 0 \leq t \leq w \), can be continued over the interval \( 0 \leq t < \infty \) and each such solution satisfies \( \lim_{t \to \infty} |u(t)| = 0 \).

We make use of this theorem to prove the asymptotic stability of (2).

If all the characteristic roots of the characteristic equation

\[
(7) \quad \alpha \beta = -k_1 - k_2 e^{-ws} = 0,
\]
then every solution of (5) is asymptotically stable and under the non-linearity condition

\[
\lim_{t \to \infty} \frac{|F(y(t), y_1(t))|}{|y(t)| + |y_1(t)|} \to 0
\]
equation (2) is also asymptotically stable.

A simple application of the above theorem yields the result.

Remarks: The above results can be extended to differential difference systems. It is possible to obtain many interesting results in the stability of non-linear differential difference systems by considering the stability of the associated equations of first variations.