CHAPTER-IV
APPLICATIONS OF SEMIOPENSETS IN
TOPOLOGICAL SPACES.
4.1. INTRODUCTION

In this chapter the notions of \( \alpha g \)-semiclosed (\( \alpha g^s \)-closed), \( \alpha g \)-closed and \( \alpha g^\alpha \)-closed sets are introduced by using \( \alpha g^s \)-open sets due to Rajmani and Vishwanathan.

We introduced and study the notions of \( \alpha g \)-neighbourhoods, \( \alpha g \)-separation axioms along with fundamental properties of \( \alpha g \)-closed sets. The concepts of weakly \( \alpha g \)-closed sets which are weaker forms of \( \alpha g \)-closed sets are introduced and their relationship between other sets is investigated.

The class of \( \alpha g \)-semiclosed sets are stronger than \( gs \)-closed, \( *gs \)-closed [100] sets and weaker than \( \alpha \)-closed set. \( \lambda \)-closed and \( g^\prime \)-closed sets [99]. The notions of \( \alpha g^s \)-continuous, pre \( \alpha g^s \)-continuous, \( \alpha g^s \)-irresolute \( \alpha g^s \)-open and \( \alpha g^s \)-closed maps are introduced. Moreover as applications of \( \alpha g^s \)-closed sets separation axioms with respect to \( \alpha g^s \)-closed sets are introduced. Finally the concepts of \( \alpha g^s \)-normal and \( \alpha g^s \)-regular spaces are introduced. \( \alpha g^s \)-normality and \( \alpha g^s \)-regularity are separation properties obtained by utilized \( \alpha g^s \)-closed sets.

In this chapter we also introduce the concept of \( \alpha g^\alpha \)-closed sets and study the some of their properties. The \( \alpha g^\alpha \)-closed are stronger than \( \alpha g^*g \)-closed, \( \alpha g^s \) closed sets and weaker than \( \alpha \) closed sets.

4.2. ON \( \alpha g \)-CLOSED SETS IN TOPOLOGICAL SPACES.

In this section, we introduce \( \alpha g \)-closed sets and study some of their properties.
DEFINITION 4.2.1: A subset A of X is called $\alpha\hat{g}$-closed if $\text{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is $\alpha g$-open in X.

From the definition, we have the following results.

THEOREM 4.2.2: If A and B are $\alpha\hat{g}$-closed sets, then $A \cup B$ is $\alpha\hat{g}$-closed in X.

PROOF: let A and B be any two $\alpha\hat{g}$-closed sets in X. Let U be any $\alpha g$-open set in X such that $A \cup B \subseteq U$. Since A and B are $\alpha\hat{g}$-closed, $\text{Cl}(A) \subseteq U$ and $\text{Cl}(B) \subseteq U$ and hence $\text{Cl}(A \cup B) = \text{Cl}(A) \cup \text{Cl}(B) \subseteq U$, which implies $A \cup B$ is $\alpha\hat{g}$-closed in X.

THEOREM 4.2.3: Every closed set is $\alpha\hat{g}$-closed in X.

PROOF: Let A be a closed set in X. Then $\text{Cl}(A) = A$, so if $A \subseteq U$ where U is $\alpha g$-open set in X, then $\text{Cl}(A) \subseteq U$, which implies A is $\alpha\hat{g}$-closed set in X.

The converse of the above theorem need not be true as seen from the following example.

EXAMPLE 4.2.4: let $X = \{a,b,c,d\}$ and $\tau = \{\emptyset,\{a\},\{b\},\{a,b\},X\}$. Then, $\{a,c\}$, $\{a,d\}$ and $\{a,b,d\}$ are $\alpha\hat{g}$-closed but not closed in X.

THEOREM 4.2.5: Every $\alpha\hat{g}$-closed set is $*g$-closed set in X.

PROOF: Let A be a set in X. Let $A \subseteq U$ where U is $\hat{g}$-open in X. Since every $\hat{g}$-open set in X is $\alpha g$-open in X, U is $\alpha g$-open in X. Hence $\text{Cl}(A) \subseteq U$. So A is $*g$-closed set in X.
The converse of the above theorem need not be true as seen from the following example.

**EXAMPLE 4.2.6:** Let \( X = \{a, b, c\} \) and \( \tau = \{\emptyset, \{a\}, X\} \). Then \( \{b\}, \{c\}, \{a, b\} \) and \( \{a, c\} \) are \(*g\)-closed but not \( \alpha g\)-closed in \( X \).

**THEOREM 4.2.7:** Every \( g\)-closed set is \( \alpha g\)-closed set in \( X \).

**PROOF:** Let \( A \) be a \( g\) -closed set in \( X \). Let \( A \subseteq U \), where \( U \) is open in \( X \). Since every open set in \( X \) is \( \alpha g\)-open in \( X \), \( U \) is \( \alpha g\)-open in \( X \). Hence \( Cl(A) \subseteq U \). So, \( A \) is \( \alpha g\) -closed set in \( X \).

Converse of the above theorem need not be true as can be seen by the following example.

**EXAMPLE 4.2.8:** Let \( X = \{a, b, c, d\} \) and \( \tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}\} \). Then \( \{a, c\}, \{b, c\}, \{a, b, c\} \) are \( \alpha g\) -closed but not \( g\)-closed sets.

**THEOREM 4.2.9:** Every \( g\#\)-closed set in \( X \) is \( \alpha g\) - closed in \( X \).

**PROOF:** Let \( A \) be a \( g\#\)-closed set in \( X \). Let \( G \) be any \( \alpha g\)-open set in \( X \) containing \( A \). Since every \( \alpha g\)-open set is \( \alpha g\)-open, \( Cl(A) \subseteq U \). So \( A \) is \( \alpha g\)-closed set in \( X \).

Converse of the above theorem need not be true as can be seen by the following example.

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EXAMPLE 4.2.10: Let $X = \{a,b,c,d\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Then $\{d\}, \{a, d\}, \{a, c\}, \{b, d\}$ and $\{a, b, d\}$ are $\alpha \hat{g}$-closed but not $g^*$-closed in $X$.

THEOREM 4.2.11: Every $g$-closed set in $X$ is $\alpha \hat{g}$-closed in $X$.

PROOF: Since every $g$-closed is $g$-closed and by the above theorem 4.2.9 $g$-closed set in $X$ is $\alpha \hat{g}$-closed in $X$.

THEOREM 4.2.12: Every $\alpha \hat{g}$-closed is $\alpha g^*$-semi-closed in $X$.

PROOF: Since $s\text{Cl}(A) \subseteq \alpha \text{Cl}(A)$ and every $\alpha g$-open set is $\alpha g$-open in $X$, the result follows from the definition.

THEOREM 4.2.13: Every $\alpha \hat{g}$-closed set in $X$ is $\alpha g^*$-closed in $X$.

PROOF: Let $A$ be any subset of $X$. Since $s\text{Cl}(A) \subseteq \text{Cl}(A)$ and every $\alpha g$-open set is $sg$-open in $X$, the result follows from the definitions.

EXAMPLE 4.2.15: Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b, c, d\}\}$. Then $\{a\}$ is $\alpha \hat{g}$-closed but not $\alpha g^*$-closed in $X$.

THEOREM 4.2.16: If $A$ is an $\alpha \hat{g}$-closed in $X$, then $\text{Cl}(A) - A$ contains no non-empty closed set in $X$.

PROOF: Suppose that $A$ is $\alpha \hat{g}$-closed in $X$. Let $F$ be a closed subset of $\text{Cl}(A) - A$, then $A \subseteq F^c$. Since every closed set is $\alpha g$-closed in $X$, $F$ is
\[ \text{ag}_\text{gs} \text{- closed subset of } \text{Cl}(A) - A. \] Therefore \( \text{Cl}(A) \subseteq F^c \). Consequently \( F \subseteq (\text{Cl}(A))^c \), we have \( F \subseteq \text{Cl}(A) \cap (\text{Cl}(A))^c = \emptyset \) and \( F \) is empty.

**THEOREM 4.2.18:** A set \( A \) is \( \alpha \hat{g} \)-closed set if and only if \( \text{Cl}(A) - A \) contains no non-empty \( \text{ag}_\text{gs} \)-closed set in \( X \).

**PROOF:** Suppose \( A \) is a \( \alpha \hat{g} \)-closed set. Let \( U \) be a subset of \( \text{Cl}(A) - A \).
Then \( A \subseteq U^c \). since \( A \) is \( \alpha \hat{g} \)-closed set, we have \( \text{Cl}(A) \subseteq U^c \).
Consequently \( U \subseteq (\text{Cl}(A))^c \), hence \( U \subseteq \text{Cl}(A) \cap (\text{Cl}(A))^c = \emptyset \). Therefore \( U \) is empty.

Conversely, suppose that \( \text{Cl}(A) - A \) contains no non-empty \( \text{ag}_\text{gs} \)-closed set. Let \( A \subseteq G \) and let \( G \) be \( \text{ag}_\text{gs} \)-pen. If \( \text{Cl}(A) \not\subseteq G \), then \( \text{Cl}(A) \cap G^c \) is a non-empty \( \text{ag}_\text{gs} \)-closed subset of \( \text{Cl}(A) - A \). therefore, \( A \) is an \( \alpha \hat{g} \)-closed set in \( X \).

**COROLLARY 4.2.19:** Let \( A \) be \( \alpha \hat{g} \)-closed, then \( A \) is \( \text{ag}_\text{gs} \)-closed if and only if \( \text{Cl}(A) - A \) is \( \text{ag}_\text{gs} \)-closed.

**PROOF:** Let \( A \) be \( \alpha \hat{g} \)-closed which is also \( \text{ag}_\text{gs} \)-closed. Then \( \text{Cl}(A) - A = \emptyset \), which is \( \text{ag}_\text{gs} \)-closed.

Conversely, let \( \text{Cl}(A) - A \) be \( \text{ag}_\text{gs} \)-closed and \( A \) be a \( \alpha \hat{g} \)-closed.
Then \( \text{Cl}(A) - A \) does not contain any non-empty subset, because \( \text{Cl}(A) - A \) is \( \text{ag}_\text{gs} \)-closed, \( \text{Cl}(A) - A = \emptyset \), which implies that \( A \) is \( \text{ag}_\text{gs} \)-closed.

**THEOREM 4.2.20:** If \( A \) is \( \alpha \hat{g} \)-closed and \( A \subseteq B \subseteq \text{Cl}(A) \), then \( B \) is \( \alpha \hat{g} \)-closed.
PROOF: Let $B \subseteq U$ where $U$ is $\alpha gs$-open. Since $A$ is $\alpha g$-closed and $A \subseteq U$, it follows that $\text{Cl}(A) \subseteq U$. By hypothesis, $B \subseteq \text{Cl}(A)$. Hence $\text{Cl}(B) \subseteq \text{Cl}(A)$. Consequently, $\text{Cl}(B) \subseteq U$ and so $B$ becomes $\alpha g$-closed.

**Lemma 4.2.21:** A subset $A$ of $X$ is $\alpha g$-closed if and only if $\text{Cl}(A) \subseteq \alpha gs$-ker $(A)$.

**Proof:** Suppose that $A$ is $\alpha g$-closed, then $\text{Cl}(A) \subseteq U$ whenever there is $A \subseteq U$ and $U$ is $\alpha gs$-open. Let $x \in \text{Cl}(A)$. If $x \notin \alpha gs$-ker $(A)$, then there is an $\alpha gs$-open set $U$ containing $A$ such that $x \notin U$. Since $U$ is an $\alpha gs$-open set containing $A$, we have $x \notin \text{Cl}(A)$, a contradiction.

Conversely, let $\text{Cl}(A) \subseteq \alpha gs$-ker $(A)$. If $U$ is any $\alpha gs$-open set containing $A$, then $\text{Cl}(A) \subseteq \alpha gs$-ker $(A) \subseteq U$. Therefore $A$ is $\alpha g$-closed.

**Lemma 4.2.22:** Let $x$ be any point of $X$. then every singleton $\{x\}$ is either nowhere dense or pre-open.

In the notion of lemma 4.2.22 we may consider the following decomposition of a given topological space $X$: $X = X_1 \cup X_2$, where $X_1 = \{ x \in X : \{x\}$ is nowhere dense $\}$ and $X_2 = \{ x \in X : \{x\}$ is pre-open $\}$.

**Theorem 4.2.23:** For any subset $A$ of $X$, $X_2 \cap \text{Cl}(A) \subseteq \alpha gs$ -ker $(A)$.

**Proof:** Let $x \in X_2 \cap \text{Cl}(A)$ and suppose that $x \notin \alpha gs$-ker $(A)$. Then there is an $\alpha gs$-open set $U$ containing $A$ such that $x \notin U$. If $F = X / U$, then $F$ is $\alpha gs$-closed. Therefore $x \notin F$ implies $\text{Cl}(A) \subseteq F$. On the other hand,
Cl(x) ⊂ Cl(A) and since x ∈ X₂, there has to be some point y ∈ A ∩ IntCl \{x\} A ∩ Cl\{x\} ⊂ A ∩ F, a contradiction.

**THEOREM 4.2.24:** A subset A of X is ᾱg - closed if and only if $X₁ ∩ Cl(A) ⊆ A$.

**PROOF:** Suppose that A is ᾱg - closed. Let $x ∈ X₁ ∩ Cl(A)$. If $x ∈ A$ and $U = X / x$, then U is an αgs-open set containing A and so Cl(A) ⊆ U, a contradiction.

Conversely, suppose that $X₁ ∩ Cl(A) ⊆ A$. Then $X₁ ∩ Cl(A) ⊆ αgs- ker(A)$ By theorem 3.22, we have $Cl(A) = X ∩ Cl(A) = (X₁ ∩ Cl(A)) αgs-ker (A)$. Thus A is ᾱg - closed by lemma 3.20.

**THEOREM 4.2.25:** An arbitrary intersection of ᾱg - closed sets is ᾱg - closed.

**PROOF:** Let $F = \{Aᵢ | i ∈ I\}$ be a family of ᾱg-closed sets and let $A = \cap Aᵢ$. Now $A ⊆ Aᵢ$ implies that $X₁ ∩ Cl(A) ⊆ A$ by theorem 4.2.24. Therefore A is ᾱg - closed in X.

**4.3.0N ᾱg - NEIGHBOURHOODS.**

**DEFINITION 4.3.1:** A subset A of a topological space (X, τ) is said to be an ᾱg - neighbourhood (in short ᾱg -nbd) of a point $x ∈ X$ if there exists ᾱg-open set U containing x such that $U ⊆ A$.  

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DEFINITION 4.3.2: A point $x \in A$ is said to be an $\alpha \hat{g}$-interior point of $A$ if $A$ is an $\alpha \hat{g}$-nbd of $x$. In other words, it means that there exists an $\alpha \hat{g}$-open set $G$ containing $x$ such that $G \subseteq A$.

The set of all $\alpha \hat{g}$-interior points of $A$ is said to be $\alpha \hat{g}$-interior of $A$ and is denoted by $\alpha \hat{g}$-int($A$).

LEMMA 4.3.3: Let $A$ be a subset of a topological space $(X, \tau)$. Then $A$ is $\alpha \hat{g}$-open if and only if $A$ contains an $\alpha \hat{g}$-nbd of each of its pts.

PROOF: Let $A$ be an $\alpha \hat{g}$-open set in $(X, \tau)$. Let $x \in A$, implies $x \in A \subseteq A$. Thus $A$ is an $\alpha \hat{g}$-nbd of $x$. Hence $A$ contains an $\alpha \hat{g}$-nbd of each of its points.

Conversely, $A$ contains an $\alpha \hat{g}$-nbd of each of its points. For every $x \in A$ there exists a neighbourhood $N_x$ of $x$ such that $x \in N_x \subseteq A$. By the definition of $\alpha \hat{g}$-nbd of $x$, there exists an $\alpha \hat{g}$-open set $G_x$ such that $x \in G_x \subseteq N_x \subseteq A$. Now we shall prove that $A = \bigcup \{G_x : x \in A\}$.

Let $x \in A$. Then there exists an $\alpha \hat{g}$-open set $G_x$ such that $x \in G_x$. Therefore, $x \in \bigcup \{G_x : x \in A\}$ implies $A \subseteq \bigcup \{G_x : x \in A\}$.

Now let $y \in \bigcup \{G_x : x \in A\}$ so that $y \in G_x$ for some $x \in A$ and hence $y \in A$. Therefore, $\bigcup \{G_x : x \in A\} \subseteq A$. Hence $A = \bigcup \{G_x : x \in A\}$.

Also, each $G_x$ is an $\alpha \hat{g}$-open set and hence $A$ is an $\alpha \hat{g}$-open set.
NOTE 43.4: Since every open set is $\alpha \hat{g}$-open set, every interior point of a set $A \subset X$ is $\alpha \hat{g}$-interior point of $A$. Thus, $\text{Int}(A) \subset \alpha \hat{g} \text{-Int}(A)$. In general $\text{Int}(A) \neq \alpha \hat{g} \text{-Int}(A)$, which is shown in the following example.

EXAMPLE 43.5: Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a, b\}, X\}$ be a topology. Let $A = \{a, c\}$. Then $\alpha \hat{g} \text{-Int}(A) = \{c\}$ and $\text{Int}(A) = \emptyset$.

THEOREM 43.6: A is $\alpha \hat{g}$-open if and only if $A = \alpha \hat{g} \text{-Int}(A)$.

PROOF: Let $A$ be an $\alpha \hat{g}$-open set. Now $A$ being $\alpha \hat{g}$ open it is identical with largest $\alpha \hat{g}$ open subset of $A$. But $\alpha \hat{g} \text{-Int}(A)$ is the largest $\alpha \hat{g}$-open subset of $A$. Hence $A = \alpha \hat{g} \text{-Int}(A)$.

Conversely, let $\alpha \hat{g} \text{-Int}(A) = A$ and by the definition, $\alpha \hat{g} \text{-Int}(A)$ is an $\alpha \hat{g}$ open set. Then it follows that $A$ is also an $\alpha \hat{g}$-open set.

LEMMA 43.7: If $A \subset B$, then $\alpha \hat{g} \text{-Int}(A) \subset \alpha \hat{g} \text{-Int}(B)$.

Easy proof of this lemma is omitted.

NOTE: $\alpha \hat{g} \text{-Int}(A) = \alpha \hat{g} \text{-Int}(B)$ does not imply that $A = B$. This shown in the following example.

EXAMPLE 43.7.8: Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$ be a topology on $X$. Let $A = \{b\}$ and $B = \{b, c\}$, then $\alpha \hat{g} \text{-Int}(A) = \alpha \hat{g} \text{-Int}(B)$ but $A \neq B$.

THEOREM 43.9: Let $A$ and $B$ be subsets of $X$. Then,
(i) \( \alpha \hat{g} \cdot \text{Int}(A) \cup \alpha \hat{g} \cdot \text{Int}(B) \subset \alpha \hat{g} \cdot \text{Int}(A \cup B) \).

(ii) \( \alpha \hat{g} \cdot \text{Int}(A \cap B) \subset \alpha \hat{g} \cdot \text{Int}(A) \cap \alpha \hat{g} \cdot \text{Int}(B) \).

**PROOF:** Follows by lemma 4.3.7.

**COROLLARY 4.3.10:** If \( A \) is an \( \alpha \hat{g} \) - closed subset of \( X \) and \( x \in X - A \), then there exists an \( \alpha \hat{g} \) - nbd \( N \) of \( x \) such that \( N \cap A = \emptyset \)

**PROOF:** If \( A \) is an \( \alpha \hat{g} \) -closed subset in \( X \), then \( X - A \) is an \( \alpha \hat{g} \) -open set. By lemma 4.3.3, \( X - A \) contains an \( \alpha \hat{g} \) -nbd of each of its points. This implies that there exists an \( \alpha \hat{g} \) -nbd \( N \) of \( x \) such that \( N \subseteq X - A \). It is clear that no point of \( N \) belongs to \( A \) and hence \( N \cap A = \emptyset \)

**DEFINITION 4.3.11:** Let \( (X, \tau) \) be a topological space and \( A \) be a subset of \( X \). Then a point \( x \in X \) is called an \( \alpha \hat{g} \) - limit point of \( A \) if and only if every \( \alpha \hat{g} \) -nbd of \( x \) contains a point of \( A \) distinct from \( x \). That is, \( (N - \{x\}) \cap A \neq \emptyset \), \( \forall \) \( \alpha \hat{g} \) -nbd \( N \) of \( x \). Alternatively, we can say that if and only if every \( \alpha \hat{g} \) -open set \( G \) containing \( x \) contains a point of \( A \) other than \( x \).

