CHAPTER -II
CHARACTERIZATIONS OF SEMIOPEN SETS.
2.1. INTRODUCTION.

In 1987 [8] the concepts of semi generalized closed (sg-closed) and semi generalized open (sg-open) sets were defined by using semiopen sets as generalizations of closed and open sets respectively. These are independent of those notions of generalized open sets due to Levine [51] some basic properties of these sg-closed and sg-open sets were studied in [8]

Using g-closed sets [51], Malghan [66] and Munashi et al [68] have studied g-closed and g-continuous mappings respectively.

Maheshwari and Prasad [57], Ganguly et al. [40] and Ganster et al [41] have defined and investigated s-regular and s-normal, semi-g-regular (sg-regular) and semi-g-normal (sg-normal) spaces respectively, Dorsett [24] defined and investigated some properties of semi normal spaces, Nori and Popa [76] further investigated the concepts of regular and g-normal spaces. Dontchev [34], Dontchev and Noiri [36] have introduced a new notion of continuity called contra continuity and contra-semi-continuity respectively. Caldas and Jafari [11] introduced and investigated contra-β-continuity. Popa [81] introduced the notion of rarely continuity.

In this chapter, we obtain some more characterizations of sg-closed sg-open sets and semi-R₀ spaces. We extend the uses of sg-closed, sg-open sets to mappings. We introduce a new class of functions called sg-closed, Sg-open, Sg-continuous functions and investigate some of its fundamental properties. The notion of sg*-closed mappings are introduced. sg*-closed maps are generalizations of pre semiclosed maps due to sundaram et al. [92]. The concepts of semi generalized closure
(interior) of a set analogous to those of semiclosure (interior) of a set are introduced. We introduce and study the generalizations of semiopen functions and semiclosed functions which are called as contra-semiopen and contra semi closed functions respectively. Contra semiopen functions are characterized by having semi closed images of open sets of function. We also introduce and study a new type of continuous functions called rarely gs-continuous functions.

In the following section we study generalized closed sets with the help of semiopen sets.

2.2. ON sg-CLOSED SETS.

Some basic properties of semi generalized closed and semi generalized open sets are studied in [8]. In this section our aim is to establish some more characterizations of these sets.

We establish the following results for sg-closed sets.

THEOREM 2.2.1: The following statements are equivalent:

i) A is sg-closed.

ii) For each \( x \in sCl(A) \), \( sCl\{x\} \cap A \neq \emptyset \).

iii) \( sCl(A) - A \) contains no nonempty semiclosed sub sets.

PROOF: (i) \( \Rightarrow \) (ii): Assume that \( x \in sCl(A) \) and \( sCl\{x\} \cap A = \emptyset \). Then \( A \subset X \) -sCl{\( x \)}, which implies \( sCl(A) \subset X - sCl\{x\} \), which contradicts that \( x \in sCl(A) \). Therefore, \( sCl\{x\} \cap A \neq \emptyset \).

(ii) \( \Rightarrow \) (iii): Suppose \( F \subset sCl(A) - A \), where \( F \) is semiclosed. If there is a point \( x \in F \), then by (b), \( \emptyset \neq sCl\{x\} \cap A \subset F \cap A \)
\[ (s\text{Cl}(A) - A), \text{Which is a contradiction. Hence we conclude} \]

\[ \text{that } F = \emptyset. \]

\( (iii) \Rightarrow (i) : \) It follows from Th. 1 of [8].

We recall the following definition.

**DEFINITION 2.2.2:** [13,40] A space \( X \) is said to be semi-symmetric if and only if for \( x \) and \( y \) in \( X \), \( x \in s\text{Cl} \{ y \} \) implies that \( y \in s\text{Cl} \{ x \} \).

We prove the following result.

**THEOREM 2.2.3:** In a semi-\( T_1 \) space, sg-closed sets are semiclosed.

**PROOF:** Let \( A \) is sg-closed set in semi-\( T_1 \) space. Then by Th. 2.2.1, we obtain \( s\text{Cl} (A) - A = \emptyset \). Hence \( s\text{Cl}(A) = A \) implies \( A \) is semiclosed.

We recall the following results.

**THEOREM 2.2.4:** [40] Every semi-\( T_1 \) space is semi-symmetric.

**THEOREM 2.2.5:** [40] A space \( X \) is semi-symmetric and semi- \( T_0 \) if and only if \( X \) is semi- \( T_1 \).

**PROOF:** Suppose \( X \) is semi- \( T_1 \). Then clearly, \( X \) is semi- \( T_0 \) by [26] and semi-symmetric by Th. 2.2.4.

Conversely, suppose \( X \) is semi-symmetric and semi- \( T_0 \). We have to prove that \( X \) is semi- \( T_1 \). Let \( x, y \in X \) with \( x \neq y \). Since \( X \) is semi- \( T_0 \), we obtain that \( x \in U \subseteq X - \{ y \} \) for some semi open set \( U \). Then \( x \in s\text{Cl} \{ y \} \) and hence \( y \notin s\text{Cl} \{ x \} \). There exists a semiopen set \( V \) such that \( y \in V \subseteq X - \{ x \} \) and hence \( X \) is semi- \( T_1 \).

We recall the following definition.
DEFINITION 2.2.6: A space $X$ is said to be semi-$R_0$ if for each semiopen set $U$ in $X$ and each $x \in U$, $sCl \{x\} \subseteq U$.

Now, we characterize the semi-$R_0$ spaces in the following.

THEOREM 2.2.7: A space $X$ is semi-$R_0$ if and only if $\{x\}$ is sg-closed for each $x \in X$.

PROOF: Suppose $X$ is semi-$R_0$ and $x \in X$. Then there is a semiopen set $G$ and $x \in G$ such that, $sCl \{x\} \subseteq U$. It implies that $\{x\}$ is sg-closed set.

Conversely, assume that $\{x\}$ is sg-closed for each $x \in X$. Then by definition of sg-closed set, $\{x\} \subseteq G$. Where $G$ is semiopen and $sCl \{x\} \subseteq G$. This implies that $X$ is semi-$R_0$.

2.3 A SEMI-G-REGULAR-AND SEMI-G-NORMAL SPACE

In the present section the sg-open and sg-closed sets are used to redefine regularity and normality axioms.

We recall the following definition.

DEFINITION 2.3.1: A space $X$ is said to be semi-regular if for each semiclosed set $F \subseteq X$ and each $x \notin F$, there exist disjoint semiopen sets $U$ and $V$ such that $x \in U$ and $F \subseteq V$.

Now we recall the stronger form of the above axiom in the following

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**DEFINITION 2.3.2:** [45] A space $X$ is said to be semi-g-regular if for every sg-closed set $G$ and each point $x \notin G$, there exist disjoint semiopen sets $M$ and $N$ such that $x \in N$ and $G \subset M$.

Since every semiclosed set is sg-closed and thus every semi-g-regular space is semi-regular. However in presence of semi - $T_{1/3}$ of [8], every semi-regular space is semi-g-regular.

**DEFINITION 2.3.3** A set $U \subset X$ is said to be sg-open neighbourhood of $x$ if $x \in U$ and $U$ is a sg-open.

Now, we prove the following.

**THEOREM 2.3.4:** For a space $X$, the following statements are equivalent.

i) $X$ is semi-g-regular.

ii) For each $x \in U$ and an sg-open neighbourhood $U$ of $x$, there exists a semiopen neighbourhood $V$ of $x$ such that $sCl(V) \subset U$.

iii) For each $x \in U$ and for every sg-closed subset $A$ not containing $x$, there is a semiopen set $U$ containing $x$ such that $sCl(U) \cap A = \emptyset$.

**PROOF:** (i) $\Rightarrow$ (ii). Let $U$ be a sg-open neighbourhood of a point $x \in X$. Then $x \in U$ and $U$ is a sg-open set. Then $X - U$ is sg-closed set and $x \notin X - U$. Since $X$ is semi-g-regular, there exist disjoint semiopen sets $V$ and $W$ such that $x \in V$ and $X - U \subset W$. As $V \cap W = \emptyset$, implies $V \subset X - W$ implies $sCl(V) \subset sCl(X - W) = X - W$. Since $X - W$ is a semiclosed set. Hence, $sCl(V) \subset X - W \subset U$ implies $sCl(V) \subset U$.

(ii) $\Rightarrow$ (i). Let $F$ be a sg-closed set and $x \notin F$. Then $X - F$ is a sg-open set and $x \in X - F$. Then by definition 4.3, $X - F$ is an sg-open neighbourhood of $x$. Then by (b), there exists a semiopen neighbourhood $V$ of $x$ such
that $sCl(V) \subset X - F$ Now, as $x \in V$ implies $x \in sCl(V)$. Now we put $W = X - sCl(V)$. Then, $F \subset W$ and $W$ is a semiopen set since $sCl(V)$ is semiclosed set and also $V \cap W = \emptyset$. Thus, $x \in V, F \subset W$ and $V \cap W = \emptyset$, and $V$ and $W$ are semiopen. Hence the space $X$ is semi-$g$-regular.

(i) $\Rightarrow$ (iii): Let $A$ be a sg-closed set of a space $X$ and $x \in X - A$. Then as $X$ is semi-$g$-regular, there exist semiopen sets $U$ and $V$ such that $x \in U$ and $A \subset V$ with $U \cap V = \emptyset$. As $U \cap V = \emptyset$ implies $sCl(U) \cap V = \emptyset$, implies $sCl(U) \subset X - V$ and $A \subset V$ implies $X - V \subset X - A$. Thus, we obtain that $sCl(U) \subset X - V \subset X - A$ implies $sCl(U) \cap A = \emptyset$.

(iii) $\Rightarrow$ (i): Let $x \in X$ and $A$ be sg-closed subset of $X$ such that $x \not\in A$. Then there is a semi-open set $U$ such that $x \in U$ and $sCl(U) \cap A = \emptyset$. Let $V = X - sCl(U)$. Then $U, V$ are disjoint semiopen sets such that $x \in U$ and $A \subset V$. Hence $X$ is semi-$g$-regular.

We prove the following lemma.

**LEMMA 2.3.5:** Let $f: X \rightarrow Y$ is an irresolute and pre semiclosed map and if $B$ is sg-closed (sg-open) sub set of $Y$, then $f^{-1}(B)$ is sg-closed (resp. sg-open) set of $X$.

**PROOF:** Let $B$ be sg-closed subset of $X$. Suppose $f^{-1}(B) \subset U$, where $U$ is semiopen in $X$. Let $V = Y - f(X - U)$. As $U$ is semiopen, $f$ is presemiclosed then $f(X - U)$ is semiclosed in $Y$. Hence $V$ is semiopen. Further, $f^{-1}(B) \cap (X - U) = \emptyset$ which implies that $B \cap f(X - U) = \emptyset$ implies $B \subset V$. Also, $f^{-1}(V) \subset U$. Since $B$ is sg-closed, $sCl(B) \subset V$. Therefore, $f^{-1}(sCl(B)) \subset f^{-1}(V) \subset U$. Now $B \subset sCl(B)$ implies $f^{-1}(B) \subset f^{-1}(sCl(B))$. As $f$ is an irresolute, $f^{-1}(sCl(B))$ is semiclosed. Thus $sCl(f^{-1}(B)) \subset sCl(f^{-1}(sCl(B)))$. Therefore, $sCl(f^{-1}(B)) \subset f^{-1}(sCl(B))$.
cU \implies s\text{Cl}(f^{-1}(B)) \subseteq U. Hence f^{-1}(B) is sg-closed in X. Similarly, by taking complements we can show the second part.

Now, we study the stronger form of the semi-normality in the following paragraph.

**DEFINITION 2.3.6:** [45] A space X is said to semi-g-normal if for every pair of disjoint sg-closed sets A and B of X, there exist disjoint semiopen sets U and V of X such that A \subseteq U and B \subseteq V.

Clearly every semi-g-normal space is semi-normal.

We recall the following result of ganguly et.al[40].

**LEMMA 2.3.7:** [40] If X is semi-normal and F \cap A = \emptyset where F is semiclosed and A is sg-closed set, then there exist disjoint semiopen sets U and V such that F \subseteq U and A \subseteq V.

Now, we give some more characterizations of semi-normality in terms of sg-open and sg-closed sets.

**THEOREM 2.3.8:** For a space X, the following statements are equivalent.

i) X is semi-normal.

ii) For every semiclosed set E and every sg-open set G containing E, there is a semi-open set M such that E \subseteq M \subseteq s\text{Cl}(M) \subseteq G.

iii) For every sg-closed set E and every semiopen set G containing E, there is a semiopen set M such that E \subseteq M \subseteq s\text{Cl}(M) \subseteq G.

iv) For every pair of disjoint sets E and G one of which is semiclosed and other is sg-closed, there exist semiopen sets M and N such that E \subseteq M, G \subseteq N and s\text{Cl}(M) \cap s\text{Cl}(N) = \emptyset.
PROOF: (i) $\Rightarrow$ (ii): Let $E$ be a semiclosed set and $G$ a sg-open set containing $E$. Then $E \cap (X - G) = \emptyset$, where $X - G$ is sg-closed. Now, by Th. 4.13, there exist semiopen sets $M$ and $N$ such that $E \subset M$, $X - G \subset N$ and $M \cap N = \emptyset$. Thus, $E \subset M \subset X - N \subset G$. Now, $X - N$ being semiclosed, we have $M \subset X - N$ implies $sCl(M) \subset sCl(X - N) = X - N \subset G$. Then, it follows that $E \subset M \subset sCl(M) \subset G$.

(ii) $\Rightarrow$ (iii): Let $E$ be a sg-closed and $G$ be semiopen set containing $E$. Then, $X - G \subset X - E$, where $X - E$ is sg-open set containing a semiclosed set $X - G$. Then by (b), there is a semi open set $H$ such that $X - G \subset H \subset sCl(H) \subset X - E$, so that $E \subset X - sCl(H) \subset X - H \subset G$. Now, let $M = X - sCl(H)$. Then $M$ is semiopen and we obtain, $E \subset M \subset sCl(M) \subset G$.

(iii) $\Rightarrow$ (iv): Let $E$ be a sg-closed and $G$ be a semiclosed set such that $E \cap G = \emptyset$. Then $E \subset X - G$. So there exists by (c), a semiopen set $H$ such that $E \subset H \subset sCl(H) \subset X - G$, since $H$ is a semiopen set containing sg-closed set $E$, then by (c) again there exists a semiopen set $M$ such that $E \subset M \subset sCl(M) \subset H \subset sCl(H) \subset X - G$. If $N = X - sCl(H)$, then $E \subset M$, $G \subset N$ and $sCl(M) \cap sCl(N) = \emptyset$.

(iv) $\Rightarrow$ (i): Let $E$ and $G$ be two disjoint semiclosed subsets of a space $X$. As every semiclosed set is sg-closed, $G$ is a sg-closed set. Hence for every pair of disjoint sets $E$ and $G$, one of which is semiclosed and other is sg-closed, from (iv) there exist semiopen sets $M$ and $N$ such that $E \subset M$, $G \subset N$ and $sCl(M) \cap sCl(N) = \emptyset$. Further more, we have by Th.1.2.9. $M \cap N \subset sCl(M) \cap sCl(N) = \emptyset$. Therefore, $X$ is semi normal. This completes the proof of the theorem.
THEOREM 2.3.9: Every semi normal and semi-symmetric space is semi regular.

PROOF: Let X be the semi normal and semi-symmetric space. Now, suppose A is a semiclosed of X and x ∈ X and x \not\in A. Then, \{x\} is a sg-closed set since X is semi-symmetric space. So lemma 2.3.6 implies that there exist disjoint semiopen sets U and V such that A \subset U and \{x\} \subset V. This implies that space X is semi regular.

Next, we characterize the semi-g-normal spaces in the following.

THEOREM 2.3.10: Every semi-g-normal and semi-symmetric space is semi-g-regular.

PROOF: Let X be a semi-g-normal and semi-symmetric space. Let A be a sg-closed subset of X and x ∈ X such that x \notin A. Then by Lemma 2.3.6, \{x\} is sg-closed set disjoint from A. Then by semi-g-normality of X, there exist disjoint semiopen sets U and V such that A \subset U and \{x\} \subset V. Hence the space X is semi-g-regular.

THEOREM 2.3.11: For a space X, the following are equivalent.

i) X is semi-g-normal.

ii) For every sg-closed set E and every sg-open set G containing E, there is a semiopen set M such that E \subset M \subset sCl(M) \subset G.

PROOF (i) ⇒ (ii): Let E be a sg- closed and G is sg- open set containing E. Then E \cap (X - G) = \emptyset. Also E and (X - G) are two disjoint sg-closed sets of X. Then by (i), there exists disjoint semiopen sets M and N such that E \subset M and X - G \subset N. Thus, E \subset M \subset X - M \subset G. Now X - N being
semiclosed we have \( M \subseteq X - N \) implies \( s\text{Cl}(M) \subseteq s\text{Cl}(X - N) = X - N \subseteq G\).

Then it follows that \( E \subseteq M \subseteq s\text{Cl}(M) \subseteq G\).