The set of all \( \alpha \hat{g} \) -limit points of \( A \) is called an \( \alpha \hat{g} \) - derived set of \( A \) and is denoted by \( \alpha \hat{g} \cdot d(A) \).

**THEOREM 4.3.12:** Let \( A \) and \( B \) be subsets of \( X \) and \( A \subseteq B \). Then \( A \subseteq B \) implies \( \alpha \hat{g} \cdot d(A) \subseteq \alpha \hat{g} \cdot d(B) \).

**PROOF:** Let \( x \in \alpha \hat{g} \cdot d(A) \) implies \( x \) is an \( \alpha \hat{g} \) -limit point of \( A \), i.e., every \( \alpha \hat{g} \) -nbd of \( x \) contains a point of \( A \) other than \( x \). Since \( A \subseteq B \),
every $\alpha \hat{g}$-nbd of $x$ contains a point of $B$ other than $x$. Consequently $x$ is an $\alpha \hat{g}$-limit point of $B$ ie $x \in \alpha \hat{g} - d (B)$. Therefore $\alpha \hat{g} - d (A) \subseteq \alpha \hat{g} - d (B)$.

**THEOREM 4.3.13:** Let $(X, \tau)$ be a topological space and let $A$ be a subset of $X$. Then $A$ is $\alpha \hat{g}$-closed if and only if $\alpha \hat{g} - d (A) \subseteq A$.

**PROOF:** Suppose $A$ is $\alpha \hat{g}$-closed in $X$. Then $X - A$ is $\alpha \hat{g}$-open in $X$. Now we show that $\alpha \hat{g} - d (A) \subseteq A$. Let $x \in \alpha \hat{g} - d (A)$ implies $x$ is an $\alpha \hat{g}$-limit point of $A$ ie. every $\alpha \hat{g}$-nbd of $x$ contains a point of $A$ other than $x$. Now suppose $x \notin A$ so that $x \in X - A$, which is $\alpha \hat{g}$-open and by definition of $\alpha \hat{g}$-open sets there exists an $\alpha \hat{g}$-nbd $N$ of $x$ such that $N \subseteq X - A$. From this we conclude that $N$ contains no point of $A$, which is a contradiction. Therefore $x \in A$ and hence $\alpha \hat{g} - d (A) \subseteq A$.

Conversely, assume that $\alpha \hat{g} - d (A) \subseteq A$, and we will show that $A$ is an $\alpha \hat{g}$-closed set in $X$. Otherwise we will show that there exists an $\alpha \hat{g}$-nbd $N$ of $x$ for each $x \in X - A$. Let $x$ be an arbitrary point of $X - A$ so that $x \notin A$ but as $\alpha \hat{g} - d (A) \subseteq A$, $x \notin A$ implies that $x \notin \alpha \hat{g} - d (A)$ ie., there exists an $\alpha \hat{g}$-nbd $N$ of $x$, which does not contain any point of $A$. That is there exists an $\alpha \hat{g}$-nbd $N$ of $x$ which contains only points of $X - A$. This means that $X - A$ is $\alpha \hat{g}$-open set and hence $A$ is $\alpha \hat{g}$-closed.

**THEOREM 4.3.14:** Let $A$ be any subset of a topological space $(X, \tau)$. Then $A \cup \alpha \hat{g} - d (A)$ is an $\alpha \hat{g}$-closed set.

**PROOF:** $A \cup \alpha \hat{g} - d (A)$ will be an $\alpha \hat{g}$-closed set in $X$ if the set $X - (A \cup \alpha \hat{g} - d (A))$ is an $\alpha \hat{g}$-open set in $X$. But by De- Morgan’s laws $X - (A \cup \alpha \hat{g} - d (A)) = (X - A) \cap (X - \alpha \hat{g} - d (A))$. Thus we will show that
(X - A) ∩ (X - α\(\hat{g}\) - d(A)) is an α\(\hat{g}\) - open set in X i.e. it contains an α\(\hat{g}\) - nbd of each of its points.

Let x ∈ (X - A) ∩ (X - α\(\hat{g}\) - d(A)) implies x ∈ X - A and x ∈ X - α\(\hat{g}\) - d(A) or x ∉ A and x ∉ α\(\hat{g}\) - d(A). Since x ∉ α\(\hat{g}\) - d(A) i.e. x is not a α\(\hat{g}\) - limit point of A, it follows that there exists an α\(\hat{g}\) - nbd N of x which contains no points of A other than possibly x. But x ∉ A so that N contains no point of A and as such N is a subset of A^c i.e. N ⊆ X - A. Again N is α\(\hat{g}\) - open set and is an α\(\hat{g}\) - nbd of each of its points, but as N does not contain any point of A, no point of N can be a limit point of A. That is, no point of N can belong to α\(\hat{g}\) - d(A). Thus N is a subset of (α\(\hat{g}\) - d(A))^c, i.e. N ⊆ X - α\(\hat{g}\) - d(A).

Therefore N ⊆ (X - A) ∩ (X - α\(\hat{g}\) - d(A)). This implies X - (A - α\(\hat{g}\) - d(A)) contains an α\(\hat{g}\) - nbd N of each of its points so that it is α\(\hat{g}\) - open by Lemma 4.3.3. Hence A ∪ α\(\hat{g}\) - d(A) is α\(\hat{g}\) - closed.

**THEOREM 4.3.15:** In any topological space (X, τ), every α\(\hat{g}\) - derived set is an α\(\hat{g}\) - closed set.

**PROOF:** Let A be a set of X and α\(\hat{g}\) - d(A) is an α\(\hat{g}\) - derived set of A. Then by theorem 4.3.13, A is an α\(\hat{g}\) - closed if and only if α\(\hat{g}\) - d(A) ⊆ A. Hence α\(\hat{g}\) - d(A) is α\(\hat{g}\) - d(A) is α\(\hat{g}\) - closed if and only if α\(\hat{g}\) - d(α\(\hat{g}\) - d(A)) ⊆ α\(\hat{g}\) - d(A) i.e., every α\(\hat{g}\) - limit point of α\(\hat{g}\) - d(A) belongs to α\(\hat{g}\) - d(A).

Let x be an α\(\hat{g}\) - limit point of α\(\hat{g}\) - d(A) i.e. x ∈ α\(\hat{g}\) - d(α\(\hat{g}\) - d(A)) so that there exists an α\(\hat{g}\) - open set G containing x such that (G - {x}) ∩ α\(\hat{g}\) - d(A) ≠ Ø implies (G - {x}) ∩ A ≠ Ø. Since every
\begin{align*}
\alpha \mathfrak{g}\text{-} \text{nhd of an element of } \alpha \mathfrak{g}\text{-}d(A) \text{ has at least one point of } A. \text{ Therefore, } x \text{ is a } \alpha \mathfrak{g}\text{-} \text{limit point of } A \text{ ie., } x \in \alpha \mathfrak{g}\text{-}d (A). \text{ Thus } x \in \alpha \mathfrak{g}\text{-}d (\alpha \mathfrak{g}\text{-}d (A)) \text{ implies } x \in \alpha \mathfrak{g}\text{-}d (A). \text{ Therefore } \alpha \mathfrak{g}\text{-}d (A) \text{ is an } \alpha \mathfrak{g}\text{-} \text{closed set in } X.
\end{align*}

4.4. \( \alpha \mathfrak{g} \) - R \text{ SPACES AND } \alpha \mathfrak{g} \text{ - R}_1 \text{ SPACES}

DEFINITION 4.4.1: \text{ Let } A \text{ be a subset of a topological space } (X, \tau). \text{ The } \alpha \mathfrak{g}\text{-kernel of } A, \text{ denoted by } \alpha \mathfrak{g}\text{ker}(A) \text{ is defined to be a set } \alpha \mathfrak{g}\text{ker}(A) = \bigcap \{ U : A \subseteq U \text{ and } U \text{ is } \alpha \mathfrak{g}\text{-open in } (X, \tau) \}.

DEFINITION 4.4.2: \text{ Let } x \text{ be a point of a topological space } (X, \tau). \text{ The } \alpha \mathfrak{g}\text{kernel of } x, \text{ denoted by } \alpha \mathfrak{g}\text{ker}(\{x\}) \text{ is defined to be the set } \alpha \mathfrak{g}\text{ker}(\{x\}) = \bigcap \{ U : x \in U \text{ and } U \text{ is } \alpha \mathfrak{g}\text{-open in } (X, \tau) \}.

THEOREM 4.3.3: \text{ Let } (X, \tau) \text{ be a topological space and } x \in X. \text{ Then } \alpha \mathfrak{g}\text{-ker}(A) = \{ x \in X : \alpha \mathfrak{g}\text{-Cl}(\{x\}) \cap A \neq \emptyset \}.

PROOF: \text{ Assume that } x \in \alpha \mathfrak{g}\text{-ker}(A) \text{ and } \alpha \mathfrak{g}\text{Cl}(\{x\}) \cap A = \emptyset. \text{ Hence } x \notin X - \alpha \mathfrak{g}\text{Cl}(\{x\}) \text{ which is an } \alpha \mathfrak{g}\text{-open set containing } A. \text{ This is impossible, since } x \in \alpha \mathfrak{g}\text{ker}(A). \text{ Hence } \alpha \mathfrak{g}\text{Cl}(\{x\}) \cap A \neq \emptyset.

Conversely, let } \alpha \mathfrak{g}\text{Cl}(\{x\}) \cap A \neq \emptyset \text{ and assume that } x \notin \alpha \mathfrak{g}\text{ker}(\{A\}). \text{ Then there exists an } \alpha \mathfrak{g}\text{-open set } U \text{ containing } A \text{ and } x \notin U. \text{ Let } y \in \alpha \mathfrak{g}\text{Cl}(\{x\}) \cap A. \text{ Hence, } U \text{ is an } \alpha \mathfrak{g}\text{nbd of } y \text{ for which } x \notin U. \text{ By this contradiction, } x \in \alpha \mathfrak{g}\text{ker}(A).
DEFINITION 4.4.4: A topological space \((X, \tau)\) is said to be \(\alpha \hat{g} - R_0\) space if and only if for each \(\alpha \hat{g}\) - open set \(G\) and \(x \in G\) implies 
\(\alpha \hat{g} \ Cl(\{x\}) \subseteq G\).

THEOREM 4.4.5: Let \((X, \tau)\) be a topological space and \(x \in X\). Then 
y \in \alpha \hat{g} \ker(\{x\}) if and only if \(x \in \alpha \hat{g} \ Cl(\{y\})\).

PROOF: Suppose that \(y \not\in \alpha \hat{g} \ker(\{x\})\). Then there exists an \(\alpha \hat{g}\) -open set \(A\) containing \(x\) such that \(y \not\in A\). Therefore we have \(x \not\in \alpha \hat{g} \ Cl(\{y\})\).

Converse is similar.

THEOREM 4.4.4: For any points \(x\) and \(y\) in a topological space \((X, \tau)\), the following statements are equivalent.

i) \(\alpha \hat{g} \ ker(\{x\}) \neq \alpha \hat{g} \ ker(\{y\})\).

ii) \(\alpha \hat{g} \ Cl(\{x\}) \neq \alpha \hat{g} \ Cl(\{y\})\).

PROOF: (i) \(\rightarrow\) (ii). Suppose that \(\alpha \hat{g} \ ker(\{x\}) \neq \alpha \hat{g} \ ker(\{y\})\), then there exists a point \(z\) in \(X\) such that \(z \in \alpha \hat{g} \ ker(\{x\})\) and \(z \not\in \alpha \hat{g} \ ker(\{y\})\). Since \(z \in \alpha \hat{g} \ ker(x)\), \(\{x\} \cap \alpha \hat{g} \ Cl(\{z\}) = \emptyset\). This implies \(x \in \alpha \hat{g} \ Cl(\{z\})\). By \(z \not\in \alpha \hat{g} \ ker(\{y\})\) we have \(\{y\} \cap \alpha \hat{g} \ Cl(\{z\}) = \emptyset\). Since \(x \in \alpha \hat{g} \ Cl(\{z\})\), \(\alpha \hat{g} \ Cl(\{x\}) \subseteq \alpha \hat{g} \ Cl(\{z\})\) and \(\{y\} \cap \alpha \hat{g} \ Cl(\{x\}) = \emptyset\). Hence \(\alpha \hat{g} \ Cl(\{x\}) \neq \alpha \hat{g} \ Cl(\{y\})\).

(ii) \(\rightarrow\) (i). Suppose that \(\alpha \hat{g} \ Cl(\{x\}) \neq \alpha \hat{g} \ Cl(\{y\})\). Then there exists a point \(z \in \alpha \hat{g} \ Cl(\{x\})\) and \(z \not\in \alpha \hat{g} \ Cl(\{y\})\). Then, there exists an \(\alpha \hat{g}\) -open set containing \(z\) and therefore \(x\), but not \(y\) i.e. \(y \not\in \alpha \hat{g} \ ker(\{x\})\). Hence \(\alpha \hat{g} \ ker(\{x\}) \neq \alpha \hat{g} \ ker(\{y\})\).
THEOREM 4.4.5: A topological space $(X, \tau)$ is an $\alpha \hat{g} - R_0$ space if and only if for any $x, y \in X$, $\alpha \hat{g} \text{Cl} \{x\} \neq \alpha \hat{g} \text{Cl} \{y\}$ implies $\alpha \hat{g} \text{Cl} \{x\} \cap \alpha \hat{g} \{y\} = \emptyset$.

PROOF: Suppose that $(X, \tau)$ is an $\alpha \hat{g} - R_0$ space and $x, y \in X$ such that $\alpha \hat{g} \text{Cl} \{x\} \neq \alpha \hat{g} \{y\}$. Then there exists a point $z \in \alpha \hat{g} \text{Cl} \{x\}$ such that $z \notin \alpha \hat{g} \text{Cl} \{y\}$ (or $z \in \alpha \hat{g} \text{Cl} \{y\}$ such that $z \notin \alpha \hat{g} \text{Cl} \{x\}$). There exists an $\alpha \hat{g}$-open set $V$ such that $y \notin V$ and $z \in V$; hence $x \in V$. Therefore, we have $x \notin \alpha \hat{g} \text{Cl} \{y\}$. Thus $x \in X - \alpha \hat{g} \text{Cl} \{y\}$, an $\alpha \hat{g}$-open set, which implies $\alpha \hat{g} \text{Cl} \{x\} \subseteq X - \alpha \hat{g} \text{Cl} \{y\}$ and $\alpha \hat{g} \text{Cl} \{x\} \cap \alpha \hat{g} \text{Cl} \{y\} = \emptyset$.

The proof for otherwise part is similar.

Conversely, let $V$ be an $\alpha \hat{g}$-open set in $(X, \tau)$ and let $x \in V$. We have to show that $\alpha \hat{g} \text{Cl} \{x\} \subseteq V$. Let $y \notin V$ i.e. $y \in X - V$. Then $x \neq y$ and $x \notin \alpha \hat{g} \text{Cl} \{y\}$. This implies $\alpha \hat{g} \text{Cl} \{x\} \neq \alpha \hat{g} \text{Cl} \{y\}$. By assumption, $\alpha \hat{g} \text{Cl} \{x\} \cap \alpha \hat{g} \text{Cl} \{y\} = \emptyset$. Hence $y \notin \alpha \hat{g} \text{Cl} \{x\}$ and therefore $\alpha \hat{g} \text{Cl} \{x\} \subseteq V$.

THEOREM 4.4.6: A topological space $(X, \tau)$ is an $\alpha \hat{g} - R_0$ space if and only if for any points $x$ and $y$ in $(X, \tau)$, $\alpha \hat{g} \ker \{x\} \neq \alpha \hat{g} \ker \{y\}$ implies $\alpha \hat{g} \ker \{x\} \cap \alpha \hat{g} \ker \{y\} = \emptyset$.

PROOF: Assume that $(X, \tau)$ is an $\alpha \hat{g} - R_0$ space. Then by theorem 4.4.4 for any points $x$ and $y$ in $X$, if $\alpha \hat{g} \ker \{x\} \neq \alpha \hat{g} \ker \{y\}$, then $\alpha \hat{g} \text{Cl} \{x\} \neq \alpha \hat{g} \text{Cl} \{y\}$. We will show that $\alpha \hat{g} \ker \{x\} \cap \alpha \hat{g} \ker \{y\} = \emptyset$. Suppose $z \in \alpha \hat{g} \ker \{x\} \cap \alpha \hat{g} \ker \{y\}$. By
Theorem 4.4.5 and $z \in \alpha \hat{g} \ker \{\{x\}\}$ implies $x \in \alpha \hat{g} \Cl \{\{z\}\}$. Since $x \in \alpha \hat{g} \Cl \{\{x\}\}$, by Theorem 4.4.5, $\alpha \hat{g} \Cl \{\{x\}\} = \alpha \hat{g} \Cl \{\{z\}\}$. Similarly, we have $\alpha \hat{g} \Cl \{\{y\}\} = \alpha \hat{g} \Cl \{\{z\}\} = \alpha \hat{g} \Cl \{\{x\}\}$, a contradiction. Hence $\alpha \hat{g} \ker \{\{x\}\} \cap \alpha \hat{g} \ker \{\{y\}\} = \emptyset$.

Conversely, let $(X, \tau)$ be a topological space such that for any two points $x$ and $y$ in $X$, $\alpha \hat{g} \ker \{\{x\}\} \neq \alpha \hat{g} \ker \{\{y\}\}$. Hence $\alpha \hat{g} \ker \{\{x\}\} \cap \alpha \hat{g} \ker \{\{y\}\} = \emptyset$. If $\alpha \hat{g} \Cl \{\{x\}\} \neq \alpha \hat{g} \Cl \{\{y\}\}$, then by Theorem 4.4.4 $\alpha \hat{g} \ker \{\{x\}\} \neq \alpha \hat{g} \ker \{\{y\}\}$. Hence $\alpha \hat{g} \ker \{\{x\}\} \cap \alpha \hat{g} \ker \{\{y\}\} = \emptyset$ implies $\alpha \hat{g} \Cl \{\{x\}\} \cap \alpha \hat{g} \Cl \{\{y\}\} = \emptyset$. Since $z \in \alpha \hat{g} \Cl \{\{x\}\}$ implies that $x \in \alpha \hat{g} \ker \{\{z\}\}$. Therefore $\alpha \hat{g} \ker \{\{x\}\} \cap \alpha \hat{g} \ker \{\{z\}\} \neq \emptyset$. By hypothesis, $\alpha \hat{g} \ker \{\{x\}\} = \alpha \hat{g} \ker \{\{z\}\}$. Then $z \in \alpha \hat{g} \Cl \{\{x\}\} \cap \alpha \hat{g} \Cl \{\{y\}\}$ implies that $\alpha \hat{g} \ker \{\{x\}\} = \alpha \hat{g} \ker \{\{z\}\} = \alpha \hat{g} \ker \{\{y\}\}$, a contradiction. Hence $\alpha \hat{g} \Cl \{\{x\}\} \cap \alpha \hat{g} \Cl \{\{y\}\} = \emptyset$. Therefore by Theorem 4.7, $(X, \tau)$ is an $\alpha \hat{g}$-space.