(ii) \( \Rightarrow \) (i): Let \( F_1 \) and \( F_2 \) be two disjoint sg-closed subsets of \( X \). Then \( F_1 \cap F_2 = \emptyset \) implies that \( F_1 \subseteq X - F_2 \) where \( X - F_2 \) is a sg-open set containing \( F_1 \). Thus by (ii) there exists a semiopen set \( M \) such that \( F_1 \subseteq M \subseteq s\text{Cl}(M) \subseteq X - F_2 \). Now, we have \( s\text{Cl}(M) \subseteq X - F_2 \) implies \( F_2 \subseteq X - s\text{Cl}(M) \) implies \( F_1 \subseteq X - M = N \) (say) which is a semiopen set and \( M \cap N = \emptyset \). Since \( s\text{Cl}(M) \cap N = s\text{Cl}(M) \cap (X - s\text{Cl}(M)) = \emptyset \) implies \( M \cap N = \emptyset \). Thus \( F_1 \) and \( F_2 \) are two disjoint sg-closed sets which are separated by disjoint semiopen sets \( M \) and \( N \) and hence the space \( X \) is semi-g-normal.

**THEOREM 2.3.12:** Let \( X \) be a semi-g-normal space and let \( f : X \to Y \) be an irresolute pre semiclosed mapping from \( X \) onto a spacy \( Y \). Then \( Y \) is semi-g-normal.

**PROOF:** Let \( F_1 \) and \( F_2 \) be disjoint sg-closed subsets of \( Y \). Then by lemma 2.3.5, it follows that \( f^\uparrow(F_1) \) and \( f^\uparrow(F_2) \) are disjoint sg closed sets of \( X \). Then, by semi-g-normality of \( X \), there exists disjoint semiopen sets \( G_1 \) and \( G_2 \) of \( X \) such that \( G_i \supseteq f^\uparrow(F_i) \), \( i = 1, 2 \). Since, \( f \) is presemiclosed map and \( G_i \) is a semiopen set \( f^\uparrow(U_{G_i}) \subseteq G_i \) for \( i = 1, 2 \) by Theorem 3.5 of [20],

\[
f^{-1}(U_{G_i}) \subseteq X - f^{-1}(f(X - G_i)) \subseteq X - (X - G_i) = G_i, \quad i = 1, 2.
\]

Furthermore, as \( G_1 \cap G_2 = \emptyset \) implies that \( U_{G_1} \cap U_{G_2} = \emptyset \). Thus, \( F_1 \subseteq U_{G_1}, F_2 \subseteq U_{G_2} \) and \( F_1 \cap F_2 = \emptyset \). Hence \( F_1 \) and \( F_2 \) are separated by disjoint semi open sets \( U_{G_1} \) and \( U_{G_2} \), and hence space \( Y \) is semi-g-normal.

**THEOREM 2.3.13:** Semi-g-normality is a semi tological property.
**PROOF:** It can be easily proved by lemma 2.3.7, Theorem 3.5 of [20] and Theorem 2.3.12.

### 2.4. ON SG-CONTINUOUS MAPPINGS

Using g-closed sets [51], Malghan [66] and Munashi et al [67] have studied g closed and g-continuous mappings in topology respectively. In this section, we extend the uses of sg-closed and sg-open sets to functions.

We define SG-continuous, sg-continuous and weakly sg-continuous maps using sg-open sets of [8] as follows.

**DEFINITION 2.4.1:** A function $f: X \to Y$ is said to be SG-continuous (written as SG.C) if and only if for any sg-open subset $A$ of $Y$, $f^{-1}(A)$ is semi open in $X$.

**DEFINITION 2.4.2:** A function $f: X \to Y$ is said to be sg-continuous (written as sg.c) if and only if for any sg-open subset $A$ of $Y$, $f^{-1}(A)$ is sg-open in $X$.

**DEFINITION 2.4.3:** A function $f: X \to Y$ is said to be weakly sg-continuous (written $\omega$.sgc) if and only if for any semiopen subset $U$ of $Y$, $f^{-1}(U)$ is sg-open in $X$.

It is observed from the above definitions that is shown in the following implication diagram.

\[SG.C \quad \text{Irresolute} \quad \omega.g.c \]

\[sg.c \]

\[\text{Irresolute} \]

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THEOREM 2.4.4: A space $X$ is said to be semi-$T_{1/2}$ if and only if every sg-open set is semiopen.

PROOF: Let $X$ be semi-$T_{1/2}$ and $A \subseteq X$ be sg-open set which implies $X - A$ is sg-closed. As $X$ is semi-$T_{1/2}$, $X - A$ is semiclosed. Then $A$ is semiopen.

Conversely, let every sg-open set is semiopen. If $A \subseteq X$ be sg-closed. Then $X - A$ is sg-open, $X - A$ is semiopen by hypothesis. Hence, $A$ is semiclosed and implies that $X$ is semi-$T_{1/2}$.

THEOREM 2.4.5: Let $f: X \rightarrow Y$ be sg.c and $X$ be semi-$T_{1/2}$. Then $f$ is SG.C map.

PROOF: Let $A \subseteq Y$ be any sg-open subset. Then $f^{-1}(A)$ is sg-open as $f$ is sg.c. Then $f^{-1}(A)$ semiopen as $X$ is semi-$T_{1/2}$ by 2.4.4. Hence $f$ is SG.C map.

THEOREM 2.4.6: Let $f: X \rightarrow Y$ be $\omega$sg.c and $X$ be semi-$T_{1/2}$. Then $f$ is an irresolute.

PROOF: Let $U$ be any semiopen subset in $Y$. then $f^{-1}(U)$ is sg-open subset of $X$ since $f$ is $\omega$sgc which is again semiopen as $X$ is semi-$T_{1/2}$. Hence it follows that $f$ is an irresolute.

THEOREM 2.4.7: Let $f: X \rightarrow Y$ be $\omega$sgc and $g: Y \rightarrow Z$ be SG.C. Then $gof: X \rightarrow Z$ is sgc.

PROOF: Let $B \subseteq Z$ be any sg-open subset. Then $g^{-1}(B)$ is semiopen subset of $Y$ as $g$ is SG.C. Further, as $g^{-1}(B)$ is semiopen and $f$ is $\omega$sgc,
THEOREM 2.4.8: The composition of two sg.c maps is sg.c.

PROOF: Let \( f: X \to Y \) and let \( g: Y \to Z \) be two sg.c. maps. Let \( U \subseteq Z \) be any sg-open sub set \( g \) is sg.c. implies \( g^{-1}(U) \) is sg-open in \( Y \). And, hence, \( f^{-1}(g^{-1}(U)) = (gof)^{-1}(U) \) is also sg-open as \( f \) is sg.c. Thus, it follows that \( gof: X \to Z \) is sg.c.

THEOREM 2.4.9: Let \( f: X \to Y \) and \( g: Y \to Z \) be two \( \omega \)-sg.c maps with semi- \( T_{1/2} \) space \( Y \). Then \( gof \) is also \( \omega \)-sg.c map.

PROOF: Let \( V \subseteq Z \) be a semiopen subset. Then \( g^{-1}(V) \) is sg-open subset in \( Y \). As \( Y \) is semi- \( T_{1/2} \) implies \( g^{-1}(V) \) is semiopen. Again \( f \) is \( \omega \)-sg.c implies \( f^{-1}(g^{-1}(V)) = (gof)^{-1}(V) \) is sg-open in \( X \) which means that \( gof \) is \( \omega \)-sg.c map.

THEOREM 2.4.10: If \( f: X \to Y \) is \( \omega \)-sg.c and \( g: Y \to Z \) is an irresolute map, then \( gof \) is \( \omega \)-sg.c.

PROOF: Let \( U \) be any semiopen subset of \( Z \). Then \( g^{-1}(U) \) is semiopen in \( Y \) as \( g \) is an irresolute. Then, as \( f \) is \( \omega \)-sg.c and \( g^{-1}(U) \) is semiopen in \( Y \) implies that \( f^{-1}(g^{-1}(U)) = (gof)^{-1}(U) \) is sg-open in \( X \). It implies that \( gof \) is \( \omega \)-sg.c.

THEOREM 2.4.11: A map \( f: X \to Y \) is SG.C if any only if for every sg-closed subset A of \( Y \), \( f^{-1}(A) \) is semiclosed set in \( X \).
PROOF: Let \( f: X \rightarrow Y \) be SGC map. And suppose that \( A \subset Y \) be any sg-closed subset. Then \( Y - A \) is g-open set and since \( f \) is SGC implies \( f^{-1}(Y - A) \) is semiopen. That is \( X - f^{-1}(A) \) is semiopen implies \( f^{-1}(A) \) is semiclosed set in \( X \).

Conversely, assume that every sg-closed subset \( B \) of \( Y \), \( f^{-1}(B) \) is semi closed. Let \( V \) be a sg-open subset of \( Y \). Then, \( Y - V \) is sg-closed. Hence put \( B = Y - V \) and by the given condition \( f^{-1}(B) \) is semi closed in \( X \). That is \( f^{-1}(Y - V) = X - f^{-1}(V) \) implies \( f^{-1}(V) \) semiopen set in \( X \). Therefore by definition, \( f \) is SGC.