**Theorem 4.4.7:** For a topological space $(X, \tau)$, the following properties are equivalent.

i) $(X, \tau)$ is an $\alpha \hat{g}$-space.

ii) For any $A \neq \emptyset$ and $G$ is an $\alpha \hat{g}$-open set in $(X, \tau)$ such that $A \cap G \neq \emptyset$, there exists an $\alpha \hat{g}$-closed set $F$ in $(X, \tau)$ such that $A \cap F \neq \emptyset$ and $F \subseteq G$.

iii) For any $\alpha \hat{g}$-open set $G$ in $(X, \tau)$, $G = \bigcup \{F : F \subseteq G, F$ is an $\alpha \hat{g}$-closed set in $(X, \tau)\}$.

iv) For any $\alpha \hat{g}$-closed set $F$ in $(X, \tau)$, $F = \cap \{G : F \subseteq G, G$ is an $\alpha \hat{g}$-open set in $(X, \tau)\}$. For any $x \in X$, $\alpha \hat{g} \Cl \{\{x\}\} \subseteq \alpha \hat{g} \ker \{\{x\}\}$.
PROOF : (i) → (ii) : Let A be any non empty set and G be an $\alpha_{g}$-open set in $(X, \tau)$ such that $A \cap G \neq \emptyset$. There exists $x \in A \cap G$. Since $x \in G$, where $G$ is $\alpha_{g}$-open set in $(X, \tau)$, $\alpha_{g} \text{Cl} \{\{x\}\} \subseteq G$. Set $F = \alpha_{g} \text{Cl} \{\{x\}\}$, then $F$ is $\alpha_{g}$-closed, $F \subseteq G$ and $A \cap F \neq \emptyset$.

(ii) → (iii). Let $G$ be an $\alpha_{g}$-open set in $(X, \tau)$, then $G \supseteq \cup\{ F : F \subseteq G, F$ is $\alpha_{g}$-closed in $X \}$. Let $x$ be any point of $G$, there exists an $\alpha_{g}$-closed set $F$ in $(X, \tau)$ such that $x \in F$ and $F \subseteq G$. Therefore, $x \in F \subseteq \cup\{ F : F \subseteq G, F$ is $\alpha_{g}$-closed in $(X, \tau)\}$ and hence $G = \cup\{ F : F \subseteq G, F$ is $\alpha_{g}$-closed in $(X, \tau)\}$.

(iii) → (iv). Proof is obvious.

(iv) → (v). Let $x$ be any point of $X$ and $y \notin \alpha_{g} \ker \{\{x\}\}$. There exists $\alpha_{g}$-open set $U$ such that $x \in U$ and $y \notin U$. Hence $\alpha_{g} \text{Cl} \{\{y\}\} \cap U = \emptyset$. By (iv), $\cap\{ G : \alpha_{g} \text{Cl} \{\{y\}\} \subseteq G, G$ is $\alpha_{g}$ open set in $X \} \cap U = \emptyset$. There exist and $\alpha_{g}$ open set $G$ such that $x \in G$ and $\alpha_{g} \text{Cl} \{\{y\}\} \subseteq G$. Therefore $\alpha_{g} \text{Cl} \{\{x\}\} \cap G = \emptyset$ and $y \notin \alpha_{g} \text{Cl} \{\{x\}\}$. Consequently, we obtain $\alpha_{g} \text{Cl} \{\{x\}\} \subseteq \alpha_{g} \ker \{\{x\}\}$.

(v) → (i). Let $G$ be an $\alpha_{g}$-open set in $(X, \tau)$ and $x \in G$. Suppose $y \in \alpha_{g} \ker \{\{x\}\}$, then $x \in \alpha_{g} \text{Cl} \{\{y\}\}$ and $y \in G$. This implies that $\alpha_{g} \text{Cl} \{\{x\}\} \subseteq \alpha_{g} \ker \{\{x\}\} \subseteq G$. Therefore $(X, \tau)$ is an $\alpha_{g}$-$R_\alpha$ space.

COROLLARY 4.4.8: The following properties are equivalent for a topological space $(X, \tau)$.

i) $(X, \tau)$ is an $\alpha_{g}$-$R_\alpha$ space.

ii) $\alpha_{g} \text{Cl} \{\{x\}\} = \alpha_{g} \ker \{\{x\}\}$ for all $x \in X$.  

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**PROOF :** (i) $\rightarrow$ (ii). Assume that $(X, \tau)$ is an $\alpha\hat{g}$-$R_\circ$ space. By theorem 4.4.7 $\alpha\hat{g} \text{Cl}(\{x\}) \subseteq \alpha\hat{g} \text{ker}(\{x\})$ for each $x \in X$. Let $y \in \alpha\hat{g} \text{ker}(\{x\})$, then $x \in \alpha\hat{g} \text{Cl}(\{y\})$ and so $\alpha\hat{g} \text{Cl}(\{y\}) = \alpha\hat{g} \text{Cl}(\{x\})$. Therefore $y \in \alpha\hat{g} \text{Cl}(\{x\})$ and hence $\alpha\hat{g} \text{ker}(\{x\}) \subseteq \alpha\hat{g} \text{Cl}(\{x\})$. This shows that $v(\{x\}) = \alpha\hat{g} \text{ker}(\{x\})$.

(ii) $\rightarrow$ (i). This is obvious by Theorem 4.4.7.

Now, we define the following.

**DEFINITION 4.4.9:** A topological space $(X, \tau)$ is $\alpha\hat{g}$-symmetric if for any points $x$ and $y$ in $X$, $x \in \alpha\hat{g} \text{Cl}(\{y\})$ implies $y \in \alpha\hat{g} \text{Cl}(\{x\})$.

**THEOREM 4.4.10:** The following properties are equivalent for a topological space $(X, \tau)$.

i) $(X, \tau)$ is an $\alpha\hat{g}$-$R_\circ$ space,

ii) $x \in \alpha\hat{g} \text{Cl}(\{y\})$ if and only if $y \in \alpha\hat{g} \text{Cl}(\{x\})$ for any points $x$ and $y$ in $(X, \tau)$.

**PROOF :** (i) $\rightarrow$ (ii). Suppose $X$ is an $\alpha\hat{g}$-$R_\circ$ space. Let $x \in \alpha\hat{g} \text{Cl}(\{y\})$ and $U$ be any $\alpha\hat{g}$-open set such that $y \in U$. Now by hypothesis, $x \in U$. Therefore, every $\alpha\hat{g}$-open set containing $y$ contains $x$. Hence $y \in \alpha\hat{g} \text{Cl}(\{x\})$.

(iiib) $\rightarrow$ (i). Let $V$ be an $\alpha\hat{g}$-open set and $x \in V$. If $y \notin V$, then $x \notin \alpha\hat{g} \text{Cl}(\{y\})$ and hence $y \notin \alpha\hat{g} \text{Cl}(\{x\})$. This implies that $\alpha\hat{g} \text{Cl}(\{x\}) \subseteq V$. Hence $(X, \tau)$ is an $\alpha\hat{g}$-$R_\circ$ space.

**REMARK 4.4.11:** The notions of $\alpha\hat{g}$-symmetric and $\alpha\hat{g}$-$R_\circ$ are equivalent by Definition 4.4.9 and Theorem 4.4.10.
THEOREM 4.4.12: For a topological space \((X, \tau)\), the following properties are equivalent.

i) \((X, \tau)\) is an \( \alpha \hat{g} \)-\( R_o \) space.

ii) If \( A \) is an \( \alpha \hat{g} \)-closed, then \( A = \alpha \hat{g} \ker(A) \).

iii) If \( A \) is an \( \alpha \hat{g} \)-closed and \( x \in A \), then \( \alpha \hat{g} \ker(\{x\}) \subseteq A \).

iv) If \( x \in X \), then \( \alpha \hat{g} \ker(\{x\}) \subseteq \alpha \hat{g} \Cl(\{x\}) \).

PROOF: (i) \(\rightarrow\) (ii): Let \( A \) be \( \alpha \hat{g} \)-closed in \( X \) and \( x \notin A \). Thus \( X - A \) is \( \alpha \hat{g} \)-open and \( x \in X - A \). Since \((X, \tau)\) is \( \alpha \hat{g} \)-\( R_o \), \( \alpha \hat{g} \Cl(\{x\}) \subseteq X - A \). Thus \( \alpha \hat{g} \Cl(\{x\}) \cap A = \emptyset \) and by theorem 4.4.3, \( x \notin \alpha \hat{g} \ker(A) \). Therefore \( \alpha \hat{g} \ker(A) = A \).

(ii) \(\rightarrow\) (iii): In general, \( U \subseteq V \) implies \( \alpha \hat{g} \ker(U) \subseteq \alpha \hat{g} \ker(V) \). Therefore \( \alpha \hat{g} \ker(\{x\}) \subseteq \alpha \hat{g} \ker(\{A\}) = A \) by (ii).

(iii) \(\rightarrow\) (iv): Since \( x \in \alpha \hat{g} \Cl(\{x\}) \) and \( \alpha \hat{g} \Cl(\{x\}) \) is \( \alpha \hat{g} \)-closed, by (3) \( \alpha \hat{g} \ker(\{x\}) \subseteq \alpha \hat{g} \Cl(\{x\}) \).

(iv) \(\rightarrow\) (i): Let \( x \in \alpha \hat{g} \Cl(\{y\}) \). Then by Lemma 4.4.3, \( y \in \alpha \hat{g} \ker(\{x\}) \). Since \( x \in \alpha \hat{g} \Cl(\{x\}) \) and \( \alpha \hat{g} \Cl(\{x\}) \) is an \( \alpha \hat{g} \)-closed, by (ii) we obtain \( y \in \alpha \hat{g} \ker(\{x\}) \subseteq \alpha \hat{g} \Cl(\{x\}) \). Therefore \( x \in \alpha \hat{g} \Cl(\{y\}) \) implies \( y \in \alpha \hat{g} \Cl(\{x\}) \). The converse is obvious and \((X, \tau)\) is an \( \alpha \hat{g} \)-\( R_o \) space.

THEOREM 4.4.13: A topological space \((X, \tau)\) is \( \alpha \hat{g} \)-\( R \), if and only if for \( x, y \in X \); \( \alpha \hat{g} \ker(\{x\}) \neq \alpha \hat{g} \ker(\{y\}) \), there exist disjoint \( \alpha \hat{g} \)-open sets \( U \) and \( V \) such that \( \alpha \hat{g} \Cl(\{x\}) \cup \alpha \hat{g} \Cl(\{y\}) \subseteq V \).

PROOF: Follows from Lemma 4.4.4.
4.5. WEAKLY $\alpha^g$-CLOSED SETS.

We introduce the definition of a weakly $\alpha^g$-closed set in a topological space and study the relationship between other such closed sets.

**DEFINITION 4.5.1:** A subset $A$ of a topological space $X$ is called a weakly $\alpha^g$-closed (briefly $w^g$-closed) set if $\text{Cl} (\text{Int}(A)) \subseteq G$ and $G$ is $\alpha$gs-open in $X$.

**REMARK 4.5.2:** Every $\alpha^g$-closed set is $w\alpha^g$-closed. But the converse of this implication is not true in general.

**EXAMPLE 4.5.3:** Let $X = \{ a, b, c \}$ and $\tau = \{ \emptyset, \{a\}, X \}$. Then the set $\{b\}$ is $w\alpha^g$-closed but not $\alpha^g$-closed in $(X, \tau)$.

**COROLLARY 4.5.4:** Every closed set is $w\alpha^g$-closed.

**THEOREM 4.5.5:** Every $w\alpha^g$-closed set is gsp-closed.

**PROOF:** Let $A$ be $w\alpha^g$-closed and $G$ be an open set containing $A$ in $X$. Then $G \supseteq \text{Cl} (A) \supseteq \text{Cl} (\text{Int}(A))$. Thus $A$ is $w\alpha^g$-closed in $X$. Then $G$ is an $\alpha$gs-open set containing $A$ and so $G \supseteq \text{Int} (\text{Cl} (\text{Int}(A)))$. Which implies $A \cup G \supseteq A \cup \text{Int} (\text{Cl} (\text{Int}(A)))$, that is $G \supseteq \text{spcl}(A)$. Thus $A$ is gsp-closed set in $X$.

Converse of the above theorem need not be true as seen from the following example.
EXAMPLE 4.5.6: Let $X = \{a, b, c\}$, and $
abla = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Then the sets $\{a\}$ and $\{b\}$ are gsp-closed but not $w\alpha g$-closed in $X$.

THEOREM 4.5.7: If a subset $A$ of a topological space $X$ is both closed and $\alpha g$-closed, then it is $w\alpha g$-closed in $X$.

PROOF: Let be an $\alpha g$-closed set in $X$ and $G$ be an open set containing $A$. Then $G$ is $\alpha g$-open containing $A$ and so $G \supseteq Cl(A) = A \cup Cl(Int(A))$. Since $A$ is closed, $G \supseteq Cl(Int(A))$ and hence $A$ is $w\alpha g$-closed in $X$.

THEOREM 4.5.8: If a subset $A$ of a topological space $X$ is both open and $w\alpha g$-closed, then it is closed.

PROOF: Since $A$ is both open and $w\alpha g$-closed, $A \supseteq Cl(Int(A)) = Cl(A)$ and hence $A$ is closed set in $X$.

COROLLARY 4.5.9: If a subset $A$ of topological space $X$ is both open and $w\alpha g$-closed, then it is both regular open and regular closed in $X$.

THEOREM 4.5.10: A set $A$ is $w\alpha g$-closed if and only if $Cl(Int(A)) - A$ contains no nonempty $\alpha gs$-closed set.

PROOF: Necessity. Let $F$ be a $\alpha gs$-closed set such that $F \subseteq Cl(Int(A)) - A$. Since $F^c$ is $\alpha gs$-open and $A \subseteq F^c$, from the definition of $w\alpha g$-closed set it follows that $Cl(Int(A)) \subseteq F^c$ i.e., $F \subseteq (Cl(Int(A))^c)$. This implies that $F \subseteq Cl(Int(A)) \cap (Cl(Int(A))^c) = \emptyset$.

Sufficiency. Let $A \subseteq G$ and $G$ be an $\alpha gs$-open set in $X$. If $Cl(Int(A))$ is not contained in $G$, then $Cl(Int(A)) \cap G^c$ is a nonempty
αg - closed set of Cl (Int (A)) – A, we obtain a contradiction. This proves the sufficiency and hence the theorem.

**COROLLARY 4.5.11:** A wαg-closed set A is regular closed set if and if Cl (Int (A)) – A is αgs-closed set and Cl (Int (A)) ⊇ A.

**PROOF:** **Necessity.** Since the set A is regular closed set, Cl (Int (A)) – A = Ø. Therefore A is regular closed set.

**Sufficiency.** By theorem 4.5.10 Cl (Int (A)) – A contains nonempty αgs-closed set. That is Cl (Int (A)) – A = Ø. Therefore, A is regular closed.

**THEOREM 4.5.12:** Let (X, τ) be a topological space and B ⊆ A ⊆ X. If B is wαg-closed set relative to A and A is both open and αg-closed subset of X, then B is wαg-closed set relative to X.

**PROOF:** Let B ⊆ G and G be an αgs-open set in X. Then B ⊆ A ∩ G. Since B is wαg-closed relative to A, Cl (Int (B)) ⊆ A ∩ G. That is A ∩ Cl (Int (B)) ∩ A ∩ G. We have A ∩ Cl (Int (B)) ∩ G and then A ∩ Cl (Int (B)) ∩ Cl (Int (B)) ∩ G ∩ (Cl (Int (B)) ∩ G). Since A is wαg-closed in X, we have Cl (Int (A)) ∩ G ∩ (Cl (Int (B)) ∩ G). Therefore Cl (Int (B)) ∩ G, since Cl (Int (B)) is not contained in Cl (Int (B)) ∩ G. Thus B is wαg-closed set relative to X.

**COROLLARY 4.5.13:** If A be wαg-closed set and F is closed set in a topological space X, then A ∩ F is wαg-closed in X.
PROOF: Let $A \cap F$ is closed set in $A$. Therefore $\text{Cl}_A(A \cap F) = A \cap F$ in $A$. Let $A \cap F \subseteq G$, where $G$ is ags-open in $A$. then $\text{Cl}(\text{Int}(A \cap F)) \subseteq G$ and hence $A \cap F$ is $w\alpha \hat{g}$-closed in $A$. By theorem 4.5.9, $A \cap F$ is $w\alpha \hat{g}$-closed in $X$.

THEOREM 4.5.14: If $A$ is $w\alpha \hat{g}$-closed and $A \subseteq B \subseteq \text{Cl}(\text{Int}(A))$, then $B$ is $w\alpha \hat{g}$-closed.

PROOF: Since $A \subseteq B$, $\text{Cl}(\text{int}(B)) - B \subseteq \text{Cl}(\text{Int}(A)) - A$. theorem 4.5.10 $\text{Cl}(\text{Int}(A)) - A$ contains no nonempty closed set and so $\text{Cl}(\text{Int}(B)) - B$ again by theorem 4.5.10, $B$ is $w\alpha \hat{g}$-closed.

THEOREM 4.5.15: Let $X$ be a topological space and $A \subseteq Y \subseteq X$. If $A$ is $w\alpha \hat{g}$-closed in $X$, then $A$ is $w\alpha \hat{g}$-closed relative to $Y$.

PROOF: Let $A \subseteq Y \cap G$ where $G$ is $\alpha g$s-open in $X$. Since $A$ is $w\alpha \hat{g}$-closed in $X$ $A \subseteq G$ implies $\text{Cl}(\text{Int}(A)) \subseteq G$. That is $Y \cap (\text{Cl}(\text{Int}(A))) \subseteq Y \cap G$, where $Y \cap \text{Cl}(\text{Int}(A))$ is closure of $A$ in $Y$. Thus $A$ is $w\alpha \hat{g}$-closed relative to $Y$.

THEOREM 4.5.16: If a subset $A$ of a topological space $X$ is nowhere dense, then it is $w\alpha \hat{g}$-closed in $X$.

PROOF: Since $\text{Int}(A) \subseteq \text{Int}(\text{Cl}(A))$ and $A$ is nowhere dense, $\text{Int}(A) = \emptyset$ and hence $A$ is $w\alpha \hat{g}$-closed in $X$.

Converse of the above theorem need not be true as seen from the following example.
EXAMPLE 4.5.17: Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$. Then the set $\{a\}$ and $\{b, c\}$ is $\omega g\hat{\cdot}$-closed in $X$ but not nowhere dense in $X$.

REMARK 4.5.18: If any subsets $A$ and $B$ of topological space $X$ are $w\alpha \hat{\cdot} g\hat{\cdot}$-closed, then their union need not be $w\alpha \hat{\cdot} g\hat{\cdot}$-closed.

EXAMPLE 4.5.19: Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, X\}$. In this topological space the sets $\{a\}$ and $\{c\}$ are $w\alpha \hat{\cdot} g\hat{\cdot}$-closed but their union $\{a, c\}$ is not $w\alpha \hat{\cdot} g\hat{\cdot}$-closed in $X$.

REMARK 4.5.20: If any subsets $A$ and $B$ of a topological space are $w\alpha \hat{\cdot} g\hat{\cdot}$-closed, then their intersection need not be $w\alpha \hat{\cdot} g\hat{\cdot}$-closed.

EXAMPLE 4.5.21: Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a, b\}, \{a, c\}, \{a\}, \{b\}, X\}$. In this topological space, the subsets $\{a, b\}$ and $\{a, c\}$ are $w\alpha \hat{\cdot} g\hat{\cdot}$-closed but their intersection $\{a\}$ is not $w\alpha \hat{\cdot} g\hat{\cdot}$-closed in $X$.

THEOREM 4.5.22: Every $g\alpha$-closed set is $w\alpha \hat{\cdot} g\hat{\cdot}$-closed but not conversely.

PROOF: Suppose $A$ is $g\alpha$-closed subset of $X$ and let $G$ be an $\alpha$-open set containing $A$. By theorem 3.2 [18] $G$ is a $\alpha gs$-open set containing $A$. Now $G \supseteq \alpha Cl(A) = Cl(Int(Cl(A))) \supseteq Cl(Int(A))$. Thus $A$ is $w\alpha \hat{\cdot} g\hat{\cdot}$-closed in $X$.

Converse of the above theorem need not be true as seen from the following example.
**EXAMPLE 4.5.23:** Let $X = \{ a, b, c \}$ and $\tau = \{ \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X \}$. Then the set $\{a\}$ is $\omega\alpha\hat{g}$-closed in $X$ but not $g\alpha$-closed set.

**REMARK 4.5.24:** $\omega\alpha\hat{g}$ closedness is independent of semi-closedness, $g$-closedness, $gs$-closedness, $^g$-closedness, $^g$-semi-closedness and $\alpha\hat{g}$-semi-closedness.

**EXAMPLE 4.5.25:** Let $X = \{ a, b, c \}$ and $\tau = \{ \emptyset, \{a\}, X \}$. Then the set $\{a, b\}$ is $g$-closed, $gs$-closed, and $^g$-closed in $X$ but not $\omega\alpha\hat{g}$-closed in $X$.

**EXAMPLE 4.5.26:** Let $X = \{ a, b, c \}$ and $\tau = \{ \emptyset, \{a\}, \{b\}, \{a, b\}, X \}$. Then the set $\{a, b\}$ is not $w\alpha\hat{g}$-closed, not semi-closed set in $X$, but it is $^g$-closed, $\alpha g$ closed and $g$-closed set in $X$.

**EXAMPLE 4.5.27:** $X = \{ a, b, c \}$ and $\tau = \{ \emptyset, \{a\}, \{b\}, \{a, b\}, X \}$. Then the set $\{a\}$ is semi-closed, $^g$-gs-closed and $gs$-closed but not is $\omega\alpha\hat{g}$-closed in $X$.

**EXAMPLE 4.5.28:** Let $X$ and $\tau$ be as in Example 4.5.26, then the set $\{a\}$ is $\omega\alpha\hat{g}$-closed but not $gs$-closed and $g$-closed in $X$.

### 4.6. ON $\alpha\hat{g}$-CLOSED SETS IN TOPOLOGICAL SPACES

In this section, we define and study the concept of $\alpha\hat{g}$-closed sets in topological spaces.
DEFINITION 4.6.1: A subset $A$ of $X$ is called an $\alpha \tilde{g}$-semiclosed (briefly $\alpha \tilde{g}s$-closed) if $s\text{Cl}(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is $\alpha gs$-open in $X$.

THEOREM 4.6.2: Every closed set is $\alpha gs$-closed set in $X$.

PROOF: Let $A$ be a closed set in $X$. Note that $s\text{Cl}(A) \subseteq \text{Cl}(A)$ always and $\text{Cl}(A) = A$ if $A$ is closed set. So, if $A \subseteq G$, where $G$ is $\alpha gs$-open set in $X$, then $s\text{Cl}(A) \subseteq \text{Cl}(A) = A \subseteq G$. That is $s\text{Cl}(A) \subseteq G$ and $G$ is $\alpha gs$-open in $X$. Hence $A$ is $\alpha gs$-closed set in $X$.

The converse of the above theorem need not be true as seen from the following example.

EXAMPLE 4.6.3: Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, X\}$ Then the sets $\{b\}$ and $\{c\}$ are $\alpha gs$-closed but not closed in $X$.

REMARK 4.6.4: Since every closed set is semiclosed set, every semiclosed set is $\alpha gs$-closed in $X$.

PROOF 4.6.5: Trivial.

THEOREM 4.6.6: Every $\alpha$-closed set is $\alpha \tilde{g}s$-closed in $X$.

PROOF: Let $A \subseteq X$ be a closed set. Note that $s\text{Cl}(A) \subseteq \alpha \text{Cl}(A)$ always, and $\alpha \text{Cl}(A) = A$, if $A$ is $\alpha$-closed set. So, if $A \subseteq G$, where $G$ is $\alpha gs$-open in $X$, then $s\text{Cl}(A) \subseteq A \subseteq G$. Hence $A$ is $\alpha \tilde{g}s$-closed in $X$.

Converse of the above theorem need not be true as shown by the following example.
**EXAMPLE 4.6.7:** Let $X = \{a, b, c\}$ and $\tau = \emptyset, \{a\}, \{b\}, \{a, b\}, X$.
Then the sets $\{a\}$ and $\{b\}$ are $\alpha\bar{g}\bar{s}$-closed but not $\alpha$-closed in $X$.

We recall the following lemma due to Rajamani et.al.

**LEMMA 4.6.8:** [85] Every $\alpha\bar{g}\bar{s}$ open set in $X$ is $\alpha\bar{g}$-open in $X$.

Now we prove the following theorem.

**THEOREM 4.6.9:** Every $g^s$-closed set is $\alpha\bar{g}$ $s$-closed set in $X$.

**PROOF:** Let $A$ be a $g^s$-closed set in $X$. Let $A \subseteq U$, where $U$ is $\alpha\bar{g}\bar{s}$-open in $X$. Every $\alpha\bar{g}\bar{s}$ – open set in $X$ is $\alpha\bar{g}$- open in $X$ by the above lemma, therefore $U$ is $\alpha\bar{g}$-open in $X$. Hence $s\text{Cl}(A) \subseteq U$, so $A$ is $\alpha\bar{g}$ $s$ – closed set in $X$.

Converse of the above theorem need not be true as seen from the following example.

**EXAMPLE 4.6.10:** Let $X = \{a, b, c, d\}$ and $\tau = \emptyset, \{a\}, \{b\}, \{a, b\}$ Then the sets $\{a, b\}, \{a, b, d\}$ are $\alpha\bar{g}\bar{s}$-closed sets but not $g^s$-semi-closed in $X$.

**THEOREM 4.6.11:** Every $\alpha\bar{g}$ $s$ – closed set is $gs$-closed set in $X$.

**PROOF:** Let $A$ be an $\alpha\bar{g}$ $s$–closed set in $X$. Let $U$ be an open set containing $A$ in $X$. Since every open set is $\alpha\bar{g}\bar{s}$-open [85], $U$ is $\alpha\bar{g}\bar{s}$-open in $X$. Hence $s\text{Cl}(A) \subseteq U$, therefore $A$ is $gs$-closed in $X$.

Converse of the above theorem need not be true as seen from the following example.
**EXAMPLE 4.6.12:** Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, X\}$ Then the sets $\{a, b\}, \{a, c\}$ are gs-closed but not $\alpha\hat{g}s$ - closed set in $X$.

**THEOREM 4.6.13:** Every $\alpha\hat{g}s$ -closed set is $\psi$- closed set in $X$.

**PROOF:** Let $A$ be $\alpha\hat{g}s$ -closed set in $X$. Let $A \subseteq U$, where $U$ is $\alpha\hat{g}s$-open in $X$. Every $\alpha\hat{g}s$-open set is sg-open in $X$, therefore $U$ is sg-open in $X$. Hence $s\text{Cl}(A) \subseteq U$. So $A$ is $\psi$- closed set in $X$.

Converse of the above theorem need not be true as seen from the following example.

**EXAMPLE 4.6.14:** Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ Then the set $\{a, b, c\}$ is $\psi$-closed but not $\alpha\hat{g}s$-closed in $X$.

**THEOREM 4.6.15:** Every $\alpha\hat{g}s$ - closed set is $^*\text{gs}$-closed set in $X$.

**PROOF:** Let $A$ be any $\alpha\hat{g}s$ - closed set in $X$. Let $A \subseteq U$, where $U$ is $\hat{g}$-open set (or $\omega$-open set ) in $X$. Every $\omega$-open ($\hat{g}$-open set ) in $X$ is $\alpha\hat{g}s$-open in $X$ [85]. Therefore $U$ is $\alpha\hat{g}s$-open in $X$. hence $s\text{Cl}(A) \subseteq U$. So $A$ is $^*\text{gs}$-closed in $X$.

Converse of the above theorem need not be true as seen from the following example.

**EXAMPLE 4.6.16:** Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ Then the set $\{a, b, c\}$ is $^*\text{gs}$ but not $\alpha\hat{g}s$-closed in $X$.

**THEOREM 4.6.17:** Every $\alpha\hat{g}$-closed set is $\alpha\hat{g}$-$s$-closed set in $X$. 
**PROOF:** Let $A$ be any $\alpha \hat{g}$-closed set in $X$ and $U$ be any $\alpha g$s-open set in $X$ containing $A$. Then $\text{Cl}(A) \subseteq U$. Since $s\text{Cl}(A) \subseteq \text{Cl}(U)$ always, we have $s\text{Cl}(A) \subseteq \text{Cl}(A) \subseteq (U)$, That is $s\text{Cl}(A) \subseteq U$. Therefore $A$ is $\alpha \hat{g}$ s–closed set in $X$.

Converse of the above theorem need not be true as seen from the following example.

**EXAMPLE 4.6.18:** Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Then the sets $\{a\}, \{b\}$ are $\alpha g$s-closed sets but not $\alpha g$s-closed sets in $X$.

In general union of two $\alpha \hat{g}$ s–closed sets is not $\alpha \hat{g}$ s–closed set in $X$. Therefore we have the following results for the union of two $\alpha \hat{g}$ s–closed set s in $X$.

**THEOREM 4.6.19:** Let $A$ and $B$ be any two $\alpha \hat{g}$ s–closed sets in $X$. such that $D[A] \subseteq D\alpha[A]$ and $D[B] \subseteq D\alpha[B]$. Then $A \cup B$ is an $\alpha \hat{g}$ s–closed set in $X$.

**PROOF:** Let $U$ be an $\alpha g$s-open in $X$ such that $A \cup B \subseteq U$, That is $A \subseteq U$ and $B \subseteq U$, then $s\text{Cl}(A) \subseteq U$ and $s\text{Cl}(B) \subseteq U$. however, for any set $E$, $D\alpha[E] \subseteq D[E]$ [8]. Therefore $\text{Cl}(A) = s\text{Cl} (A)$ and $Cl (B) = s\text{Cl} (B)$, and this shows that $\text{Cl} (A \cup B) = Cl(A) \cup Cl(B) \subseteq U$ , That is $s\text{Cl} (A) \subseteq s\text{Cl} (B) \subseteq U$. since $s\text{Cl} (A \cup B) \subseteq \text{Cl} (A \cup B)$. Always we have , $s\text{Cl} (A \cup B) \subseteq \text{Cl} (A \cup B) = s\text{Cl} (A) \cup s\text{Cl} (B) \subseteq U$. That is. $s\text{Cl} (A \cup B) \subseteq U$. Hence $A \cup B$ is $\alpha \hat{g}$ s–closed set in $X$.

We recall the following result.
LEMMA 4.5.20: [85] Every closed set is αgs-closed in X.

THEOREM 4.5.21: Let $B \subseteq A$ where $A$ is open and $\alpha \dot g$ s-closed. Then $B$ is $\alpha \dot g$ s-closed relative to $A$ if and only if $B$ is $\alpha \dot g$ s-closed.

PROOF: Necessity. Since $A$ is open and $\alpha \dot g$ s-closed, $s\text{Cl}_A(A) \subseteq A$. Since $B \subseteq A$, $s\text{Cl}_A(B) = s\text{Cl}_A(A) = A$. Now by [8], $A \cap s\text{Cl}_A(B) = s\text{Cl}_A(B)$.

If $B$ is $\alpha \dot g$ s-closed relative to $A$, then $s\text{Cl}_A(B) \subseteq O$, whenever $B \subseteq O$ and $O$ is αgs-open in $A$. The openness of $A$ in $X$ gives that $O$ is also αgs-open in $X$. Therefore $B \subseteq O$ implies $s\text{Cl}_X(B) \subseteq O$, i.e. $B$ is αgs-closed.

Sufficiency. If $B$ is $\alpha \dot g$ s-closed, then $s\text{Cl}_X(B) \subseteq O$, whenever $B \subseteq O$ and $O$ is αgs-open. This implies that $A \cap s\text{Cl}_X(B) \subseteq A \cap O$ whenever $B \subseteq A \cap O$.

Now since $A \cap U$ is αgs-open in $X$ (Since arbitrary intersection of two αgs-open set is αgs-open[ ] ). It is αgs-open in $A$ and therefore $s\text{Cl}_A(B) \subseteq A \cap O$. $B$ is $\alpha \dot g$ s – relative to $A$.

THEOREM 4.5.22: If $A$ is $\alpha \dot g$ s-closed in $X$ and $A \subseteq B \subseteq s\text{Cl}(A)$, then $B$ is $\alpha \dot g$ s-closed in $X$.

PROOF: Let $B \subseteq U$, where $U$ is αgs-open in $X$. Since $A$ is $\alpha \dot g$ s-closed and $A \subseteq U$, it follows that $s\text{Cl}(A) \subseteq U$. By hypothesis, $B \subseteq s\text{Cl}(A)$ and hence $s\text{Cl}(B) \subseteq s\text{Cl}(A) \subseteq U$. Consequently $s\text{Cl}(B) \subseteq U$. So $B$ become $\alpha \dot g$ s-closed.
THEOREM 4.5.23: A subset A of X is \(\alpha_{gs}\)-s-closed if and only if \(s\text{Cl}(A) \subseteq \alpha_{gs}\text{-ker}(A)\).

PROOF: Suppose that A is \(\alpha_{gs}\)-s-closed. Then \(s\text{Cl}(A) \subseteq U\), whenever U is \(\alpha_{gs}\)-open. Let \(x \in s\text{Cl}(A)\). If \(x \notin \alpha_{gs}\text{-ker}(A)\), then there is a \(\alpha_{gs}\)-open set in U containing A such that \(x \notin U\). Since U is an \(\alpha_{gs}\)-open set containing A, we have \(x \notin s\text{Cl}(A)\), a contradiction. Conversely, let \(s\text{Cl}(A) \subseteq \alpha_{gs}\text{-ker}(A)\). If U is any \(\alpha_{gs}\)-open set containing A, then \(s\text{Cl}(A) \subseteq \alpha_{gs}\text{ker}(A) \subseteq U\). Therefore A is \(\alpha_{gs}\)-s-closed.

THEOREM 4.5.24: If A is \(\alpha_{gs}\)-s-closed in X, then \(s\text{Cl}(A) - A\) contains no non empty closed set in X.

PROOF: Suppose that A is \(\alpha_{gs}\)-s-closed set. Let F be closed subset of \(s\text{Cl}(A) - A\), then \(A \subseteq F^c\), \(F^c\) is open. Therefore \(F^c\) is \(\alpha_{gs}\)-open in X [13], since A is \(\alpha_{gs}\)-s-closed, we have \(s\text{Cl}(A) \subseteq F^c\), consequently \(F \subseteq (s\text{Cl}(A))^c\), we have \(F \subseteq s\text{Cl}(A)\). Thus \(F \subseteq s\text{Cl}(A) \cap (s\text{Cl}(A))^c = \emptyset\) and therefore F is empty.

The converse of the above theorem need not be true as seen from the following example.

EXAMPLE 4.5.25: Let \(X = \{a, b, c\}\) and \(\tau = \{\emptyset, \{a\}, X\}\) Let \(A = \{a, b\}\). Then \(s\text{Cl}(A) - A\) does into contain any non empty closed set. But A is not \(\alpha_{gs}\)-closed in X.

THEOREM 4.5.26: A set A is \(\alpha_{gs}\)-s-closed set if and only if \(s\text{Cl}(A) - A\) contains no non empty \(\alpha_{gs}\)-closed set in X.
PROOF: Suppose that $A$ is an $\alpha \cdot \check{g}$ s-closed set. Let $U$ be an $\alpha g s$-closed subset of $sCl(A) - A$, then $A \subseteq U^c$. Since $A$ is $\alpha \cdot \check{g}$ s-closed, $sCl(A) \subseteq U^c$ which implies $U \subseteq (sCl(A))^c$. Hence $U \subseteq sCl(A) \cap (sCl(A))^c = \emptyset$, which implies $U = \emptyset$.

Conversely, suppose $A$ is a subset of $X$ such that $sCl(A) - A$ does not contain any non empty $\alpha g s$-closed set. Let $U$ be an $\alpha g s$-open set in $X$ such that $A \subseteq U$. If $sCl(A) \not\subseteq U$, then $sCl(A) \subseteq U$ and $sCl(A) \cap U^c$ is a non empty $\alpha g s$-closed subset of $sCl(A) - A$. Therefore $A$ is an $\alpha \cdot \check{g}$ s-closed set in $X$.

COROLLARY 4.5.27: Let $A$ be a $\alpha \cdot \check{g}$ s-closed set of $X$. Then $A$ is $\alpha g s$-closed iff $sCl(A) - A$ is $\alpha g s$-closed.

PROOF: Let $A$ be $\alpha \cdot \check{g}$ s-closed which is also $\alpha g s$-closed. Then $sCl(A) - A = \emptyset$. That is $sCl(A) - A$ does not contain any non empty $\alpha g s$-closed subset of $X$. Since $sCl(A) - A$ is $\alpha g s$-closed, $sCl(A) - A = \emptyset$, which implies that $A$ is $\alpha g s$-closed.

4.6. $\alpha g s$-OPEN SETS

DEFINITION 4.6.1: A set $A$ is called $\alpha \cdot \check{g}$ s-open set if and only if $A^c$ is $\alpha \cdot \check{g}$ s-closed set.

THEOREM 4.6.2: A set $A$ is $\alpha \cdot \check{g}$ s-open if and only if $F \subseteq sInt A$ whenever $F$ is $\alpha g s$-closed and $F \subseteq A$. 

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PROOF: Let $A$ be an $\alpha \check{g}$ s-open set. Suppose $F \subset A$ where $F$ is $\alpha g$s-closed. By definition $X - A$ is $\alpha \check{g}$ s-closed set. Also $X - A$ contained in the $\alpha g$s-open set $X - F$. This implies $sCl(X - A) \subset X - F$. Now $sCl(X - F) = X - \text{sint} A \subset X - F$. Now $sCl(X - A) = X - \text{sint} A$ [8]. Hence, $X - \text{sint} A \subset X - F$. That is $F \subset \text{sint} A$.

Conversely, if $F$ is an $\alpha g$s-closed set with $F \subset \text{sint} A$ whenever $F \subset A$, it follows that $X - A \subset X - F$ and $X - \text{sint} A \subset X - F$; that is $sCl (X-A) \subset X - F$. Hence $X - A$ is $\alpha \check{g}$ s-closed set and $A$ becomes $\alpha \check{g}$ s-open. This proves the theorem.

REMARK 4.6.3: Every semi-open set is $\alpha \check{g}$ s-open but converse is not true.

EXAMPLE 4.6.4: Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a,b,c\}, \{b,c,d\}, \{b,c\}\}$. Then the sets $\{b\}$ and $\{c\}$ are $\alpha g$s-open but not semi-open sets in $X$.

THEOREM 4.6.5: Let $A$ and $B$ be semi-separated $\alpha \check{g}$ s-open sets, then $A \cup B$ is $\alpha \check{g}$ s-open.

PROOF: Let $A$ and $B$ be two semi-separated $\alpha \check{g}$ s-open sets, then we have $sCl (A) \cap B = A \cap sCl (B)$. If $F$ is an $\alpha \check{g}$ s-closed set such that $F \subset A \cup B$, then $F \cap sCl(A) \subset sCl(A) \cap ( A \cup B ) = ( sCl(A) \cap A ) \cup ( sCl(A) \cap B ) = A \cup \emptyset = A$. Similarly, $F \cap sCl(B) \subset B$. Hence by Theorem 4.6.2 we have, $F \subset \text{sint} A$ and $F \subset \text{sint} B$ that is, $F \cap sCl(A) \subset F \cap \text{sint} A = \text{sint} A$ (since $F \subset B$) and $F \cap sCl(B) \subset \text{sint} (B)$. Now $F = F \cap ( A \cup B ) = ( F \cap A ) \cup ( F \cap B ) \subset ( F \cap sCl(A) ) \subset \text{sint} A \cup \text{sint} B \subset \text{sint} ( A \cup B )$ [4].
Hence by Theorem 4.6.2, \( A \cup B \) is \( \alpha \hat{g} \) s-open.

**THEOREM 4.6.6:** If slnt \( A \subset B \subset A \) and \( A \) is \( \alpha \hat{g} \) s-open, then \( B \) is \( \alpha \hat{g} \) s-open.

**PROOF:** By hypothesis, \( A^c \subset B^c \subset (\text{slnt}(A))^c \) implies \( A^c \subset B^c \subset [X - \text{sCl}(A)]^c = \text{sCl}(A)^c \).

Now \( A^c \) is \( \alpha \hat{g} \) s-closed set and hence by Theorem 4.5.22, \( B^c \) is \( \alpha \hat{g} \) s-closed set. Therefore \( B \) is \( \alpha \hat{g} \) s-open.

**LEMMA 4.6.7:** For any \( A \subset X \), slnt \((\text{sCl}(A) - A) = \emptyset\).

**THEOREM 4.6.8:** If a set \( A \) is \( \alpha \hat{g} \) s-closed set, then \( \text{sCl}(A) - A \) is \( \alpha \hat{g} \) s-open.