In [8], it is proved that if \( A \) is an open sg-closed subset of \( X \) then \( A \cap F \) is also sg-closed whenever \( F \) is semi closed set in \( X \). Hence, we prove the following.

**THEOREM 2.4.12:** Let \( X \) and \( Y \) be two spaces and \( A \) be a non empty semiclosed subset of \( X \). If \( f : X \rightarrow Y \) is sg.c, then restriction \( f/A : A \rightarrow Y \) is sg.c. map.

**PROOF:** Let \( f \) be sg.c from \( X \) into \( Y \). \( A \) be any non empty semiclosed subset of \( X \). For any sg-closed subset \( U \) of \( Y \), \( f/A)^{-1}(U) = A \cap f^{-1} (U) \). But as being sg.c, \( f^{-1} (U) \) is sg-closed and then \( A \cap f^{-1} (U) \) is a sg-closed subset of \( X \). Also, \( A \cap f^{-1} (U) \subset A \subset X \) Therefore, \( A \cap f^{-1} (U) \) is sg-closed set in \( A \). That is \( (f/A)^{-1}(U) \) is sg-closed subset of \( A \), implies \( f/A \) is sg.c.

Next, we study the SG-open and SG-closed mappings in the following.
DEFINITION 2.4.13: A mapping $f: X \to Y$ is said to be SG-open (SG-closed) if the image of every sg-open (resp. sg-closed) subset of $X$ is a sg-open (resp. sg-closed) subset of $Y$.

Clearly, a bijective map $f$ is SG-open if and only if $f$ is SG-closed.

THEOREM 2.4.14: Let $f: X \to Y$ be SG-open (resp. SG-closed) mapping and $Y$ is semi-$T_{\frac{1}{2}}$ -space. Then $f$ is presemiopen (resp. presemiclosed) mapping.

PROOF. Case (i): To prove that $f$ is pre semiopen: Let $f: X \to Y$ be SG-open map from a space $X$ into semi-$T_{\frac{1}{2}}$ space $Y$. Let $A \subset X$ be any semiopen subset. We know that every semiopen set is sg-open and $f$ is SG-open implies that $f(A)$ is sg-open in $Y$. But $Y$ is semi-$T_{\frac{1}{2}}$. It follows that $f(A)$ is semiopen by 2.4.4. Hence $f$ is pre semiopen. Similarly we can prove the case (ii) that is $f$ is presemiclosed.

Now, we define the following.

DEFINITION 2.4.15: Let $X$ and $Y$ be any two spaces. Then $X$ and $Y$ are said to be sg-homeomorphic to each other if and only if there exists a function $f: X \to Y$ such that $f$ is one-one, onto, sg.c and SG-open. Such a function $f$ is called SG-homeomorphism.

THEOREM 2.4.16: If $f: X \to Y$ and $g: Y \to Z$ two SG-open mappings, then $gof: X \to Z$ is also SG-open.

Next, we recall the following result from [21].

THEOREM 2.4.17: [21] If $f: X \to Y$ is a semi homeomorphism. Then,

1. $sCl(f(B)) = f(sCl(B))$ for all $B \subset X$.
2. $f(sInt(B)) = sInt(f(B))$ for all $B \subset X$.
3. $f^{-1}(sInt(B)) = sInt(f^{-1}(B))$ for all $B \subset X$. 

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LEMMA 2.4.18: A function $f: X \to Y$ is an irresolute if and only if, for every semiclosed set $H$ of $Y$, $f^{-1}(H)$ is semiclosed in $X$.

We prove the following.

THEOREM 2.4.19: If $f: X \to Y$ be a semi homeomorphism then $f$ is SG-open.

PROOF: Let $B \subset X$ be any sg-open subset. Then there is a semiclosed set $F$ such that $F \subset B$ and $F \subset sInt(B)$. Then we have, $f(F) \subset f(sInt(B)) = sInt(f(F))$, as $f$ is semi homeomorphism and by Th.2.2.17.

Also, by Remark-4.8, $f(F)$ is semiclosed $f(F) \subset f(B)$ as $F \subset B$. Thus we obtain $f(F) \subset sInt(f(F))$ with $f(F) \subset f(B)$ and $f(F)$ is semiclosed in $Y$. Thus, $f(B)$ is sg-open in $Y$, it follows that $f$ is SG-open map.

LEMMA 2.4.20: If $f: X \to Y$ be a semi homeomorphism then $f$ is sg.c

PROOF: Let $B \subset Y$ be any sg-open subset. Then there is a semiclosed set $F \subset Y$ such that $F \subset B$ and $F \subset sInt(B)$. Thus, we have $f^{-1}(F) \subset f^{-1}(sInt(B)) = sInt(f^{-1}(B))$, $f^{-1}(F) \subset f^{-1}(B)$ and $f^{-1}(F)$ is semiclosed by Th.2.4.11 and Th.2.4.11. Hence $f^{-1}(B)$ is sg-open in $X$ for a sg-open set $B \subset Y$. It implies that $f$ is sg.c.

Finally, we conclude that every semi homeomorphism is SG-homeomorphism in view of Th. 2.4.19 and Lemma. 2.4.20.

THEOREM 2.4.21: Let $f: X \to Y$ is SGC and SG-open map and $g: Y \to Z$ be any map. Then, $gof: X \to Z$ is irresolute if and only if $g$ is $\omega$ sg.c.
THEOREM 2.4.22: Let \( f: X \to Y \) is SGC and SG-open map and \( g: X \to Z \) be any map. Then, \( g \circ f: X \to Z \) is SG.C if and only if \( g \) is sg.c.

THEOREM 2.4.23: Let \( f: X \to Y \) be a one-one, onto, SG-closed mapping and \( X \) is semi-symmetric space. Then \( Y \) is also semi-symmetric space.

2.5. Sg-CLOSED AND Sg-OPEN MAPPINGS

In this section, we study another class of mappings called sg-closed and sg-open mappings and also we study some preserving properties by these maps with semi-normality and covering axioms.

DEFINITION 2.5.1: A function \( f: X \to Y \) is called sg-closed (sg-open) if for each closed (resp. open) set \( F \) (resp. set \( G \)) of \( X \), \( f(F) \) (resp. \( f(G) \)) is a sg-closed (resp. sg-open) set in \( Y \).

It is evident that every semiclosed [78] (resp. semiopen [10]) function is sg-closed (resp. sg-open).

THEOREM 2.5.2: A function \( f: X \to Y \) is sg-closed if and only if for each sub set \( V \) of \( Y \) and each open set \( U \) containing \( f^{-1}(V) \) there is a sg-open set \( W \) of \( Y \) such that \( f^{-1}(W) \subseteq U \).

PROOF: Suppose \( f \) is sg-closed map. Let \( V \subseteq Y \) be any set and \( U \) is an open set of \( X \) such that \( f^{-1}(V) \subseteq U \). Then \( W = Y - f(X - U) \) is a sg-open set containing \( V \) such that \( f^{-1}(W) \subseteq U \).

Conversely, suppose that \( F \) be a closed set of \( X \). Then \( f^{-1}(Y(F)) \subseteq X - F \) and \( X - F \) is an open set. Then by hypothesis, there is a
sg-open set V of Y such that Y - f(F) ⊂ V and f⁻¹(V) ⊂ X - F. Therefore, F ⊂ X - f⁻¹(V). Hence, Y - V ⊂ f(F) ⊂ f(X - f⁻¹(V)) ⊂ Y - V, which means f(F) = Y - V. Since Y - V is a sg-closed, f(F) is sg-closed set and thus f is a sg-closed map.

**THEOREM 2.5.3:** If f: X → Y is irresolute and SG-closed and A is a sg-closed subset of X, then f(A) is sg-closed set of Y.

**PROOF:** Let f(A) ⊂ U, where U is a semiopen set of Y. Since f is an irresolute map, f⁻¹(U) is a semiopen set containing A. Hence sCl(A) ⊂ f⁻¹(U) as A is sg-closed set. Since f is sg-closed, f(sCl(A)) is also sg-closed set in Y contained in the semiopen set U since sCl(A) is semiclosed and hence sg-closed. Then, if follows that sCl(f(sCl(A))) ⊂ U. Thus, as A ⊂ sCl(A) implies f(A) ⊂ f(sCl(A)) which implies sCl(sCl(f(A))) ⊂ sCl(f(sCl(A))) ⊂ U implies sCl(f(A)) ⊂ U implies f(A) is sg-closed.

**THEOREM 2.5.4:** If f: X → Y be sg-closed and g: Y → Z is irresolute and SG-closed. Then gof: X → Z is sg-closed map.

**PROOF:** Let A ⊂ X be any closed set. Then f(A) is sg-closed set in X. But as g is irresolute and sg-closed implies g(f(A)) = gof(A) is also sg-closed set in Z by Th.2.5.3. Hence gof: X → Z is sg-closed.