**PROOF:** If \( A \) is an \( \alpha \hat{g} \) s-closed set and \( F \) is an \( \alpha \text{gs} \)-closed set such that \( F \subset \text{sCl}(A) - A \), then by Theorem 4.5.26, \( F = \emptyset \). Hence \( F \subset \text{sint} (\text{sCl}(A)- A) \) and by Theorem 4.6.2, \( \text{sCl}(A) - A \) is \( \alpha \hat{g} \) s-open.

### 4.7. \( \alpha \hat{g} \text{s} \)-SEPARATION AXIOMS

**DEFINITION 4.7.1:** A topological space \((X, \tau)\) is said to be \( \alpha \hat{g} \text{s} \)-semi-\( T_0 \) (briefly \( \alpha \hat{g} \text{s}-T_0 \)) if and only if to each pair of distinct points \( x, y \) of \( X \), there exists an \( \alpha \hat{g} \) s-open set containing one of the points but not the other.
EXAMPLE 4.7.2: Let \( X = \{ a, b, c \} \), \( \tau = (\emptyset, \{ a \}, X) \). Then the space \((X, \tau)\) is \( \alpha \hat{g} s-T_0 \).

REMARK 4.7.4 (i) Every \( \alpha- T_0 \) space is \( \alpha\hat{g}s - T_0 \) space. Since every \( \alpha \)-open set is \( \alpha\hat{g}s \) open.

(ii) Every \( \alpha\hat{g}s-T_0 \) space is \( gs - T_0 \) space. Since every \( \alpha\hat{g}s \) open set is \( sg \) open.

(iii) Every \( \alpha\hat{g}s-T_0 \) space is \( ^*gs - T_0 \) space. Since every \( \alpha\hat{g}s \) open set is \( ^*gs \) open.

We characterize \( \alpha \hat{g}s-T_0 \) spaces in the following.

THEOREM 4.7.5: A topological space \((X, \tau)\) is \( \alpha \hat{g}s-T_0 \) space if and only if \( \alpha \hat{g}s \) closures of distinct points are distinct.

PROOF. Let \( x, y \in X \) with \( x \neq y \) and \((X, \tau)\) is \( \alpha \hat{g}s- T_0 \) space. We will show that \( \alpha \hat{g}s Cl(\{x\}) \neq \alpha \hat{g}s Cl(\{y\}) \). Since \((X, \tau)\) is \( \alpha \hat{g}s- T_0 \), there exists a \( \alpha \hat{g}s \) open set \( G \) such that \( x \in G \) but \( y \notin G \). Also \( x \notin X - G \) and \( y \in X - G \), where \( X - G \) is \( \alpha \hat{g}s \) closed set in \((X, \tau)\). Now by Definition \( \{y\} \) is the intersection of all \( \alpha \hat{g}s \) closed sets which contain \( y \). Hence \( y \in \alpha \hat{g}s Cl(\{y\}) \) but \( x \notin \alpha \hat{g}s Cl(\{y\}) \) as \( x \notin X - G \). Therefore that \( \alpha \hat{g}s Cl(\{x\}) \neq \alpha \hat{g}s Cl(\{y\}) \).

Conversely, for any pair of distinct points \( x, y \in X \) and \( \alpha \hat{g}s Cl(\{x\}) \neq \alpha \hat{g}s Cl(\{y\}) \). Then there exists at least one point \( z \in X \) such that \( z \in \alpha \hat{g}s Cl(\{x\}) \) but \( z \notin \alpha \hat{g}s Cl(\{y\}) \). We claim that \( x \notin \alpha \hat{g}s Cl(\{y\}) \) because if \( x \in \alpha \hat{g}s Cl(\{y\}) \), then \( \{x\} \subseteq \alpha \hat{g}s Cl(\{y\}) \) implies \( \alpha \hat{g}s Cl(\{x\}) \subseteq \alpha \hat{g}s Cl(\{y\}) \). So, \( z \in \alpha \hat{g}s Cl(\{y\}) \), which is a contradiction. Hence \( x \notin \alpha \hat{g}s Cl(\{y\}) \). Now \( x \notin \alpha \hat{g}s Cl(\{y\}) \) implies \( x \notin X - \alpha \hat{g}s Cl(\{y\}) \).
which is an $\alpha \mathcal{g}$ s-open set in $(X, \tau)$ containing $x$ but not $y$. Hence $(X, \tau)$ is a $\alpha \mathcal{g}$ s- $T_0$ space.

**THEOREM 4.7.6:** Every subspace of $\alpha \mathcal{g}$ s-$T_0$ space is a $\alpha \mathcal{g}$ s-$T_0$ space. In other words the property of being a $\alpha \mathcal{g}$ s-$T_0$ space is a hereditary property.

**PROOF.** Let $(X, \tau)$ be a topological space and $(Y, \tau^*)$ be a subspace of $(X, \tau)$ where $\tau^*$ is a relative topology. Let $y_1, y_2$ be two distinct points of $Y$ and as $Y \subseteq X$, therefore these two are distinct points of $X$. Since $(X, \tau)$ is a $\alpha \mathcal{g}$ s-$T_0$ space, there exists a $\alpha \mathcal{g}$ s-open set $G$ such that $y_1 \in G$ and $y_2 \in G$. Then by definition $G \cap Y$ is an $\alpha \mathcal{g}$ s-open set in $(Y, \tau^*)$ which contains $y_1$ but not $y_2$. Hence $(Y, \tau^*)$ is a $\alpha \mathcal{g}$ s-$T_0$ space.

We define the following map analogous to always open $,\alpha$-open map defined in [1].

**DEFINITION 4.7.7:** A map $f: (X, \tau) \to (Y,\sigma)$ is said to be always $\alpha \mathcal{g}$ s-open map if the image of every $\alpha \mathcal{g}$ s-open set is $\alpha \mathcal{g}$ s-open.

**EXAMPLE 4.7.8:**

1. Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\}$. Let $f$ be the identity map. Then the map $f$ is always an $\alpha \mathcal{g}$ s-open.

2. Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$ and $\sigma = \{\emptyset, \{a, b\}, Y\}$. Let $f: (X, \tau) \to (Y, \sigma)$ be the identity map. Then the map $f$ is not always an $\alpha \mathcal{g}$ s-open since $f(\{a, c\}) = \{a, c\}$ which is not an $\alpha \mathcal{g}$ s-open set in $(Y, \sigma)$.

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**THEOREM 4.7.9:** The property of a space being an $\alpha\hat{g}$ s-To space is preserved under one-one, onto, always an $\alpha\hat{g}$ s-open mapping and hence is a topological property.

**PROOF.** Let $(X, \tau)$ be a an $\alpha\hat{g}$ s-$T_0$ space and $(Y, \sigma)$ be any other topological space. Let $f : (X, \tau) \to (Y, \sigma)$ be a one-one, onto, always an $\alpha\hat{g}$ s-open mapping from $X$ to $Y$. Let $y_1, y_2 \in Y$ with $y_1 \neq y_2$ and since $f$ is one-one, onto, there exist distinct points $x_1, x_2 \in X$ such that $f(x_1) = y_1$, $f(x_2) = y_2$. Since $(X, \tau)$ is a an $\alpha\hat{g}$ s-$T_0$ space, there exists an $\alpha\hat{g}$ s-open set $G$ in $(X, \tau)$ such that $x_1 \in G$ but $x_2 \notin G$. Since $f$ is always an $\alpha\hat{g}$ s-open, $f(G)$ is an $\alpha\hat{g}$ s-open set containing $f(x_1)$ but not containing $f(x_2)$.

Thus there exists an $\alpha\hat{g}$ s-open set $f(G)$ in $(Y, \sigma)$ such that $y_1 \in f(G)$ but $y_2 \notin f(G)$ and hence $(Y, \sigma)$ is a an $\alpha\hat{g}$ s-$T_0$ space.

Again as the property of being an $\alpha\hat{g}$ s-To space is preserved under one-one, onto mapping it is also preserved under homoeomorphism and hence it is a topological property.

We define the following.

**DEFINITION 4.7.10:** A topological space $(X, \tau)$ is said to an $\alpha\hat{g}$-generalized semi-$T_1$ (briefly an $\alpha\hat{g}$ s-$T_1$) if and only if to each pair of distinct points $x, y$ of $X$, there exists a pair of an $\alpha\hat{g}$ s-open sets, one containing $x$ but not $y$ and the other containing $y$ but not $x$.

**EXAMPLE 4.7.12:**
1. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$. Then the space $(X, \tau)$ is not $\alpha\hat{g}$ s-$T_1$.
2. Let $X = \{ a, b, c \}$, $\tau = \{ \emptyset, \{a\}, \{b\}, \{a, b\}, X \}$. Then the space $(X, \tau)$ is an $\alpha \hat{g} s-$ $T_1$ space.

**REMARK 4.7.16:**

i. Every $\alpha$-$T_1$ space is an $\alpha \hat{g}$ $s-$ $T_1$ space. Since $\alpha$-open set is $\alpha \hat{g}$ $s$-open.

ii. Every $\alpha \hat{g}$ $s-$ $T_1$ space is a $\alpha \hat{g}$ $s$-$T_0$ space.

iii. Every $\alpha \hat{g}$ $s-$ $T_1$ space is a $gs$-$T_1$ space. Since every $\alpha \hat{g}$ $s$-open set is $gs$-open in $X$.

The converse of the above remark need not be true as seen from the following examples.

**EXAMPLE 4.7.12:** Let $X = \{ a, b, c \}$, $\tau = \{ \emptyset, \{a\}, \{b\}, \{a, b\}, X \}$. Then the space $(X, \tau)$ is an $\alpha \hat{g}$ $s-$ $T_1$ space but not $\alpha$-$T_1$ space.

**EXAMPLE 4.7.12:** Let $X = \{ a, b, c \}$ and $\tau = \{ \emptyset, \{a\}, X \}$. Then the space $(X, \tau)$ is $gs$-$T_1$ but not $\alpha \hat{g}$ $s-$ $T_1$ space.

**THEOREM 4.7.13:** If every singleton subset $\{x\}$ of $X$ is $\alpha \hat{g}$ $s$-closed set, then $(X, \tau)$ is an $\alpha \hat{g}$ $s-$ $T_1$ space.

**PROOF.** Let $x, y$ be any two distinct points of $X$ so that $\{x\}$ and $\{y\}$ are $\alpha \hat{g}$ $s$- closed sets and as such $\{x\}^c$ and $\{y\}^c$ are $\alpha \hat{g}$ $s$- open. Thus $y \in \{x\}^c$ but $x \not\in \{x\}^c$ and $x \in \{y\}^c$ but $y \not\in \{y\}^c$. Hence by Definition, $(X, \tau)$ is an $\alpha \hat{g}$ $s$-$T_1$ space.

**THEOREM 4.7.15:** The product space of two $\alpha \hat{g}$ $s-$ $T_0$ spaces is an $\alpha \hat{g}$ $s-$ $T_0$ space.
PROOF. Let $(X, \tau_1)$ and $(Y, \tau_2)$ be two topological spaces and $(X \times Y, \tau)$ be their product space. Let $x$ and $y$ be two distinct points of $X$. Then $(X, \tau_1)$ is $\alpha \tilde{g} s$. $T_0$ space if and only if there exists an $\alpha \tilde{g} s$-open set $G$ such that it contains only one of these two and not the other. We claim that $(X \times Y, \tau)$ is an $\alpha \tilde{g} s$. $T_0$ space. Let $(x_1, y_1)$ and $(x_2, y_2)$ be any two distinct points of $X \times Y$ then either $x_1 \neq x_2$ or $y_1 \neq y_2$. If $x_1 \neq x_2$ and $(X, \tau_1)$ being $\alpha \tilde{g} s$. $T_0$ space, then there exists an $\alpha \tilde{g} s$-open set $G$ in $(X, \tau_1)$ such that $x_1 \in G$, $x_2 \notin G$. Hence $G \times Y$ is an $\alpha \tilde{g} s$-open set in $(X \times Y, \tau)$ containing $(x_1, y_1)$ but not containing $(x_2, y_2)$. Similarly if $y_1 \neq y_2$ and $(Y, \tau_2)$ being $\alpha \tilde{g} s$. $T_0$ space then there exists an $\alpha \tilde{g} s$-open set $H$ in $(Y, \tau_2)$ such that $X \times H$ is an $\alpha \tilde{g} s$-open set in $(X \times Y, \tau)$ containing $(x_1, y_1)$ but not $(x_2, y_2)$. Hence corresponding to distinct points of $X \times Y$, there exists an $\alpha \tilde{g} s$-open set containing one but not the other so that $(X \times Y, \tau)$ is also an $\alpha \tilde{g} s$. $T_0$ space.

DEFINITION 4.7.16: A topological space $(X, \tau)$ is said to be $\alpha \tilde{g} s$. $T_2$ (briefly $\alpha \tilde{g} s$. $T_2$) if and only if to each pair of distinct points $x, y$ of $X$, there exists a pair of disjoint $\alpha \tilde{g} s$-open sets, one containing $x$ and the other containing $y$.

REMARK 4.7.17: i. Every $\alpha$-$T_2$ space is a $\alpha \tilde{g} s$. $T_2$ space.

ii. Every $\alpha \tilde{g} s$. $T_2$ space is a $gs$-$T_2$ space.

iii. Every $\alpha \tilde{g} s$. $T_2$ space is an $\alpha \tilde{g} s$. $T_1$ spaces.

Next we have the following invariant properties.
THEOREM 4.7.18: Let $f : (X, \tau) \to (Y, \sigma)$ be $\alpha \cdot g$- irresolute and injective

i) If $(Y, \sigma)$ is $\alpha \cdot g$-T$_1$, then $(X, \tau)$ is $\alpha \cdot g$-T$_1$ space.

ii) If $(Y, \sigma)$ is $\alpha \cdot g$-T$_2$, then $(X, \tau)$ is $\alpha \cdot g$-T$_2$ space.

PROOF: (i). For each pair of points $x, y \in Y$ with $x \neq y$ and $(Y, \sigma)$ is $\alpha \cdot g$-T$_1$, there exists a pair of $\alpha \cdot g$-open sets $U, V$ such that $x \in U, y \in V$ and $x \not\in V, y \not\in U$. Since $f$ is injective, $\alpha \cdot g$-irresolute, to each pair of distinct points $f^{-1}(x), f^{-1}(y)$ in $X$, there exists a pair of $\alpha \cdot g$-open sets $f^{-1}(U), f^{-1}(V)$ such that $f^{-1}(x) \in f^{-1}(U), f^{-1}(y) \in f^{-1}(V)$ and $f^{-1}(x) \not\in f^{-1}(V), f^{-1}(y) \not\in f^{-1}(U)$. Hence $(X, \tau)$ is $\alpha \cdot g$-T$_1$.

(ii) Proof is similar to (i).

THEOREM 4.7.19 In a topological space $(X, \tau)$, the following statements are equivalent:

(i) $(X, \tau)$ is $\alpha \cdot g$-T$_2$.

(ii) Let $x \in X$. For each $y \neq x$, there exists an $\alpha \cdot g$-open set $U$ such that $x \in U$ and $Y \not\in \alpha \cdot g$ Cl$(U)$.

(iii) For each $x \in X$, $\cap \{ \alpha \cdot g$ Cl$(U) \mid U$ is $\alpha \cdot g$-open and $x \in U \} = \{x\}$.

(iv) The diagonal $\Delta = \{ (x, x) \mid x \in X \}$ is $\alpha \cdot g$-s-closed set in $X \times X$. 

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PROOF. (i) → (ii). Assume that the topological space \((X, \tau)\) is \(\alpha \hat{g} s\text{-}T_2\). Let \(x \in X\) and \(y \neq x\), then there are disjoint \(\alpha \hat{g} s\text{-}open\) sets \(U\) and \(V\) such that \(x \in U\), \(y \in V\). Clearly, \(V^c\) is \(\alpha \hat{g} s\text{-}closed\) set, \(\alpha \hat{g} s\text{Cl}(U) \subseteq V^c\), \(y \notin V^c\) and therefore \(y \notin \alpha \hat{g} s\text{Cl}(U)\).

(ii) → (iii). For each \(y \neq x\), there exists an \(\alpha \hat{g} s\text{-}open\) \(U\) such that \(x \in U\) and \(y \notin \alpha \hat{g} s\text{Cl}(U)\). So \(y \notin \cap \{ \alpha \hat{g} s\text{Cl}(U) / U \text{ is } \alpha \hat{g} s\text{-}open \text{ in } X \text{ and } x \in U \} = \{ x \} \).

(iii) → (iv). We claim that \(\Delta^c\) is \(\alpha \hat{g} s\text{-}open\) in \(X \times X\). Let \((x, y) \notin \Delta\). Then \(y \neq x\) and since \(\cap \{ \alpha \hat{g} s\text{Cl}(U) / U \text{ is } \alpha \hat{g} s\text{-}open \text{ in } (X, \tau) \text{ and } x \in U \} = \{ x \}\), there is some \(\alpha \hat{g} s\text{-}open\) set \(U\) in \((X, \tau)\) with \(x \in U\) and \(y \notin \alpha \hat{g} s\text{Cl}(U)\).

Since \(U \cap \{ \alpha \hat{g} s\text{Cl}(U) \}^c = \emptyset\), \((U \times \{ \alpha \hat{g} s\text{Cl}(U) \})^c\) is an \(\alpha \hat{g} s\text{-}open\) set such that \((x, y) \in U \times \{ \alpha \hat{g} s\text{Cl}(U) \}^c \subseteq \Delta^c\).

(iv) → (i). If \(y \neq x\), then \((x, y) \notin \Delta\) and thus there exist \(\alpha \hat{g} s\text{-}open\) sets \(U\) and \(V\) such that \((x, y) \in U \times V\) and \((U \times V) \cap \Delta = \emptyset\). Thus for the \(\alpha \hat{g} s\text{-}open\) sets \(U\) and \(V\) we have \(x \in U\), \(y \in V\) and \(U \cap V = \emptyset\). Hence \((X, \tau)\) is \(\alpha \hat{g} s\text{-}T_2\).

**THEOREM 4.7.20:** Let \(X\) be an arbitrary space, \(R\) an equivalence relation in \(X\) and \(p: X \rightarrow X / R\), the identification map. If \(R \subseteq X \times X\) is \(\alpha \hat{g} s\text{-}closed\) in \(X \times X\) and \(p\) is an always \(\alpha \hat{g} s\text{-}open\) map, then \(X / R\) is \(\alpha \hat{g} s\text{-}T_2\), and \(V\) we have \(x \in U\), \(y \in V\) and \(U \cap V = \emptyset\). Hence \((X, \tau)\) is \(\alpha \hat{g} s\text{-}T_2\).

**PROOF:** Let \(p(x), p(y)\) be distinct members of \(X / R\). Since \(x\) and \(y\) are not related, \(R \subseteq X \times X\) is \(\alpha \hat{g} s\text{-}closed\) set in \(X \times X\), there are \(\alpha \hat{g} s\text{-}open\) sets \(U\) and \(V\) such that \(x \in U\), \(y \in V\) and \(U \times V \subseteq R^c\). Thus \(p(U), p(V)\) are
disjoint and also $\alpha \hat{g}$ s-open in $X / R$ since $p$ is always $\alpha \hat{g}$ s-open. Hence $X / R$ is an $\alpha \hat{g}$ s-$T_2$ space.

**DEFINITION 4.7.21:** A topological space $(X, \tau)$ is called;

i. $\alpha$ generalized semi-$R_0$ (briefly $\alpha \hat{g}$ s-$R_0$) if for each $\alpha \hat{g}$ s-open set $U$ and $x \in U$ implies $\alpha \hat{g}$ s$\text{Cl}(x) \subseteq U$,

ii. $\alpha$-generalized semi-$R_1$ (briefly $\alpha \hat{g}$ s-$R_1$) if for $x, y \in X$ with $\alpha \hat{g}$ s$\text{Cl}({x}) \neq \alpha \hat{g}$ s$\text{Cl}({y})$, there exist disjoint $\alpha \hat{g}$ s-open sets $U$ and $V$ such that $\alpha \hat{g}$ s$\text{Cl}({x}) \subseteq U$ and $\alpha \hat{g}$ s$\text{Cl}({x}) \subseteq V$.

**EXAMPLE 4.7.22:**

i. Let $X = \{a, b, c\}, \tau = \{ \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$. Then the space $(X, \tau)$ is not a $\alpha \hat{g}$ s-$R_0$ space.

ii. Let $X = \{a, b, c\}, \tau = \{ \emptyset, \{a\}, \{b, c\}, X\}$. Then the space $(X, \tau)$ is $\alpha \hat{g}$ s-$R_1$ space.

iii. Let $X = \{a, b, c\}, \tau = \{ \emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Then the space $(X, \tau)$ is $\alpha \hat{g}$ s-$R_0$, and $\alpha \hat{g}$ s-$R_1$ space.

**REMARK 4.7.23** i) Every $\alpha$-$R_0$ space is an $\alpha \hat{g}$ s-$R_0$ space.

ii) Every $\alpha$-$R_1$ space is an $\alpha \hat{g}$ s-$R_1$ space.

iii) Every $\alpha \hat{g}$ s-$R_1$ space is an $\alpha \hat{g}$ s-$R_0$ space.

**THEOREM 4.7.24:** For a topological space $(X, \tau)$, the following statements are equivalent,

i. $(X, \tau)$ is an $\alpha \hat{g}$ s-$T_2$ space.

ii. $(X, \tau)$ is $\alpha \hat{g}$ s-$R_1$ and $\alpha \hat{g}$ s-$T_0$ space.

iii. $(X, \tau)$ is $\alpha \hat{g}$ s-$R_1$ and $\alpha \hat{g}$ s-$T_0$ space.
4.8. αġs-CONTINUOUS MAPS IN TOPOLOGICAL SPACES

In this section, we introduce αġs-continuous maps, pre-αġs continuous maps in topological spaces and study some of their properties.

DEFINITION 4.8.1: A map \( f : (X, \tau) \rightarrow (Y, \sigma) \) is said to be αġs – semi continuous (αġs -continuous) if the inverse image of every closed set in \( Y \) is αġs-closed in \( X \).

EXAMPLE 4.8.2: Let \( X = Y = \{a, b, c\} \), \( \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\} \) and \( \sigma = \{\emptyset, \{a\}, \{a, b\}, Y\} \). Let \( f \) be identity map then \( f \) is αġs-continuous.

THEOREM 4.8.2: A map \( f : (X, \tau) \rightarrow (Y, \sigma) \) αġs-continuous if and only if \( f^{-1}(V) \) is αġs-open in \( X \) for every open set \( V \) in \( Y \).

PROOF: Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be αġs-continuous and \( V \) be an open set in \( Y \). Then \( V^c \) is closed set in \( Y \) and since \( f \) is αġs-continuous, \( f^{-1}(V^c) \) is αġs-closed set in \( X \). But \( f^{-1}(V^c) = (f^{-1}(V))^c \) so \( f^{-1}(V) \) is αġs-open in \( X \).

Conversely, assume that \( f^{-1}(V) \) is αġs-open in \( X \) for each open set \( V \) in \( Y \). Let \( F \) be a closed set in \( Y \). Then \( F^c \) is open in \( Y \) and by assumption \( f^{-1}(F^c) \) is αġs-open in \( X \), since \( f^{-1}(F^c) = (f^{-1}(F))^c \), we have \( f^{-1}(F) \) is αġs-closed set in \( X \) and so \( f \) is αġs-continuous.

DEFINITION 4.8.5: A space \( (X, \tau) \) is said to be \( T_{αġs} \) space if every αġs-closed set in it is closed set.
EXAMPLE 4.8.6: Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b,c\}, X\}$. Then $(X, \tau)$ is $T_{\alpha\alpha}$ space.

EXAMPLE 4.8.7: Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Then the set $\{c, d\}$ is $\alpha\beta$-closed but not closed in $(X, \tau)$.

THEOREM 4.8.8: Let $X$ and $Z$ be topological spaces and $Y$ be a $T_{\alpha\alpha}$ space. Then the composition $gof : X \rightarrow Z$ of the $\alpha\beta$-continuous maps $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ is $\alpha\beta$-continuous.

PROOF: Let $F$ be any closed set in $Z$. As $g$ is $\alpha\beta$-continuous and $Y$ is a $T_{\alpha\alpha}$ space, then $g^{-1}(F)$ is closed set in $Y$. Since $f$ is $\alpha\beta$-continuous and $g^{-1}(F)$ is closed set in $Y$, $f^{-1}(g^{-1}(F))$ is $\alpha\beta$-closed set in $X$. But $f^{-1}(g^{-1}(F)) = (gof)^{-1}(F)$ and so $gof$ is $\alpha\beta$-continuous.

THEOREM 4.8.9: If $f : X \rightarrow Y$ is $\alpha\beta$-continuous and $g : Y \rightarrow Z$ is continuous, then their composition $gof : X \rightarrow Z$ is $\alpha\beta$-continuous.

PROOF: Let $F$ be any closed set in $Z$. Since $g$ is continuous, $g^{-1}(F)$ is closed set in $Y$. Since $f$ is $\alpha\beta$-continuous and $g^{-1}(F)$ is closed set in $Y$, $f^{-1}(g^{-1}(F)) = (gof)^{-1}(F)$ is $\alpha\beta$-closed set in $X$ and $f$ is $\alpha\beta$-continuous.

DEFINITION 4.8.10: A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is is called pre-$\alpha\beta$-continuous if $f^{-1}(V)$ is $\alpha\beta$-closed set in $X$ for every $\alpha$-closed set $V$ of $Y$.
EXAMPLE 4.8.11: Let \( X = Y = \{a, b, c\} \), \( \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\} \) and \( \sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\} \). Define \( f : (X, \tau) \to (Y, \sigma) \) as an identity map. Then \( f \) is \( \preorder {\alpha \hat{g}} \) s-continuous.

THEOREM 4.8.12: If a map \( f : (X, \tau) \to (Y, \sigma) \) is \( \alpha \)-irresolute, then it is \( \preorder {\alpha \hat{g}} \) s-continuous, but not conversely.

PROOF: Let \( V \) be any \( \alpha \)-closed set in \( Y \). Since \( f \) is \( \alpha \)-irresolute, \( f^{-1}(V) \) is \( \alpha \)-closed set in \( X \). Every \( \alpha \)-closed set is \( \preorder {\alpha \hat{g}} \) s-closed set in \( X \). So \( f^{-1}(V) \) is \( \preorder {\alpha \hat{g}} \) s-closed set in \( X \). Therefore \( f \) is \( \preorder {\alpha \hat{g}} \) s-continuous.

However, the converse of the above theorem need not be true as seen from the following example.

EXAMPLE 4.8.13: Let \( X = Y = \{a, b, c\} \), \( \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\} \), \( \sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, Y\} \). Define \( f : (X, \tau) \to (Y, \sigma) \) be an identity map. Then \( f \) is \( \preorder {\alpha \hat{g}} \) s-continuous, but not \( \alpha \)-irresolute because the inverse image of the \( \alpha \)-closed set \( \{b\} \) in \( Y \) is \( \{b\} \) which is not \( \alpha \) closed set in \( X \).

REMARK 4.8.14: The composition of two \( \preorder {\alpha \hat{g}} \) s-continuous maps need not be \( \preorder {\alpha \hat{g}} \) s-continuous as seen from the following example.

EXAMPLE 4.8.15: Let \( X = Y = Z = \{a, b, c\} \), \( \tau = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}\}, X\) \( \sigma = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, Y\} \) and \( \eta = \{\emptyset, \{a\}, Z\} \). Let \( f : (X, \tau) \to (Y, \sigma) \) by \( f(a)=c, f(b)=b, f(c)=a \) be the identity map. Then \( f \) and \( g \) are \( \preorder {\alpha \hat{g}} \) s-continuous but their composition map \( g \circ f : (X, \tau) \to (Z, \eta) \) is not \( \preorder {\alpha \hat{g}} \) s-continuous because \( \emptyset = \{c\} \) is
\( \alpha \)-closed set in \( Z \), but \((g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V)) = f^{-1}(g^{-1}(\{c\})) = f^{-1}(\{a\}) = a \) is not \( \alpha g s \)-closed set in \( X \).

**THEOREM 4.8.16:** Let \( X \) and \( Z \) be topological spaces and \( Y \) be a \( T_{\alpha g s} \)-space. Then the composition map \( g \circ f : X \to Z \) of the pre \( \alpha g s \)-continuous maps \( f : X \to Y \) and \( g : Y \to Z \) is pre-\( \alpha g s \)-continuous.

**PROOF:** Let \( F \) be any \( \alpha \)-closed set in \( Z \). As \( g \) is pre-\( \alpha g s \)-continuous and \( Y \) is a \( T_{\alpha g s} \) space, then \( g^{-1}(F) \) is \( \alpha g s \)-closed set in \( Y \). Since \( Y \) is \( T_{\alpha g s} \)-space every \( \alpha g s \)-closed is closed in \( Y \). Hence \( g^{-1}(F) \) is \( \alpha \)-closed in \( Y \). Hence every closed set in \( \alpha \)-closed set in \( Y \), because \( g^{-1}(F) \) is \( \alpha \)-closed set in \( X \). Since \( f \) is pre-\( \alpha g s \)-continuous and \( g^{-1}(F) \) is a \( \alpha \)-closed set in \( Y \), \( f^{-1}(g^{-1}(F)) \) is \( \alpha g s \)-closed set in \( X \). But \( f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F) \) and so \( g \circ f \) is pre-\( \alpha g s \)-continuous.

3. 9. \( \alpha g s \)-IRRESOLUTE MAPS

In this section, we introduce and study the concept of \( \alpha g s \)-irresolute maps in topological spaces.

**DEFINITION 4.9.1:** A map \( f : X \to Y \) is called \( \alpha g s \)-irresolute map if the inverse image of every \( \alpha g s \)-closed set in \( Y \) is \( \alpha g s \)-closed set in \( X \).

**REMARK 4.9.2:** The following examples show that the notions of irresolute maps and \( \alpha g s \) irresolute maps are independent.

**EXAMPLE 4.9.3:** Let \( X = Y = \{a, b, c\}, \tau = \{\emptyset, \{a, b\}, X\}, \sigma = \{\emptyset, \{a\}, \{a, b\}, Y\} \). Then the identity map on \( X \) is \( \alpha g s \)-irresolute map, but
it is not irresolute. Since $f^{-1}(b, c) = \{b, c\}$ which is not semiclosed in $X$ where as $\{b, c\}$ is semiclosed in $Y$.

**EXAMPLE 4.9.3:** Let $X = Y = \{a, b, c\}$, $\mathcal{T} = / \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X \}$, $\mathcal{S} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\}$. Then the identity map $f: (X, \mathcal{T}) \to (Y, \mathcal{S})$ is irresolute but not $\alpha \cdot \alpha \cdot s$-irresolute.

**THEOREM 4.9.4:** A map $f: (X, \mathcal{T}) \to (Y, \mathcal{S})$ is $\alpha \cdot \alpha \cdot s$-irresolute map if and only if the inverse image of every $\alpha \cdot \alpha \cdot s$ open set in $Y$ is $\alpha \cdot \alpha \cdot s$ open set in $X$.

**PROOF:** Let $f: (X, \mathcal{T}) \to (Y, \mathcal{S})$ be $\alpha \cdot \alpha \cdot s$-irresolute and $U$ be an $\alpha \cdot \alpha \cdot s$-open set in $Y$. Then $U^c$ is $\alpha \cdot \alpha \cdot s$-closed set in $Y$ and since $f$ is $\alpha \cdot \alpha \cdot s$ irresolute, $f^{-1}(U^c)$ is $\alpha \cdot \alpha \cdot s$-open in $X$. But $f^{-1}(U^c) = (f^{-1}(U))^c$ and so $f^{-1}(U)$ is $\alpha \cdot \alpha \cdot s$-open in $X$.

Conversely, assume that $f^{-1}(U)$ is $\alpha \cdot \alpha \cdot s$-open in $X$. For each $\alpha \cdot \alpha \cdot s$-open set $U$ in $Y$, let $F$ be $\alpha \cdot \alpha \cdot s$-closed set in $Y$. Then $F^c$ is $\alpha \cdot \alpha \cdot s$-open in $Y$ and by assumption $f^{-1}(F^c)$ is $\alpha \cdot \alpha \cdot s$-open in $X$. Since $f^{-1}(F^c) = (f^{-1}(F))^c$, we have $f^{-1}(F)$ is $\alpha \cdot \alpha \cdot s$ closed in $X$ and so $f$ is $\alpha \cdot \alpha \cdot s$-irresolute.

**THEOREM 4.9.5:** If a map $f: (X, \mathcal{T}) \to (Y, \mathcal{S})$ $\alpha \cdot \alpha \cdot s$-irresolute, then it is $\alpha \cdot \alpha \cdot s$-continuous but not conversely.

**PROOF:** Trivial.

However, the converse of the above theorem need not be true as seen from the following example.
EXAMPLE 4.9.6: Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$, $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\}$. Then the identity map $f: (X, \tau) \rightarrow (Y, \sigma)$ is $\alpha\hat{\gamma}$ s-continuous but not $\alpha\hat{\gamma}$ s-irresolute.

THEOREM 4.9.7: Let $(X, \tau)$ be any topological space, $(Y, \sigma)$ be a $T_{\alpha\hat{g}s}$ space and $f: (X, \tau) \rightarrow (Y, \sigma)$ be a map. Then the following are equivalent:

(i) $f$ is $\alpha\hat{\gamma}$ s-irresolute.

(ii) $f$ is $\alpha\hat{\gamma}$ s-continuous.

PROOF: Every $\alpha\hat{\gamma}$ s-irresolute map is $\alpha\hat{\gamma}$ s-continuous. Hence (i) $\Rightarrow$ (ii). Let $F$ be an $\alpha\hat{\gamma}$ s-closed set in $(Y, \sigma)$. Since $(Y, \sigma)$ is a $T_{\alpha\hat{g}s}$ space, $F$ is closed set in $(Y, \sigma)$ and by hypothesis $f^{-1}(F)$ is $\alpha\hat{g}s$-closed set in $X$. Therefore $f$ is $\alpha\hat{\gamma}$ s-irresolute. Hence (ii) $\Rightarrow$ (i).

DEFINITION 4.9.8: A map $f: (X, \tau) \rightarrow (Y, \sigma)$ is pre-$\alpha\hat{g}s$-open if $f(U)$ is $\alpha\hat{g}s$ in $Y$ for every $\alpha\hat{g}s$ open set $U$ in $X$.

THEOREM 4.9.9: If $f: (X, \tau) \rightarrow (Y, \sigma)$ is bijective, pre-$\alpha\hat{g}s$ open and pre-$\alpha\hat{\gamma}$ s continuous, then $f$ is $\alpha\hat{\gamma}$ s-irresolute.

PROOF: Let $A$ be $\alpha\hat{\gamma}$ s-closed set in $Y$. Let $U$ be any $\alpha\hat{g}s$-open set in $X$ such that $A \subseteq f(U)$. Since $A$ is $\alpha\hat{\gamma}$ s-closed set and $f(U)$ is $\alpha\hat{\gamma}$ s-open in $Y$, $sCl(A) \subseteq f(U)$ holds and hence $f^{-1}(sCl(A)) \subseteq U$. Since $f$ is pre-$\alpha\hat{\gamma}$ s-continuous and $sCl(A)$ is semiclosed set in $Y$ is $\alpha\hat{\gamma}$ s-closed set in $X$, we have $sCl(f^{-1}(sCl(A))) \subseteq U$ and so $sCl(f^{-1}(A)) \subseteq U$. Therefore $f^{-1}(A)$ is $\alpha\hat{g}s$-closed set in $X$ and hence $f$ is $\alpha\hat{g}s$ irresolute.
COROLLARY 4.9.10: If $f : (X, \tau) \to (Y, \sigma)$ is bijective, pre-$\alpha\beta\gamma$ and $\alpha$-irresolute. Then $f$ is $\alpha\beta\gamma$-irresolute.

PROOF: Follows from theorem 4.8.12 and theorem 4.9.9.

THEOREM 4.9.11: If $f : (X, \tau) \to (Y, \sigma)$ is bijective, $\alpha$-closed set and $\alpha\beta\gamma$ irresolute maps, then the inverse map $f^{-1} : (Y, \sigma) \to (X, \tau)$ is $\alpha\beta\gamma$-irresolute.

PROOF: Let $A$ be $\alpha\beta\gamma$-closed set in $X$. Let $(f^{-1})(A) = f(A) \subseteq U$ where $U$ is $\alpha\beta\gamma$-open in $Y$. Then $A \subseteq f^{-1}(U)$ holds. Since $f^{-1}(U)$ is $\alpha\beta\gamma$ open in $X$ and $A$ is $\alpha\beta\gamma$-closed set in $X$, $sCl(A) \subseteq f^{-1}(U)$ and hence $f(sCl(A)) \subseteq U$. Since $f$ is $\alpha$-closed and $sCl(A)$ is $\alpha$-closed set in $X$, $f(sCl(A))$ is closed set in $Y$. So $f(sCl(A))$ is $\alpha\beta\gamma$-closed set in $Y$. Therefore $sCl(f(sCl(A))) \subseteq U$ and hence $sCl(f(A)) \subseteq U$. Thus $f(A)$ is $\alpha\beta\gamma$-closed set in $Y$ and so $f^{-1}$ is $\alpha\beta\gamma$-irresolute.

THEOREM 4.9.12: Let $X, Y, Z$ be topological spaces and $f : (X, \tau) \to (Y, \sigma); \quad g : (Y, \sigma) \to (Z, \eta)$ be two maps. Their composition $gof : (X, \tau) \to (Z, \eta)$ is $\alpha\beta\gamma$-continuous

(i) if $f$ is $\alpha\beta\gamma$-irresolute and $g$ is $\alpha\beta\gamma$-continuous or,

(ii) if $f$ is $\alpha\beta\gamma$-continuous and $g$ is continuous.

PROOF: (i) Let $V$ be an open set in $Z$. Then $g^{-1}(V)$ is $\alpha\beta\gamma$-open. Since $f$ is $\alpha\beta\gamma$ irresolute $f^{-1}(g^{-1}(V))$ is $\alpha\beta\gamma$-open. But $f^{-1}(g^{-1}(V)) = (gof)^{-1}(V)$. Therefore $(gof)$ is $\alpha\beta\gamma$-continuous.
(iii) Let \( U \) be a closed set in \( Z \). Then \( g^{-1}(U) \) be a closed set in \( Y \) because \( g \) is continuous. Since \( f \) is \( \alpha \, \varepsilon \, s \)-continuous \( f^{-1}(g^{-1}(U)) \) is \( \alpha \, \varepsilon \, s \)-closed set in \( X \). But \( f^{-1}(g^{-1}(U)) = (gof)^{-1}(U) \). Therefore \( (gof) \) is \( \alpha \, \varepsilon \, s \)-continuous.

**THEOREM 4.9.13:** Let \( X, Y, Z \) be topological spaces and if \( f : (X, \tau) \rightarrow (Y, \sigma) \) and \( g: (Y, \sigma) \rightarrow (Z, \eta) \) be two maps. Their composition \( gof : (X, \tau) \rightarrow (Z, \eta) \) is \( \alpha \, \varepsilon \, s \)-irresolute if \( f \) and \( g \) are \( \alpha \, \varepsilon \, s \)-irresolute maps.

**PROOF:** Trivial.

### 3.10. ON \( \alpha \, \varepsilon \, s \)-OPEN MAPS AND \( \alpha \, \varepsilon \, s \)-CLOSED MAPS

In this section, we introduce and study \( \alpha \, \varepsilon \, s \)-open maps and \( \alpha \, \varepsilon \, s \)-closed maps in topological spaces and study some of their properties.

**DEFINITION 4.10.1:** A map \( f : X \rightarrow Y \) is said to be \( \alpha \, \varepsilon \, s \)-open (\( \alpha \, \varepsilon \, s \)-closed set) if \( f(V) \) is \( \alpha \, \varepsilon \, s \)-open (\( \alpha \, \varepsilon \, s \)-closed set) in \( Y \) for every open (closed) set \( V \) in \( X \).