**THEOREM 2.5.5:** If f: X → Y is closed and g: Y → Z is sg-closed then gof: X → Z is sg-closed.

The following results are proved for sg-open function.
**THEOREM 2.5.6:** A function \( f: X \to Y \) is sg-open if and only if for each subset \( V \) of \( Y \) and for each closed set \( F \) containing \( f^{-1}(V) \) there is a sg-closed set \( U \) of \( Y \) such that \( V \subseteq U \) and \( f^{-1}(U) \subseteq F \).

**PROOF:** Suppose \( f \) is sg-open. Let \( V \) be a subset of \( Y \) and \( F \) is a closed set of \( X \) such that \( f^{-1}(V) \subseteq F \). Then, \( U = Y - f(X - F) \) is a sg-closed set containing \( V \) such that \( f^{-1}(U) \subseteq F \).

Conversely, suppose that \( G \) is an open set of \( X \). Then \( f^{-1}(Y-f(G)) \) and \( X-G \) is closed. By hypothesis, there is sg-closed set \( F \) of \( Y \) such that \( Y-f(G) \subseteq F \) and \( f^{-1}(F) \subseteq X-G \). Therefore, \( G \subseteq f^{-1}(F) \). Hence, we obtain, \( Y-F \subseteq f(G) \). Hence, \( Y-F \subseteq f(X-f^{-1}(F)) \subseteq Y-F \), which implies \( f(G) \) is sg-open and thus \( f \) is a sg-open map.

**THEOREM 2.5.7:** If \( f: X \to Y \) is open and \( g: Y \to Z \) is sg-open, then \( g \circ f: X \to Z \) is sg-open.

Next, we define a class of mappings namely, sg*-closed which are generalization of presemiclosed maps due to [42].

**DEFINITION 2.5.8:** A function \( f: X \to Y \) is said to be sg*-closed if for each semiclosed set \( F \) of \( X \), \( f(F) \) is a sg-closed set of \( Y \).

Clearly, every presemiclosed function is sg*-closed. Similarly, we define functions which generalize the class of presemiopen functions of [15].

**DEFINITION 2.5.9:** A function \( f: X \to Y \) is called sg*-open if for each semiopen set \( U \) of \( X \), \( f(U) \) is sg-open set of \( Y \).

Clearly, every presemiopen map is sg*-open map.
**THEOREM 2.5.10:** A function $f: X \to Y$ is $sg^*$-closed if and only if for each subset $B$ of $Y$ and each semiopen set $U$ containing $f^{-1}(B)$, there is a $sg$-open set $V$ of $Y$ such that $B \subset V$ and $f^{-1}(V) \subset U$.

**THEOREM 2.5.11:** A function $f: X \to Y$ is $sg^*$-open if and only if for each subset $B$ of $Y$ and for each semiclosed set $F$ containing $f^{-1}(B)$, there is a $sg$-closed set $H$ of $Y$ such that $B \subset H$ and $f^{-1}(H) \subset F$.

The proofs of the Th.2.5.10 and Th.2.5.11 are similar to those of Th.2.5.2 and Th.2.5.6 respectively.

Next, we characterize semi-normality as defined in [24].

**THEOREM 2.5.12:** If $f: X \to Y$ is an irresolute and $sg^*$-closed map from a semi-normal space $X$ onto a space $Y$. Then $Y$ is semi-normal.

**PROOF:** Let $A$ and $B$ disjoint semiclosed sets of $Y$. Then $f^{-1}(A)$ and $f^{-1}(B)$ are also disjoint semiclosed sets of $X$ as $f$ is irresolute map. As $X$ is semi-normal there are disjoint semiopen sets $U$ and $V$ in $X$ such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. Since $f$ is $sg^*$-closed and by Th.2.5.12, there are $sg$-open sets $G$ and $H$ in $Y$ such that $A \subset G$, $B \subset H$ and $f^{-1}(G) \subset U$ and $f^{-1}(H) \subset V$. Since $U$ and $V$ are disjoint, then we obtain, $f^{-1}(G) \cap f^{-1}(H) \subset U \cap V$ implies $f^{-1}(G \cap H) \subset U \cap V$ implies $G \cap H = \emptyset$. Then, we have, $s\text{Int}(G)$ and $s\text{Int}(H)$ are disjoint semiopen sets since $s\text{Int}(G) \cap s\text{Int}(H) \subset G \cap H = \emptyset$. And, as $G$ is $sg$-open $A$ is semiclosed, then $A \subset s\text{Int}(G)$. Similarly, $H$ is $sg$-open, $B$ is semiclosed, then $B \subset s\text{Int}(H)$. Hence space $X$ is semi-normal.

We recall the following result.
**LEMMA 2.5.13:** [29] For a space $X$ the following are equivalent.

1) $X$ is semi-regular,

2) For each point $x \in X$ and each semi-open set $V$ containing $x$, there exists a semi-open set $U$ such that $x \in U \subseteq \text{sCl}(U) \subseteq V$.

Now, we prove the following.

**THEOREM 2.5.14:** If $f: X \to Y$ be pre semiopen, Quasi-irresolute and $sg^*$-closed function from a semi-regular space $X$ onto a space $Y$. Then $Y$ is semi-regular.

**PROOF:** Let $y$ be a point of $Y$ and $U$ be a semi-regular (and hence semiopen) set of $Y$ containing $y$. Let $x$ be a point of $X$ such that $f(x) = y$. Since $f$ is quasi-irresolute, $f^{-1}(V)$ is semiopen set in $X$ containing $x$. As $X$ is semi-regular, there exists a semiopen set $U$ such that $x \in U \subseteq \text{sCl}(U) \subseteq f^{-1}(V)$ by Lemma-2.5.13. Hence, we obtain, $y = f(x) \in f(U) \subseteq f(\text{sCl}(U)) \subseteq V$. As $\text{sCl}(U)$ is semiclosed and $f$ is $sg^*$-closed, $f(\text{sCl}(U))$ is $sg$-closed set in $y$. Hence, $\text{sCl}(f(\text{sCl}(U))) \subseteq V$, which implies $\text{sCl}(f(U)) \subseteq V$ since $U \subseteq \text{sCl}(U)$ implies $f(U) \subseteq f(\text{sCl}(U))$ implies again that $\text{sCl}(f(U)) \subseteq \text{sCl}(f(\text{sCl}(U))) \subseteq V$. Therefore, $y \in f(U) \subseteq \text{sCl}(f(U)) \subseteq V$ where $f(U)$ is semiopen as $f$ is presemiopen. This implies that space $Y$ is semi-regular again by Lemma-2.5.13

**2.6 SEMI-GENERALIZED NEIGHBOURHOODS**

In this section we define and study the concepts of semi-generalized closure (interior) of a set analogous to those of semiclosure (interior) of a set.
**DEFINITION 2.6.1:** The semi-generalized closure (written as sg-closure) of a subset $A$ of a space $X$ is the intersection of all sg-closed sets containing $A$ and is denoted $sgCl(A)$.

Since every semiclosed set is sg-closed and hence, $A \subseteq sgCl(A) \subseteq sCl(A)$. Also, if $A$ is a sg-closed set, then $A = sgCl(A)$. It is observed that $sgCl(A)$ may not be sg-closed.

**DEFINITION 2.6.2:** A point $x \in X$ is called a sg-limit point of a subset $A$ of $X$, if for each sg-open set $U$ containing $x$, $A \cap (U - \{x\}) \neq \emptyset$.

The set of all sg-limit points of $A$ will be denoted by $D_{sg}(A)$, called the sg-derived set of $A$.

Then, we also redefine the $sgCl(A)$ as $A \cup D_{sg}(A)$, for every subset $A$ of $X$.

From the definition, it is clear that $X \in sgCl(A)$ if and only if every sg-open set containing $x$ contains a point of $A$.

**LEMMA 2.6.3:** If $A \subseteq B$ then $D_{sg}(A) \subseteq D_{sg}(B)$.

Now, we prove the following.

**THEOREM 2.6.4:** If $A$ and $B$ are subsets of a space $X$. Then

i) $D_{sg}(A) \cup D_{sg}(B) \subseteq D_{sg}(A \cup B)$

ii) $sgCl(A \cup B) \supseteq sgCl(A) \cup sgCl(B)$

iii) $sgCl(A) = sgCl(SgCl(A))$.

iv) $D_{sg}(A \cap B) \subseteq D_{sg}(A) \cap D_{sg}(B)$
Proof (i) Since \( A \subseteq B \) implies \( D_{sg}(A) \subseteq D_{sg}(B) \). Again, we have \( A \subseteq A \cup B \) and \( B \subseteq A \cup B \). Then we obtain \( D_{sg}(A) \subseteq D_{sg}(A \cup B) \) and \( D_{sg}(B) \subseteq D_{sg}(A \cup B) \). Hence \( D_{sg}(A) \cup D_{sg}(B) \subseteq D_{sg}(A \cup B) \).

(ii) As \( A \subseteq B \Rightarrow D_{sg}(A) \subseteq D_{sg}(B) \). Then \( A \cup D_{sg}(A) \subseteq B \cup D_{sg}(B) \) \( \Rightarrow \) \( sgCl(A) \subseteq sgCl(B) \). Now, \( A \subseteq B \) implies \( A \subseteq A \cup B \) and \( B \subseteq A \cup B \). Then \( sgCl(A) \subseteq sgCl(A \cup B) \) and \( sgCl(B) \subseteq sgCl(A \cup B) \) implies \( sgCl(A) \cup sgCl(B) \subseteq sgCl(A \cup B) \).

(iii) We have, \( sgCl(A) = A \cup D_{sg}(A) \). Then \( gCl[sgCl(A)] = sgCl[A \cup D_{sg}(A)] = [A \cup D_{sg}(A)] \cup D_{sg}[A \cup D_{sg}(A)] = [A \cup D_{sg}(A)] \cup [A \cup D_{sg}(A)] = [A \cup D_{sg}(A)] = sgCl(A) \).

(iv) Since \( A \cap B \subseteq A \) and \( A \cap B \subseteq B \) as \( A \subseteq B \). Then \( D_{sg}(A \cap B) \subseteq D_{sg}(A) \) and \( D_{sg}(A \cap B) \subseteq D_{sg}(B) \). Then \( D_{sg}(A \cap B) \subseteq D_{sg}(A) \cap D_{sg}(B) \).

Now, we define the following.

Definition 2.6.5: The \( sg \)-interior of a subset \( A \) of a space \( X \) is the union of all \( sg \)-open sets contained in \( A \) and is denoted by \( sgInt(A) \).

It is clear that if \( a \) is \( sg \)-open then \( A = sgInt(A) \). And, as every semiopen set is \( sg \)-open, it follows that \( sgInt(A) \supseteq sInt(A) \).

Lemma 2.6.6: \( X - sgInt(A) = sgCl(X - A) \).

Proof: Trivial.

We define the following.
DEFINITION 2.6.7: A space $X$ is said to be sg-compact if each sg-open cover of $X$ has a finite sub cover, or equivalently, a space $X$ is said to be sg-compact if each collection of sg-closed sets with finite intersection property has a non-empty intersection.

Since every semiopen set is sg-open and hence if follows that every sg-compact space is s-compact space (= a space $X$ is said to be S-compact if every s cover of $X$ has a finite cover (where s-cover is a cover by semiopen sets of $X$ whose union is $X$). However, we prove the following.

LEMMA 2.6.8: Every s-compact and semi-$T_1$ space is sg-compact.

PROOF: Let $u = \{U_\alpha \mid \alpha \in \Lambda\}$ be an sg-open cover of a s-compact, semi-$T_1$ space $X$. We have every semi-$T_1$ space is semi-$T_{1/2}$ and in a semi-$T_{1/2}$ space every sg open set is semiopen. Hence, $u = \{U_\alpha \mid \alpha \in \Lambda\}$ be an s-cover of $x$. As $x$ is s-compact, there exists a finite subset $\mathcal{V}_0$ of $\mathcal{V}$ such that $X = \bigcup \{U_\alpha \mid \alpha \in \mathcal{V}_0\}$; It follows that $X$ is sg-compact.

We recall the result from [8], let $B \subseteq A$ where $A$ is open and sg-closed, then $B$ is sg-closed relative to $A$ if and only if $B$ is sg-closed relative to $X$. Now, we prove the following.

THEOREM 2.6.9: Let $A$ be an open sg-closed subset of a sg-compact space $X$. Let $\{F_\alpha \mid \alpha \in \Lambda\}$ be a collection of sg-closed set with f.i.p. Then by above quoted result it follows that each $F_\alpha$ is also sg-closed in $X$. Since $X$ is sg-compact, the intersection of the family $\{F_\alpha \mid \alpha \in \Lambda\}$ is non-empty. Hence $a$ is sg-compact.
THEOREM 2.6.10: If $f: X \to Y$ is $\omega$-sg.c and presemiclosed map and $B$ is a sg-closed (sg-open) set of $Y$ then $f^{-1}(B)$ is sg-closed (sg-open) set in $X$.

PROOF: Let $B$ be a sg-closed subset of $Y$. Let $f^{-1}(B) \subseteq U$ and $U$ is semiopen set in $X$. Since $f$ is presemiclosed there is a semiopen set $V$ such that $B \subseteq V$ and $f^{-1}(V) \subseteq U$ by Th.3.5(i)[16]. Since $B$ is sg-closed, $sCl(B) \subseteq V$. Hence, $f^{-1}(sCl(B)) \subseteq f^{-1}(V) \subseteq U$. Since, $f$ is $\omega$-sg.c., $(f^{-1}(sCl(B)))$ is sg-closed in $x$. Hence $sCl(f^{-1}(sCl(B))) \subseteq U$ implies that $sCl(f^{-1}(B)) \subseteq U$ and $f^{-1}(B)$ is sg-closed (since $B \subseteq sCl(B)$). Hence, $f^{-1}(B) \subseteq f^{-1}(sCl(B))$ implies $sCl(f^{-1}(B)) \subseteq sCl(f^{-1}(sCl(B))) \subseteq U$ implies $f^{-1}(B)$ is sg-closed set. By taking the complements we obtain if $B$ is sg-open, $f^{-1}(B)$ is sg-open.

Now, we prove the following.

THEOREM 2.6.11: If $f: X \to Y$ is a $\omega$-sg.c, presemiclosed map from a sg-compact space $X$ onto a space $Y$, then $Y$ is sg-compact.

PROOF: Let $\{U_{\alpha} \mid \alpha \in \Lambda\}$ be a sg-open cover of $Y$. Then by Th.2.4.9, $\{f^{-1}(U_{\alpha}) \mid \alpha \in \Lambda\}$ is also sg-open cover of $X$. Since $X$ is sg-compact, there exist $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $\Lambda$ such that $\{f^{-1}(U_{\alpha_i}) \mid i = 1, 2, \ldots, n\}$ is a finite sub cover of $X$. Hence, $\{U_{\alpha_i} \mid i = 1, 2, \ldots, n\}$ is a finite sub cover of $\{U_{\alpha} \mid \alpha \in \Lambda\}$. It follows that $Y$ is sg-compact.

2.7 CONTRA-SEMI OPEN FUNCTIONS.

DEFINITION 2.7.1: A function $f : X \to Y$ is said to contra semiopen if the image of open subset of $X$ is semiclosed set in $Y$. 
Clearly every contra semiopen function is contra semipreopen since every semi closed set is semipreclosed.

**NOTE:** Contra semiopen and semiopen function are independent of each other.

The following theorem establishes two conditions that are equivalent to contra semi openness.

**THEOREM 2.7.2:** For a function $f: X \to Y$, the following statements are equivalent.

i) A function $f$ is contra-semiopen.

ii) For every subset $B$ of $Y$ and for every closed set $F$ of $X$ with $f^{-1}(B) \subseteq F$, there exists a semiopen set $V$ of $Y$ with $B \subseteq V$ and $f^{-1}(V) \subseteq F$.

iii) For every $y \in Y$ and for every closed subset $F$ of $X$ $f^{-1}(y) \subseteq F$, there exists a semi open set $V$ of $y$ with $y \subseteq V$ and $f^{-1}(V) \subseteq F$.

**PROOF:** (i) $\to$ (ii): Let $B \subseteq Y$ and $F$ be a closed subset of $X$ with $f^{-1}(B) \subseteq F$. Since $f$ is contra-semiopen, $f(X - F)$ is semiclosed. If we set $V = Y - f(X - F)$ it can be easily seen that $B \subseteq V$ and $f^{-1}(V) \subseteq F$.

(ii) $\to$ (iii): The result follows by substituting $B = \{y\}$ in (ii).

(iii) $\to$ (i): Let $U$ be an open subset of $X$. Let $y \in Y - f(U)$ and let $F = X - U$. Then by (iii) there exist a semiopen subset $V$ of $Y$ with $y \in V, f^{-1}(V) \subseteq F$. Then $y \in V \subseteq Y - f(U)$ and hence $f(U)$ is semiclosed and therefore $f$ is contra semiopen function.

We, recall the following.

**DEFINITION 2.7.3:** A function $f: X \to Y$ is said to be weakly semiclosed if $sCl(f(int(F)) \subseteq (F)$ for each closed set $F$ of $X$. 

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**THEOREM 3.7. 4:** If \( f: X \to Y \) is contra semiopen then \( f \) is weakly semiclosed.

**PROOF:** Obvious.

**DEFINITION 2.7. 5:** A function \( f: X \to Y \) is said to be contra-open (resp. contra semiopen, contra \( \alpha \)-open) if the image of each open subset of \( X \) is closed (resp. semiclosed \( \alpha \)-closed) subset of \( Y \).