**THEOREM 4.10.2:** A map \( f : X \rightarrow Y \) is \( \alpha \, \varepsilon \, s \)-closed set if and if for each subset \( S \) of \( Y \) and for each open set \( U \) containing \( f^{-1}(S) \) there is an \( \alpha \, \varepsilon \, s \)-open set \( V \) of \( Y \) such that \( S \subseteq V \) and \( f^{-1}(V) \subseteq U \).

**PROOF.** Assume that \( f \) is \( \alpha \, \varepsilon \, s \)-closed. Let \( S \) be a subset of \( Y \) and \( U \) be an open set of \( X \) such that \( S \subseteq f(U) \). Let \( f^{-1}(S) \subseteq U \). Now, \( U^c \) is a closed set in \( Y \). Then \( f(U^c) \) is \( \alpha \, \varepsilon \, s \)-closed set in \( X \), since \( f \) is \( \alpha \, \varepsilon \, s \)-closed set. So
Y–f(U°) is αgs-open in X. Thus V = f(U°) is an αgs-open set containing S such that f⁻¹(V) ⊆ U.

Conversely, suppose that F is a closed set in X. Then f⁻¹( Y – f(F)) ⊆ X – F and X – F is open. By hypothesis, there is an αgs-open set V of Y such that Y – f (F) ⊆ V and f⁻¹ (V) ⊆ X – F and so f ⊆ X – f⁻¹ (V) . Hence Y – V ⊆ f (F) ⊆ f ( X – f⁻¹ (V)) ⊆ Y – V which implies that f (F) = Y – V. Since Y – V is αgs-closed, f(F) is αgs-closed and thus f is αgs-closed set.

**PROPOSITION 4.10.3:** If f : (X, x) → (Y, σ) is αgs irresolute , pre semi closed and A is an αgs-closed subset of X, then f (A) is αgs-closed set.

**PROOF:** Let U be a αgs-open set in Y such that f (A ) ⊆ U. Since f is αgs irresolute, f⁻¹(U) is a αgs-open set containing A ie. A ⊆ f⁻¹ (U). Hence sCl(A) ⊆ f⁻¹ (U), as A is αgs-closed set in X. Since f is pre semi closed, f( sCl(A)) is semi-closed set. So, f( sCl(A)) is an αgs-closed set contained in the αgs-open set U ie. f( sCl (A)) ⊆ U. Now sCl f( (A)) ⊆ sCl ( f( sCl (A))) = f( sCl (A)) ⊆ U. Hence sCl f(A) ⊆ U. Therefore f(A) is an αgs-closed set in Y.

The composition of two αgs-closed maps need not be αgs-closed as seen from the following example.

**EXAMPLE 4.10.5:** Let X = Y = Z = { a, b, c} , x = { ∅, {a}, {b}, { a,b} , X } , σ = { ∅, {a}, {b} , {a,c} , {a, b}, Y }, and η= { ∅, {a,b}, Z }. Let f : (X, τ) → (Y, σ) be the identity map and define g : (Y,σ) → (Z,η) by g(a) = g(b, g(b)=a, g(c)=c The both f and g are αgs-closed maps but their composition g ° f : (X, τ)→ (Z, η) is not an αgs-closed map . Since for
the closed set \( \{c\} \) in \((X, \tau)\), \(g \circ f(\{c\}) = g(f(\{c\})) = g(\{c\}) = \{a\}\), which is not \(\alpha\)gs – closed set in \((Z, \eta)\).

**COROLLARY 4.10.6:** Let \(f : (X, \tau) \to (Y, \sigma)\) be an \(\alpha\)gs closed map and \(g : (Y, \sigma) \to (Z, \eta)\) be pre-\(\alpha\) -closed map and \(\alpha\)gs irresolute , then their composition \(g \circ f : (X, \tau) \to (Z, \eta)\) is \(\alpha\)gs – closed .

**PROOF:** Let \(A\) be a closed set of \((X, \tau)\). Then by hypothesis \(f(A)\) is an \(\alpha\)gs closed set in \((Y, \sigma)\). Since \(g\) is pre semi closed and irresolute , by proposition 4.10.3. \(g(f(A)) = (g \circ f)(A)\) is \(\alpha\)gs- closed in \((Z, \eta)\). Hence \((g \circ f)\) is \(\alpha\)gs- closed map.

**THEOREM 4.10.7:** Let \(f : (X, \tau) \to (Y, \sigma)\) and \(g : (Y, \sigma) \to (Z, \eta)\) be \(\alpha\)gs-closed maps and \((Y, \sigma)\)be a \(T_{\alpha\)gs space . Then their composition \(g \circ f\) is \(\alpha\)gs – closed map .

**PROOF:** Let \(A\) be a closed set of \((X, \tau)\). Then by hypothesis \(f(A)\) is an \(\alpha\)gs- closed set in \((Y, \sigma)\). Since \((Y, \sigma)\) is a \(T_{\alpha\)gs space , \(f(A)\) is closed in \((Y, \sigma)\). Also by assumption \(g(f(A))\) is \(\alpha\)gs- closed map in \((Z, \eta)\). Hence \((g \circ f)\) is \(\alpha\)gs – closed map .

**THEOREM 4.10.8:** The composition of a closed map \(f : Z \to Y\) and an \(\alpha\)gs - closed map \(g : Y \to Z\) is an \(\alpha\)gs – closed map from \(X \to Z\).

**PROOF:** Let \(A\) be a closed set of \((X, \tau)\). Then by hypothesis \(f(A)\) is an \(\alpha\)gs – closed set in \((Y, \sigma)\). Since \((Y, \sigma)\) is a \(T_{\alpha\)gs space, \(f(A)\) is closed in \((Y, \sigma)\). Also by assumption \(g(f(A))\) is \(\alpha\)gs- closed map in \((Z, \eta)\). Hence \((gof)\) is \(\alpha\)gs-closed map .
THEOREM 4.10.9: The composition of a closed map \( f: X \to Y \) and an \( \alpha g s \) -closed \( \alpha g s \) -closed map from \( X \to Z \).

PROOF: Trivial.

THEOREM 4.10.11: Let \( f: (X, \tau) \to (Y, \sigma) \) and \( g: (Y, \sigma) \to (Z, \eta) \) be two mappings such that their composition \( g \circ f: (X, \tau) \to (Z, \eta) \) is an \( \alpha g s \) -closed mapping. Then the following statements are true.

i) If \( f \) is continuous and surjective, then \( g \) is \( \alpha g s \)-closed map

ii) If \( g \) is irresolute and injective: \( (X, \tau) \to (Y, \sigma) \), then \( g \) is \( \alpha g s \)-closed map.

PROOF: (i). Let \( A \) be a closed set in \( (Y, \sigma) \). Then \( f^{-1}(A) \) is closed in \( (X, \tau) \) as \( f \) is continuous. Since \( g \circ f \) is \( \alpha g s \) -closed map and \( f \) is surjective, \( (g \circ f)(f^{-1}(A)) = g(A) \) is \( \alpha g s \) -closed in \( (Z, \eta) \). Therefore \( g \) is an \( \alpha g s \) -closed map in \( X \).

(ii). Let \( H \) be a closed set of \( (X, \tau) \). Since \( g \circ f \) is \( \alpha g s \) closed map, \( (g \circ f)(H) \) is an \( \alpha g s \) -closed set in \( (Z, \eta) \). Since \( g \) is \( \alpha g s \) – irresolute, \( g^{-1}(g \circ f)(H) = (g \circ f(H)) = f(H) \) is \( \alpha g s \)-closed in \( (Y, \sigma) \), since \( g \) is injective. Thus \( f \) is an \( \alpha g s \)-closed map in \( X \).

PROPOSITION 4.10.12: For any bijection \( f: (X, \tau) \to (Y, \sigma) \), the following statements are equivalent.

(i) inverse of \( f \) is \( \alpha g s \)-continuous .

(ii) \( f \) is an \( \alpha g s \)-open map.

(iii) \( f \) is an \( \alpha g s \)-closed map.

PROOF. (i) \( \to \) (ii): Let \( U \) be an open set of \( (X, \tau) \). By assumption \( (f^{-1})^{-1}(U) = f(U) \) is \( \alpha g s \) –open in \( (Y, \sigma) \) and so \( f \) is \( \alpha g s \)-open.
(ii) \rightarrow (iii): Let \( F \) be a closed set of \( (X, \tau) \). Then \( F^c \) is open in \( (X, \tau) \). By assumption \( f(F^c) \) is \( \alpha\tilde{g}s \)-open in \( (Y, \sigma) \) and therefore \( f(F) \) is \( \alpha\tilde{g}s \)-closed map in \( (Y, \sigma) \). Hence \( f \) is \( \alpha\tilde{g}s \)-closed map.

(iii) \rightarrow (i). Let \( F \) be a closed set in \( (X, \tau) \). By assumption \( f(F) \) is \( \alpha\tilde{g}s \) - closed in \( (Y, \sigma) \). But \( f(F) = (f^{-1})^{-1}(F) \) and therefore \( f \) is \( \alpha\tilde{g}s \) - continuous.

3.11. ON \( \alpha\tilde{g}s \) - NORMAL AND REGULAR SPACES.

In this section we introduce and study \( \alpha\tilde{g}s \)-normal and \( \alpha\tilde{g}s \) regular spaces.

**DEFINITION 4.11.1:** For every set \( A \subseteq X \), we define the \( \alpha\tilde{g}s \) closure of intersection of all \( \alpha\tilde{g}s \)-closed sets containing \( A \). In symbols, \( \alpha\tilde{g}s \) Cl \( (A) = \cap \{ F: A \subseteq F \text{ where } F \text{ is } \alpha\tilde{g}s \text{ closed in } (X, \tau) \} \)

4.12. \( \alpha\tilde{g}s \) - NORMAL SPACES.

**DEFINITION 4.12.1:** A space \( (X, \tau) \) is said to be \( \alpha\tilde{g}s \) - normal if for any pair of disjoint \( \alpha\tilde{g}s \)-closed sets \( A \) and \( B \) in \( X \), there exist disjoint open sets \( U \) and \( V \) in \( X \) such that \( A \subseteq U \) and \( B \subseteq V \).

**REMARK 4.12.2:** It is obvious that every \( \alpha\tilde{g}s \) -normal space is normal. However the converse is not true as seen from the following example

**EXAMPLE 4.12.3:** Let \( X = \{ a, b, c, d \} \) and \( \tau = \{ \emptyset, \{ a, d \}, \{ b, c \}, X \} \). Then the space \( (X, \tau) \) is normal, but not \( \alpha\tilde{g}s \) -normal.
**THEOREM 4.12.4:** The following are equivalent for a space \((X, \tau)\).

(i) \((X, \tau)\) is normal.

(ii) For any disjoint closed sets \(A\) and \(B\), there exist disjoint \(\alpha\)-\(\alpha\)-open sets \(U, V\) such that \(A \subseteq U\) and \(B \subseteq V\).

(iii) For any closed set \(A\) and any open set \(V\) containing \(A\), there exists an \(\alpha\)-\(\alpha\)-open set \(U\) of \(X\) such that \(A \subseteq U \subseteq sCl(U) \subseteq V\).

**PROOF.** (i) \(\Rightarrow\) (ii). Follows from the fact that every open set is \(\alpha\)-\(\alpha\)-open.

(ii) \(\Rightarrow\) (iii). Let \(A\) be a closed set and \(V\) be an open set containing \(A\). Then \(A\) and \(X - V\) are disjoint closed sets. There exist disjoint \(\alpha\)-\(\alpha\)-open sets \(U\) and \(W\) such that \(A \subseteq U\) and \(X - V \subseteq W\), since \(X - V\) is closed, it is \(\alpha\)-\(\alpha\)-closed, we have \(X - V \subseteq sInt(W)\) by result 2.6 and \(U \cap sInt(W) = \emptyset\) and so we have \(sCl(U) \cap sInt(W) = \emptyset\) and hence \(A \subseteq U \subseteq sCl(U) \subseteq X - sInt(W) \subseteq V\).

(iii) \(\Rightarrow\) (i). Let \(A, B\) be disjoint closed sets of \(X\). Then \(A \subseteq X - B\) and \(X - B\) is open. There exists an \(\alpha\)-\(\alpha\)-open set \(G\) of \(X\) such that \(A \subseteq G \subseteq sCl(G) \subseteq X - B\). Since \(A\) is closed, it is \(\alpha\)-\(\alpha\)-closed, we have \(A \subseteq sInt(G)\) Put \(U = Int(Cl(\ Int(\ sInt(G)))\)) and \(V = Int(Cl(X - sCl(G)))\). Then \(U\) and \(V\) are disjoint open sets of \(X\) such that \(A \subseteq U\) and \(B \subseteq V\). Therefore \((X, \tau)\) is normal.

**THEOREM 4.12.5:** For a space \((X, \tau)\) the following are equivalent:

i) \((X, \tau)\) is \(\alpha\)-normal.
ii) For every pair of disjoint closed sets $A$ and $B$, there exists $\alpha\hat{g}s$-open sets $U$ and $V$ such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$.

**PROOF:** (i) $\Rightarrow$ (ii). Assume that $X$ is $\alpha$-normal. Let $A$ and $B$ be disjoint closed subsets of $X$. By hypothesis, there exist disjoint $\alpha$-open sets (and hence $\alpha\hat{g}s$-open sets) $U$ and $V$ such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$.

(ii) $\Rightarrow$ (i). Let $A$ and $B$ be closed subsets of $X$. Then by assumption, $A \subseteq G$, $B \subseteq H$ and $G \cap H = \emptyset$, where $G$ and $H$ are disjoint $\alpha\hat{g}s$-open sets. Since $A$ and $B$ are $\alpha\hat{g}s$-closed sets in $X$, by result 4.6.2 $A \subseteq \text{sInt}(G)$, $B \subseteq \text{sInt}(H)$ and $\text{sInt}(G) \cap \text{sInt}(H) = \emptyset$. Hence $(X, \tau)$ is $\alpha$-normal.

**THEOREM 4.12.6:** If $(X, \tau)$ is $\alpha$-normal and $F \cap A = \emptyset$, where $F$ is closed and $A$ is $\alpha\hat{g}s$-closed, then there exist disjoint $\alpha$-open sets $U$ and $V$ such that $F \subseteq U$ and $A \subseteq V$.

**PROOF.** Since $F$ is closed and $F \cap A = \emptyset$, we have $A \subseteq F^c$ and so $\text{Cl}(A) \subseteq F^c$. Thus $\text{Cl}(A) \cap F = \emptyset$. Since $F$ and $\text{Cl}(A)$ are closed and $(X, \tau)$ is $\alpha$-normal, there exists $\alpha$-open sets $U$ and $V$ such that $\text{Cl}(A) \subseteq U$ and $F \subseteq V$. $A \subseteq U$ and $F \subseteq V$.

**THEOREM 4.12.7:** If $(X, \tau)$ is $\alpha$-normal, the following statements are true.

i) For each closed set $A$ and every $\alpha\hat{g}s$-open set $B$ such that $A \subseteq B$, there exists an $\alpha$-open set $U$ such that $A \subseteq U \subseteq \alpha\text{Cl}(U) \subseteq B$.

ii) For every $\alpha\hat{g}s$-closed set $A$ and every open set $B$ containing $A$, there exists an $\alpha$-open set $U$ such that $A \subseteq U \subseteq \alpha\text{Cl}(U) \subseteq B$.
iii. In addition that the space \((X, \tau)\) is an \(\alpha\)-space, for every pair of disjoint sets \(A\) and \(B\), one of which is closed and the other is \(\alpha\)-\(\alpha\)s-closed, there exist \(\alpha\)-open sets \(U\) and \(V\) such that \(A \subseteq U, B \subseteq V\) and \(\alpha\text{Cl}(U) \cap \alpha\text{Cl}(V) = \emptyset\).

**PROOF.** (i) Let \(A\) be closed and \(B\) be an \(\alpha\)\(\alpha\)s-open set such that \(A \subseteq B\). Then \(A \cap B^c = \emptyset\), where \(A\) is closed and \(B^c\) is \(\alpha\)\(\alpha\)s-closed. Therefore, by theorem 3.6, there exist \(\alpha\)-open sets \(U\) and \(V\) such that \(A \subseteq U, B^c \subseteq V\) and \(U \cap V = \emptyset\). Thus \(A \subseteq U \subseteq V^c \subseteq B\). Since \(V^c\) is \(\alpha\)-closed, \(\alpha\text{Cl}(U) \subseteq V^c\) and so \(A \subseteq U \subseteq \alpha\text{Cl}(U) \subseteq B\).

(ii) Let \(A\) be an \(\alpha\)\(\alpha\)s-closed set and \(B\) be an open set such that \(A \subseteq B\). Then \(B^c \subseteq A^c\). Since \((X, \tau)\) is \(\alpha\)-normal and \(A^c\) is \(\alpha\)\(\alpha\)s-open set containing the closed set \(B^c\), we have by (1), there exists an \(\alpha\)-open set \(G\) such that \(B^c \subseteq G\) and \(\alpha\text{Cl}(G) \subseteq A^c\). Thus \(A \subseteq (\alpha\text{Cl}(G))^c \subseteq G^c \subseteq B\). Let \(U = (\alpha\text{Cl}(G))^c\). Then \(U\) is \(\alpha\)-open and \(A \subseteq U \subseteq \alpha\text{Cl}(U) \subseteq B\).

(iii) Let \(A\) be \(\alpha\)\(\alpha\)s-closed set and \(B\) be a closed in \(X\) such that \(A \cap B = \emptyset\), then \(A \subseteq B^c\) and \(B^c\) is open. Since \((X, \tau)\) is \(\alpha\)-normal, we have by (2), there exists an \(\alpha\)-open set \(S\) such that \(A \subseteq S \subseteq \alpha\text{Cl}(S) \subseteq B^c\). Since \(A\) is \(\alpha\)\(\alpha\)s-closed and \((X, \tau)\) is an \(\alpha\)-space and so \(S\) is open, we have again by (2), there exists an \(\alpha\)-open set \(U\) such that \(A \subseteq U \subseteq \alpha\text{Cl}(U) \subseteq S \subseteq \alpha\text{Cl}(S) \subseteq B^c\). Let \(V = (\alpha\text{Cl}(S))^c\). Thus \(V\) is \(\alpha\)-open, \(B \subseteq V\) and \(\alpha\text{Cl}(U) \cap \alpha\text{Cl}(V) = \emptyset\).

**THEOREM 4.12.8:** The following statements are equivalent for a topological space \((X, \tau)\).

(i). \((X, \tau)\) is \(\alpha\)\(\alpha\)s-normal.
(ii) For each αgs-closed set A and for each αgs-open set U containing A,
there exists an open set V containing A such that Cl(V) ⊆ U.

(iii) For each pair of disjoint αgs-closed sets A and B in (X, τ), there
exists an open set U containing A such that Cl(U) ∩ B = ∅.

(iv) For each pair of disjoint αgs-closed sets A and B in (X, τ), there
exist an open set containing A and an open set V containing B such that
Cl(U) ∩ Cl(V)B = ∅.

PROOF. (i) → (ii). Let A be an αgs-closed set and U be an αgs-open
set such that A ⊆ U. Then A ∩ Uc = ∅. Since (X, τ) is αgs-normal,
there exist open sets V and W such that A ⊆ V, Uc ⊆ W and V ∩ W = ∅,
which implies that Cl(V) ∩ W = ∅. Now Cl(V) ∩ Uc = Cl(V) ∩ W = ∅ and
so Cl(V) ⊆ U.

(ii) → (iii). Let A and B be disjoint αgs-closed sets of (X, τ). Since A ∩
B = ∅, A ⊆ Bc and Bc is αgs-open. By assumption, there exists an
open set U containing A such that Cl(U) ⊆ Bc and so Cl(U) ∩ B = ∅.

(iii) → (iv). Let A and B be disjoint αgs-closed sets of (X, τ). Then by
assumption, there exists an open set U containing A such that Cl(U) ∩ B
= ∅. Since Cl(U) is closed, it is αgs-closed and so Cl(U) and B are
disjoint αgs-closed sets in X. Therefore again by assumption, there exists
an open set V containing B such that Cl(U) ∩ Cl(V) = ∅.