We prove the following.

**THEOREM 2.7. 6:** If a function \( f: X \to Y \) is contra semiopen and \( X \) is regular then \( f \) is semi open function.

**PROOF:** Let \( y \) be an arbitrary point of \( Y \) and \( V \) be an open set of \( X \) containing \( f^{-1}(y) \). Since \( X \) is regular there exists an open set \( W \) in \( X \) containing \( f^{-1}(y) \) such that \( \text{Cl}(W) \subseteq V \). Since \( f \) is contra semiopen, by theorem (3.2) there exists \( U \in \text{SO}(X,x) \) such that \( f(U) \subseteq \text{Cl}(W) \). Then \( f(U) \subseteq V \). This shows that \( f \) is semiopen function.

**DEFINITION 2.7. 7:** A topological space \( X \) is said to be semi-\( T_2 \) if for each pair of distinct points \( x \) and \( y \) in \( X \) there exist disjoint semiopen sets containing them.

We state the following theorems.

**THEOREM 2.7. 8:** If \( f \) is a contra semiopen surjective function of an urysohn space \( X \) on to a space \( Y \) then \( Y \) is semi-\( T_2 \).

**THEOREM 2.7. 9:** Let a function \( f : X \to Y \) is contra semiopen and \( g : Y \to Z \) is presemi closed function then \( g\circ f : X \to Z \) is contra semiopen.
**PROOF:** Let $V \subset X$ be an open set. Since $f$ is contra-semiopen function, $f(V)$ is semi closed set in $Y$. Again, $g$ is pre-semiclosed function, so $g(f(V)) = gof(V)$ is semiclosed set in $Z$. Thus, $gof$ is contra-semiopen function.

2.8. CONTRA-SEMICLOSED FUNCTIONS

**DEFINITION 2.8.1:** A function $f: X \rightarrow Y$ is said to contra semiclosed if the image of closed subset of $X$ is semiclosed set in $Y$.

The following result shows that contra semi closedness and contra semiopenness are independent of each other.

**THEOREM 2.8.2:** If $f: X \rightarrow Y$ is bijective then $f$ is contra semiopen if and only if $f$ is contra semiclosed.

**THEOREM 2.8.3:** If $f: X \rightarrow Y$ is contra semiclosed if and only if for every open subset $B$ of $Y$ and for every open subset $U$ of $X$ with $f^{-1}(B) \subset U$ there exists a semiclosed subset $F$ of $Y$ with $B \subset F$ and $f^{-1}(F) \subset U$.

**PROOF:** Assume that $f$ is contra semiclosed $B \subset T$ and let $U$ be an open subset of $x$ with $f^{-1}(B) \subset U$ since $f$ is contra semiclosed function, $f(x-U)$ is semiopen. If we set $F = Y - f(x-U)$ then one can easily check that $F$ is a semiclosed set satisfying the conditions $B \subset F$ and $f^{-1}(F) \subset U$.

Conversely, assume that for every set $B \subset Y$ and for every open set $U \subset X$ with $f^{-1}(B) \subset U$ there exists a closed set $F \subset Y$ with $B \subset F$ and $f^{-1}(F) \subset U$. Let $H$ be a closed subset of $X$. Then let $B = Y - f(H)$ and let $U = X - H$. Then, we observe that $f^{-1}(B) \subset U$. By our assumption there
exists a semiclosed set $F \subset Y$ for which $B \subset F$ and $f^{-1}(F) \subset U$. It follows that $B = F$, that is $Y = f(H) = F$ then $f(H)$ is semiopen and hence $f$ is contra semiclosed function.

**NOTE:** Contra semiclosed and semiclosed functions are independent of each other.

**THEOREM 2.8.4:** If a function $f : X \to Y$ is contra semiclosed then for each $y \in Y$ and for an open subset $U$ of $X$ with $f^{-1}(y) \subset U$, there exists a semiclosed set $F$ of $Y$ with $y \in F$ and $f^{-1}(F) \subset U$.

We recall the following.

**DEFINTION 2.8.5:** A function $f : X \to Y$ is said to be a weakly semiopen if $f(U) \subset S \text{Int}(f(\text{Cl}(U)))$ for each open set $U$ of $X$.

We state the following results.

**THEOREM 2.8.6:** If $f : X \to Y$ is contra semiclosed then $f$ is weakly semiopen.

**THEOREM 2.8.7:** If $f : X \to Y$ is contra semiclosed and $g : Y \to Z$ is pre semiclosed and $g : Y \to Z$ is presemiopen function then $gof : X \to Z$ is contra semiclosed.

**THEOREM 2.8.8:** If $f : X \to Y$ is contra semiclosed and $g : Y \to Z$ is pre semiopen function and $g : Y \to Z$ is pre semiopen function then $gof : X \to Z$ is contra semiclosed.

**THEOREM 2.8.9:** If $X$ is regular space and $f : X \to Y$ is contra semiclosed then $f$ is semi open function.
2.9. ON RARELY gs-CONTINUOUS FUNCTIONS

Popa [81] introduced the notion of rarely continuity as a generalization of weak continuity which has been further investigated by Long and Herrington [53] and Jafari [46]. In this section we introduce the concept of rare gs-continuity in topological spaces as a new type of continuity. We investigate several properties of rarely gs-continuous functions. The notion of I.gs-continuity is also introduced which is weaker than gs-continuity and stronger than rare gs-continuity. It is shown that when the co-domain of a function is regular, then the notions of rare gs-continuity and I.gs-continuity are equivalent.

We recall the following definition from [81].

DEFINITION 2.9.1: A rare set is a set \( R \) such that \( \text{Int}(R) = \emptyset \). A nowhere dense set, is a set \( R \) which \( \text{In}(\text{Cl}(R)) = \emptyset \) if \( \text{Cl}(R) \) is codense.

We define rarely gs-continuous functions as follows.

DEFINITION 2.9.1: A function \( f : X \rightarrow Y \) is called rarely gs-continuous if for each \( x \in Y \) and each \( G \in \mathcal{O}(Y, f(x)) \), there exist a rare set \( R_G \) with \( G \cap \text{Cl}(R_G) = \emptyset \) and \( U \in \text{GSO}(X, x) \) such that \( f(U) \subseteq G \cup R_G \).

EXAMPLE 2.9.2: Let \( X \) and \( Y \) be the real line with indiscrete and discrete topologies respectively. The identity function is rarely gs-continuous.

Note that, every weakly continuous function is rarely continuous and every rarely continuous function is rarely gs-continuous.
THEOREM 2.9.3: The following statements are equivalent for a function \( f : X \rightarrow Y \),

i) The function \( f \) is rarely gs-continuous at \( x \in X \).

ii) For each set \( G \in O(Y, f(x)) \), there exists \( U \in GSO(X, x) \) such that \( \text{Int}[ f(U) \cap (Y \setminus G)] = \emptyset \).

iii) For each set \( G \in O(Y, f(x)) \), there exists \( U \in GSO(X, x) \) such that \( \text{Int}[f(U) \subset \text{Cl}(G)] \).

PROOF: (i) \( \rightarrow \) (ii): Let \( G \in O(Y, f(x)) \), By \( f(x \in G \subset \text{Int}(\text{Cl}(G)) \) and the fact that \( \text{Int}(\text{Cl}(G)) \in O(Y, f(x)) \), there exist a rare set \( R_G \) with \( \text{Int}(\text{Cl}(G)) \cap \text{Cl}(R_G) = \emptyset \) and a gs-open set \( U \subset X \) containing \( x \) such that \( f(U) \subset \text{Int}(\text{Cl}(G)) \cup R_G \). We have \( \text{Int}[f(U) \cap (Y - G)] = \text{Int}[f(U) \cap \text{Int}(Y - G) \subset \text{Int}[\text{Cl}(G) \cup R_G] \cap (Y - \text{Cl}(G)) \subset (\text{Cl}(G) \cup \text{Int}(R_G)) \cap (Y - \text{Cl}(G)) = \emptyset \).

(ii) \( \rightarrow \) (iii): It is straightforward.

(iii) \( \rightarrow \) (i): Let \( G \in O(Y, f(x)) \), Then by (3), there exits \( U \in GSO(X, x) \) such that \( \text{Int}[f(U)] \subset \text{Cl}(G) \). We have \( f(U) = f(U) - \text{Int}(f(U)) \cup \text{Int}(f(U)) \subset [f(U) - \text{Int}(f(U))] \cup \text{Cl}(G) = [f(U) - \text{Int}(f(U))] \cup G \cup (\text{Cl}(G) - G) \). set \( R^* = [f(U) - \text{Int}(f(U))] \cap (Y - G) \) and \( R^{**} = (\text{Cl}(G) - G) \). Then \( R^* \) and \( R^{**} \) are rare sets. Moreover \( R_G = R^* \cup R^{**} \) is a rare set such that \( \text{Cl}(R_G) \cap G = \emptyset \) and \( f(U) \subset G \cup R_G \). This shows that \( f \) is rarely-gs-continuous.

We define the following notion which is a new generalization of gs-continuity.
**DEFINITION 2.9.4:** A function \( f : X \to Y \) is I.gs-continuous at \( x \in X \) if for each set \( G \in O(Y, f(x)) \), there exists \( U \in GSO(X, x) \) such that \( \text{Int}[f(U)] \subset G \).

If \( f \) has this property at each point \( x \in X \), then we say that \( f \) is I.gs-continuous on \( X \).

**EXAMPLE 2.9.5:** Let \( X = Y = \{a, b, c\} \) and \( \tau = \sigma = \{X, \emptyset, \{a\}\} \). Then a function \( f : X \to Y \) defined by \( f(a) = f(b) = a \) and \( f(c) \) is I.gs-continuous.

**REMARK 2.9.6:** Since, if \( f : X \to Y \) is gs-continuous, then for each point \( x \in X \) and each open set \( V \) containing \( f(x) \), there exists \( U \in GSO(X, x) \) such that \( f(U) \subset V \). Then, it should be noted that I.gs-continuity is weaker than gs-continuity and stronger than rare gs-continuity.

**THEOREM 2.9.7:** Let \( Y \) be a regular space. Then the function \( f : X \to Y \) is I.gs-continuous on \( X \) if and only if \( f \) is rarely gs-continuous on \( X \).

**PROOF.** We prove only the sufficient condition since the necessity condition is evident (Remak)

Let \( f \) be rarely gs-continuous on \( X \) and \( x \in X \). Suppose that \( f(x) \in G \), where \( g \) is an open set in \( Y \). By the regularity of \( Y \), there exists an open set \( G_1 \in O(Y, f(x)) \) such that \( \text{Cl}(G_1) \subset G \). Since \( f \) is rarely gs-continuous, then there exists \( U \in GSO(X, x) \) such that \( \text{Int}[f(U)] \subset \text{Cl}(G_1) \). This implies that \( \text{Int}[f(U)] \subset G \) and therefore \( f \) is I.gs-continuous on \( X \).
We say that a function \( f: X \rightarrow Y \) is r.gs-open if the image of a gs-open set is open.

**Theorem 2.9.8:** If \( f: X \rightarrow Y \) be an r.gs-open rarely gs-continuous function, then \( f \) is weakly gs-continuous.

**Proof.** Suppose that \( x \in X \) and \( G \in O(Y, f(x)) \). Since \( f \) is rarely gs-continuous, there exist a rare set \( R_G \) with \( \text{Cl}(R_G) \cap U = \emptyset \) where \( U \in \text{GSO}(X,x) \) such that \( f(U) \subset G \cup R_G \). This means that \( (f(U) \cap (Y \setminus \text{Cl}(G))) \subset R_G \). Since the function \( f \) is r.gs-open, then \( f(U) \cap (Y \setminus \text{Cl}(G)) \) is open. But the rare set \( R_G \) has no interior points. Then \( f(U) \cap (Y \setminus \text{Cl}(G)) = \emptyset \). This implies that \( f(U) \subset \text{Cl}(G) \) and thus \( f \) is weakly gs-continuous.

**Theorem 2.9.9:** Let \( \text{GSO}(X, \tau) \) closed under finite intersections. If \( f: X \rightarrow Y \) is rarely g-continuous function, then the graph function \( g: X \rightarrow X \times Y \), defined by \( g(x) = (X, f(x)) \) for every \( x \) in \( X \) is rarely g-continuous.

**Proof.** Suppose that \( x \in X \) and \( W \) is any open set containing \( g(x) \). It follows that there exist open sets \( U \) and \( V \) in \( X \) and \( Y \), respectively, such that \( (X, f(x)) \in U \times V \subset W \). Since \( f \) is rarely gs-continuous, there exists \( G \in \text{GSO}(X,x) \) such that \( \text{Int} [f(G)] \subset \text{Cl}(V) \). Let \( E = U \cap G \). It follows that \( E \in \text{GSO}(X,x) \) and we have \( \text{Int}[g(E)] \subset \text{Int}(U \times f(G)) \subset U \times \text{Cl}(V) \subset \text{Cl}(W) \). Therefore, \( g \) is rarely gs-continuous.

**Definition 2.9.10:** Let \( A = \{ G_i \} \) be a class of subsets of \( X \). By rarely union sets [53,54] of \( A \) we mean \( \{ G_i \cup R_{G_i} \} \), where each \( R_{G_i} \) is a rare set such that each of \( \{ G_i \cup \text{Cl}(R_{G_i}) \} \) is empty.
Recall that, a subset \( B \) of \( X \) is said to be rarely almost compact relative to \( X \) if every open cover of \( B \) by open sets of \( X \), there exists a finite subfamily whose rarely union sets cover \( B \).

A topological space \( X \) is said to be rarely almost compact if the set \( X \) is rarely almost compact relative to \( X \).

A topological space \( X \) is called GSO-compact if every cover of \( X \) by gs-open sets has a finite subcover.

**THEOREM 2.9.11:** Let \( f : X \rightarrow Y \) be rarely gs-continuous and \( K \) an GSO-compact set relative to \( X \). Then \( f(K) \) is rarely almost compact subset relative to \( Y \).

**PROOF:** Suppose that \( \Omega \) is an open cover of \( f(K) \). Let \( B \) be the set of all \( V \) in \( \Omega \) such that \( V \cap f(K) \neq \emptyset \). Then \( B \) is an open cover of \( f(K) \). Hence for each \( k \in K \), there is some \( V_k \in B \) such that \( f(k) \in V_k \). Since \( f \) is rarely gs-continuous there exist a rare set \( R_{V_k} \) with \( V_k \cap \text{Cl}(R_{V_k}) = \emptyset \) and a gs-open set \( U_k \) containing \( k \) such that \( f(U_k) \subset V_k \cup R_{V_k} \). Hence there is a finite subfamily \( \{U_k\}_{k \in \Delta} \) which covers \( K \), where \( \Delta \) is a finite subset of \( K \). The subfamily \( \{V_k \cup R_{V_k}\}_{k \in \Delta} \) also covers \( f(K) \).

**COROLLARY 2.9.12:** Let \( f: X \rightarrow Y \) be rarely continuous and \( K \) be an GSO-compact set in \( X \). Then \( f(K) \) is rarely almost compact subset of \( Y \).

**LEMMA 2.9.13:** (Long and Herrington [53]). If \( g : Y \rightarrow Z \) is continuous and one-to-one, then \( g \) preserves rare sets.

**THEOREM 2.9.14:** If \( f : X \rightarrow Y \) is rarely gs-continuous and \( g : Y \rightarrow Z \) is continuous and one-to-one, then \( g \circ f : X \rightarrow Z \) is rarely gs-continuous.
**PROOF:** Suppose that $x \in X$ and $(g \circ f)(x) \in V$, where $V$ is an open set in $Z$. By hypothesis, $g$ is continuous, therefore there exists an open set $G \subset Y$ containing $f(x)$ such that $g(G) \subset V$. Since $f$ is rarely gs-continuous, there exists a rare set $R_G$ with $G \cap \text{Cl}(R_G) = \emptyset$ and an gs-open set $U$ containing $x$ such that $f(U) \subset G \cup R_G$. It follows from Lemma 2.10 that $g(R_G)$ is a rare set in $Z$. Since $R_G$ is a subset of $Y \setminus G$ and $g$ is injective, we have $\text{Cl}(g(R_G)) \cap V = \emptyset$. This implies that $(g \circ f)(U) \subset V \cup g(R_G)$. Hence the result.

Recall, that a function $f : X \rightarrow Y$ is called pre-gs-open if $f(U)$ is gs-open in $Y$ for every gs-open set $U$ of $X$.

**THEOREM 3.6.15:** Let $f : X \rightarrow Y$ be pre-gs-open and $g : Y \rightarrow Z$ a function such that $g \circ f : X \rightarrow Z$ is rarely gs-continuous. Then $g$ is rarely gs-continuous.

**PROOF:** Let $y \in Y$ and $x \in Y$ such that $f(x) = y$. Let $G \in \mathcal{O}(Z,(g \circ f)(x))$. Since $g \circ f$ is rarely gs-continuous, there exists a rare set $R_G$ with $G \cap \text{Cl}(R_G) = \emptyset$ and $U \in \text{GSO}(X,x)$ such that $(g \circ f)(U) \subset G \cup R_G$. But $f(U)$ (say $V$) is a gs-open set containing $f(x)$. Therefore, there exists a rare set $R_G$ with $G \cap \text{Cl}(R_G) = \emptyset$ and $V \in \text{GSO}(Y,y)$ such that $g(V) \subset G \cup R_G$, i.e., $g$ is rarely gs-continuous.