(iv) → (i). Let A and B be any two disjoint αgs-closed sets of (X, τ).
By assumption, there exist open sets U containing A and V containing B
such that Cl(A) ∩ Cl(V) = ∅, we have U ∩ V = ∅. Hence (X, τ) is αgs-
normal.
THEOREM 4.12.9: A topological space $X$ is $\alpha\beta$-normal if and if for any disjoint $\alpha\beta$-closed sets $A$ and $B$ of $X$, there exist open sets $U$ and $V$ such that $A \subseteq U$, $B \subseteq V$ and $\text{Cl}(U) \cap \text{Cl}(V) = \emptyset$.

PROOF. Follows from theorem 4.12.8.

THEOREM 4.12.10: If $(X, \tau)$ is $\alpha\beta$-normal space and $Y$ is a $\alpha\beta$-closed subset of $X$, then the subspace $Y$ is $\alpha\beta$-normal.

PROOF. Let $A$ and $B$ be any disjoint $\alpha\beta$-closed sets of $Y$. By [Theorem 4.5.21], $A$ and $B$ are $\alpha\beta$-closed in $(X, \tau)$. Since $(X, \tau)$ is $\alpha\beta$-normal, there exist disjoint open sets $U$ and $V$ of $X$ such that $A \subseteq U$ and $B \subseteq V$. Therefore $U \cap Y$ and $V \cap Y$ are disjoint open subsets of the subspace $Y$ such that $A \subseteq U \cap Y$ and $B \subseteq V \cap Y$. This shows that the subspace $Y$ is $\alpha\beta$-normal.

COROLLARY 4.12.11: The property of being $\alpha\beta$-normal is closed hereditary.

THEOREM 4.12.12: If $f: (X, \tau) \to (Y, \sigma)$ is a pre-$\alpha\beta$ open, pre-$\alpha\beta$-continuous, bijection and open and $(X, \tau)$ is $\alpha\beta$-normal, then $(Y, \sigma)$ is $\alpha\beta$-normal.

PROOF. Let $A$ and $B$ be any disjoint $\alpha\beta$-closed sets of $(Y, \sigma)$. Since $f$ is pre-$\alpha\beta$ open, pre-$\alpha\beta$-continuous and bijective, $f$ is $\alpha\beta$-irresolute [Theorem 4.9.9] and hence $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint $\alpha\beta$-closed sets of $(X, \tau)$. Since $(X, \tau)$ is $\alpha\beta$-normal, there exists disjoint open sets $U$ and $V$ such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$. Since $f$ is open and
bijective, we obtain $A \subseteq f(U)$, $B \subseteq f(V)$, $f(U) \cap f(V) = \emptyset$ and also $f(U)$ and $f(V)$ are open in $(Y, \sigma)$. This shows that $(Y, \sigma)$ is $\alpha\gamma$-normal.

**THEOREM 4.12.13**: If $f : (X, \tau) \to (Y, \sigma)$ is $\alpha\gamma$ irresolute, pre-semi-closed, continuous injection and $(Y, \sigma)$ is $\alpha\gamma$-normal, then $(X, \tau)$ is $\alpha\gamma$-normal.

**PROOF**. Let $A$ and $B$ be disjoint $\alpha\gamma$-closed sets of $(X, \tau)$. Since $f$ is irresolute pre-semi-closed, $f(A)$ and $f(B)$ are disjoint $\alpha\gamma$-closed sets of $(Y, \sigma)$ [Theorem 4.10.3]. Since $(Y, \sigma)$ is $\alpha\gamma$-normal, then there exist disjoint open sets $U$ and $V$ such that $f(A) \subseteq U$ and $f(B) \subseteq V$. Thus, we obtain $A \subseteq f^{-1}(U)$, $B \subseteq f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Since $f$ is continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are open in $(X, \tau)$. This shows that $(X, \tau)$ is $\alpha\gamma$-normal.

**THEOREM 4.12.14**: If $f : (X, \tau) \to (Y, \sigma)$ is weakly continuous $\alpha\gamma$-closed injection and $(Y, \sigma)$ is $\alpha\gamma$-normal, then $(X, \tau)$ is normal.

**PROOF**. Let $A$ and $B$ be any disjoint closed sets of $(X, \tau)$. Since $f$ is $\alpha\gamma$-closed and injective, $f(A)$ and $f(B)$ are disjoint $\alpha\gamma$-closed sets of $(Y, \sigma)$. Since $(Y, \sigma)$ is $\alpha\gamma$-normal, by Theorem 4.12.8, there exist open sets $U$ and $V$ such that $f(A) \subseteq U$, $f(B) \subseteq V$ and $\text{Cl}(U) \cap \text{Cl}(V) = \emptyset$. Since $f$ is weakly continuous, it follows from [5, Theorem 1] that $A \subseteq f^{-1}(U) \subseteq \text{Int}[f^{-1}(\text{Cl}(U))]$, $B \subseteq f^{-1}(V) \subseteq \text{Int}[f^{-1}(\text{Cl}(V))]$ and $\text{Int}[\text{Cl}(U)] \cap \text{Int}[f^{-1}(\text{Cl}(V))] = \emptyset$. This shows that $(X, \tau)$ is normal.

**4.13. ON $\alpha\gamma$-REGULAR SPACES.**
In this section we introduce \(\alpha\gamma\delta\) regular spaces using \(\alpha\gamma\delta\) closed sets in topological spaces, some characterizations of \(\alpha\gamma\delta\) regular spaces are obtained.

**DEFINITION 4.13.1:** A topological space \((X, \tau)\) is said to be \(\alpha\gamma\delta\)-regular if for each \(\alpha\gamma\delta\)-closed set \(F\) of \(X\) and each point \(x \in F\), there exist disjoint open sets \(U\) and \(V\) of \(X\) such that \(x \in U\) and \(F \subseteq V\).

**REMARK 4.13.2:** It is obvious that every \(\alpha\gamma\delta\) regular space is regular. However, the converse is not true as seen from the following example.

**EXAMPLE 4.13.3:** Let \(X = \{a, b, c, d\}\) and \(\tau = \{\emptyset, \{a, d\}, \{b, c\}, X\}\). Then the space \((X, \tau)\) is regular, but not \(\alpha\gamma\delta\) regular.

**THEOREM 4.13.4:** The following are equivalent for a space \((X, \tau)\).

(i) \((X, \tau)\) is \(\alpha\gamma\delta\)-regular.

(ii) \(\text{Cl}_\emptyset(A) = \alpha\gamma\delta\text{Cl}(A)\) for each subset \(A\) of \((X, \tau)\).

(iii) \(\text{Cl}_\emptyset(A) = A\) for each \(\alpha\gamma\delta\)-closed set \(A\).

**PROOF.** (i) \(\rightarrow\) (ii). Assume that \((X, \tau)\) is \(\alpha\gamma\delta\)-regular. For any subset \(A\) of \((X, \tau)\), we always have \(A \subseteq \alpha\gamma\delta\text{Cl}(A) \subseteq \text{Cl}_\emptyset(A)\). Suppose that \(x \in (\alpha\gamma\delta\text{Cl}(A))^c\). Then there exists an \(\alpha\gamma\delta\)-closed set \(F\) such that \(x \in X - F\) and \(A \subseteq F\). Since \((X, \tau)\) is \(\alpha\gamma\delta\)-regular, there exist disjoint open sets \(U\) and \(V\) such that \(x \in U\) and \(F \subseteq V\). Therefore, we have \(x \in \text{Cl}(U) \subseteq X - V \subseteq X - F \subseteq X - A\) and hence \(\text{Cl}(U) \cap A = \emptyset\). Therefore, we have \(X - \text{Cl}_\emptyset(A)\) and hence \(\text{Cl}_\emptyset(A) = \alpha\gamma\delta\text{Cl}(A)\).

(ii) \(\rightarrow\) (iii). The proof is trivial.

(iii) \(\rightarrow\) (i). Let \(F\) be any \(\alpha\gamma\delta\)-closed set and \(x \notin F\). Since \(F\) is \(\alpha\gamma\delta\)-closed, by assumption \(x \in X - \text{Cl}_\emptyset(F)\) and there exists an open set \(U\) such that
\[ x \in U \text{ and } \text{Cl}(U) \cap F = \emptyset. \text{ Therefore we obtain that } F \subseteq X - \text{Cl}(U) \text{ ie. } F \subseteq V \text{ where } V = X - \text{Cl}(U) \text{ and also } U \text{ and } V \text{ are disjoint. This shows that } (X, \tau) \text{ is } \alpha\gamma\delta\text{-regular.} \]

**THEOREM 4.13.5:** Let \((X, \tau)\) be a topological space. Then the following are equivalent:

(i) \((X, \tau)\) is an \(\alpha\gamma\delta\text{-regular space.}\)

(ii) For each \(x \in X\) and each \(\alpha\gamma\delta\)-open neighbourhood \(A\) of \(x\) there exists an open neighbourhood \(V\) of \(x\) such that \(\text{Cl}(V) \subseteq A\).

**PROOF.** (i) \(\rightarrow\) (ii). Let \(A\) be any \(\alpha\gamma\delta\)-open neighbourhood \(A\) of \(x\). Then there exists an \(\alpha\gamma\delta\)-open set \(G\) such that \(x \in G \subseteq A\). Since \(X - G\) is \(\alpha\gamma\delta\) closed and \(x \in X - G\), by hypothesis, there exist open sets \(U\) and \(V\) such that \(X - G \subseteq U\), \(x \in V\) and \(U \cap V = \emptyset\) and so \(V \subseteq X - U\). Now \(\text{Cl}(V) \subseteq \text{Cl}(X - U) = X - U\) and \(X - G \subseteq U\) implies \(X - U \subseteq G \subseteq A\). Therefore \(\text{Cl}(V) \subseteq A\).

(iii) \(\rightarrow\) (i). Let \(F\) be any \(\alpha\gamma\delta\)-closed set of \(X\) and \(x \notin F\). Then \(x \in X - F\) and \(X - F\) is \(\alpha\gamma\delta\)-open and so \(X - F\) is an \(\alpha\gamma\delta\) neighbourhood of \(x\). By hypothesis, there exists an open neighbourhood \(V\) of \(x\) such that \(x \in V\) and \(\text{Cl}(V) \subseteq X - F\), which implies \(F \subseteq X - \text{Cl}(V)\). Then \(X - \text{Cl}(V)\) is an open set containing \(F\) and \(V \cap (X - \text{Cl}(V)) = \emptyset\). Therefore \((X, \tau)\) is \(\alpha\gamma\delta\text{-regular.}\)

We recall the following definition.

**THEOREM 4.13.7:** A space \((X, \tau)\) is semi-symmetric if and only if \(\{x\}\) is \(\alpha\gamma\delta\)-closed for each \(x\) in \(X\).
PROOF. Sufficiency. Suppose \( x \in \text{sCl}(y) \), but \( y \notin \text{sCl}(x) \). Then \( \{y\} \subseteq X - \text{sCl}(x) \) and thus \( \text{sCl}(y) \subseteq X - \text{sCl}(x) \). Then \( x \in X - \text{sCl}(x) \), a contradiction.

Necessity. Suppose \( \{x\} \subseteq U \in \alpha\text{gsO}(X, \tau) \), but \( \text{sCl}(x) \subseteq U \). Then \( \text{sCl}(x) \cap U^c \neq \emptyset \). Take \( y \in \text{sCl}(x) \cap U^c \). Therefore, \( x \in \text{sCl}(y) \subseteq U^c \) and \( x \notin U \), a contradiction. Hence \( \{x\} \) is \( \alpha\text{gs} \)-closed in \((X, \tau)\).

THEOREM 4.13.8: Every \( \alpha \)-normal, semi-symmetric space \((X, \tau)\) is \( \alpha \)-regular.

PROOF. Let \( F \) be a closed subset of \((X, \tau)\) and \( x \in X \) such that \( x \notin F \). Since \((X, \tau)\) is semi-symmetric space, by theorem 4.13.7, \( \{x\} \) is \( \alpha\text{gs} \)-closed. Since \( F \) is closed and \((X, \tau)\) is \( \alpha \)-normal, we have by theorem 4.12.6, there exists disjoint \( \alpha \)-open sets \( U \) and \( V \) such that \( F \subseteq U \) and \( \{x\} \subseteq V \). Therefore \((X, \tau)\) is \( \alpha \)-regular.

THEOREM 4.13.9: A topological space \((X, \tau)\) is \( \alpha\text{gs} \)-regular if and only if for each \( \alpha\text{gs} \) closed set \( F \) of \( X \) and each point \( x \in F \), there exist open sets \( U \) and \( V \) of \( X \) such that \( x \in U \), \( F \subseteq V \) and \( \text{Cl}(U) \cap \text{Cl}(V) = \emptyset \).

PROOF. Necessity. Let \( F \) be an \( \alpha\text{gs} \)-closed set of \( X \) and \( x \in F \). There exist open sets \( U_0 \) and \( V \) of \( X \) such that \( x \in U_0 \), \( F \subseteq V \) and \( U_0 \cap V = \emptyset \); hence \( U_0 \cap \text{Cl}(V) = \emptyset \). Since \((X, \tau)\) is \( \alpha\text{gs} \) regular, there exist open sets \( G \) and \( H \) of \( X \) such that \( x \in G \), \( \text{Cl}(V) \subseteq H \) and \( G \cap H = \emptyset \); hence \( \text{Cl}(G) \cap H = \emptyset \). Now put \( U = U_0 \cap G \), then \( U \) and \( V \) are open sets of \( X \) such that \( x \in U \), \( F \subseteq V \) and \( \text{Cl}(U) \cap \text{Cl}(V) = \emptyset \).

Sufficient: Sufficiency is obvious.

COROLLARY 4.13.10: If a space \((X, \tau)\) is \( \alpha\text{gs} \) regular and semi-symmetric, then it is Uryshon.
PROOF: Similar to Corollary 4.11. Let \( x \) and \( y \) be any district point of \( X \).

Since \( X \) is semi- symmetric, \( \{x\} \) is \( \alpha\gs \)-closed by theorem 4.13.9 where exists open sets \( U \) and \( V \) such that \( x \in U \) and \( y \in V \) and \( \text{Cl}(U) \cap \text{Cl}(V) = \emptyset \).

**THEOREM 4.13.11:** If \((X, \tau)\) is an \( \alpha\gs \)-regular space and \( Y \) is an open set and \( \alpha\gs \)-closed subset of \((X, \tau)\), then the subspace \( Y \) is \( \alpha\gs \)-regular.

**PROOF:** Let \( A \) be any \( \alpha\gs \)-closed subset of \( Y \) and \( y \not\in A \). By [Theorem 4.5.21], \( A \) is \( \alpha\gs \)-closed in \((X, \tau)\). Since \((X, \tau)\) is \( \alpha\gs \)-regular, there exists disjoint open sets \( U \) and \( V \) of \( X \) such that \( y \in U \) and \( A \subseteq V \). Therefore \( U \cap Y \) and \( V \cap Y \) are disjoint open sets of the subspace \( Y \) such that \( y \in U \cap Y \) and \( A \subseteq V \cap Y \). This shows that the subspace \( Y \) is \( \alpha\gs \)-regular.

**THEOREM 4.13.12:** If \( f : (X, \tau) \to (Y, \sigma) \) is \( \alpha\gs \)-irresolute, pre-semi-closed, continuous injection and \((Y, \sigma)\) is \( \alpha\gs \)-regular, then \((X, \tau)\) is \( \alpha\gs \)-regular.

**PROOF:** Let \( F \) be any \( \alpha\gs \)-closed set of \( X \) and \( x \in X - F \). Since \( f \) is irresolute pre-\( \alpha \)-closed, \( f(F) \) is \( \alpha\gs \)-closed in \( Y \) [Theorem 4.10.3]. And \( f(x) \in Y - f(F) \). Since \((Y, \sigma)\) is \( \alpha\gs \)-regular, there exist disjoint open sets \( U \) and \( V \) such that \( f(x) \in U \) and \( f(F) \subseteq V \). Thus, we obtain \( x \in f^{-1}(U) \), \( F \subseteq f^{-1}(V) \) and \( f^{-1}(U) \cap f^{-1}(V) = \emptyset \). This shows that \((X, \tau)\) is \( \alpha\gs \)-regular.

**THEOREM 4.13.13:** If \( f : (X, \tau) \to (Y, \sigma) \) is weakly continuous \( \alpha\gs \)-closed injection and \((Y, \sigma)\) is \( \alpha\gs \)-regular, then \((X, \tau)\) is regular.
PROOF. Let A be any αgs-closed set of X and x ∈ A. Since f is αgs-closed, f(A) is αgs-closed in Y and f(x) ∈ Y - f(A). Since Y is αgs-regular, by theorem 4.13.9, there exist disjoint open sets U and V such that f(x) ∈ U, f(A) ⊆ V and Cl(U) ∩ Cl(V) = ∅. Since f is weakly continuous, it follows from [58 theorem 1] that x ∈ f⁻¹(U) ⊆ Int[f⁻¹(Cl(U))], A ⊆ f⁻¹(V) ⊆ Int[f⁻¹(Cl(V))] and Int[f⁻¹(Cl(U))] ∩ Int[f⁻¹(Cl(V))] = ∅. Therefore (X, τ) is regular.

4.14. On αgα-closed sets in topological spaces

DEFINITION 4.14.1: A subset A of X is called α-generalized αgs-closed set (briefly αgα-closed set) if αCl(A) ⊆ U whenever A ⊆ U and U is αgs-open set in X.

THEOREM 4.14.2: If A and B are αgα-closed sets, then A ∪ B is αgα-closed set in X.

PROOF: If A ∪ B ⊆ G and G is αgs-open, then A ⊆ G and B ⊆ G. Since A and B are αgα-closed sets, αCl(A) ⊆ G and αCl(B) ⊆ G and hence αCl(A) ∪ αCl(B) ⊆ G, since αCl(A ∪ B) = αCl(A) ∪ αCl(B). We have αCl(A ∪ B) = αCl(A) ∪ αCl(B) ⊆ G, thus A ∪ B is a αgα-closed set in X.

THEOREM 4.14.3: Every closed set is αgα-closed set in X.

PROOF: Let A be a closed set in X. Note that αCl(A) ⊆ αCl(A) always and Cl(A) = A, if A is closed set. So if A ⊆ G where G is αgs-open set in X, then αCl(A) ⊆ G. Hence A is αgα-closed set in X.

Converse is not true.
**EXAMPLE 4.14.4:** Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, X\}$ Then the sets $\{c\}, \{b\}$ are $\alpha \alpha$-closed sets in $X$ but not closed in $X$.

**THEOREM 4.14.5:** Every $\alpha \hat{\alpha}$-closed set is $\alpha \hat{\alpha} \hat{s}$-closed set in $X$.

**PROOF:** Let $A$ be any $\alpha \hat{\alpha}$-closed set in $X$ and $G$ be any $\alpha \hat{s} \hat{g}$-open set containing $A$. Then $\alpha \text{Cl}(A) \subseteq G$. Hence $\hat{s}\text{Cl}(A) \subseteq G$. Thus $A$ is $\alpha \hat{\alpha} \hat{s}$-closed set in $X$.

Converse is not true.

**EXAMPLE 4.14.6:** Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ Then the sets $\{a\}$ and $\{b\}$ are $\alpha \hat{\alpha} \hat{s}$-closed set but not $\alpha \hat{\alpha} \alpha$-closed sets in $X$.

**THEOREM 4.14.7:** Every $\alpha \hat{\alpha}$-closed set is $\alpha \hat{g}$-closed set in $X$.

**PROOF:** Let $A$ be an $\alpha \hat{\alpha}$-closed set in $X$. Let $A \subseteq U$, where $U$ is open in $X$. Every open set in $X$ is $\alpha \hat{g} \hat{s}$-open in $X$. Therefore $U$ is $\alpha \hat{g} \hat{s}$-open in $X$. Hence $\alpha \text{Cl}(A) \subseteq U$. So $A$ is $\alpha \hat{g}$-closed set in $X$.

Converse is not true.

**EXAMPLE 4.14.8:** Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, X\}$ Then the sets $\{a, b\}, \{b, c\}$ are $\alpha \hat{g}$-closed sets but not $\alpha \hat{\alpha} \alpha$-closed sets in $X$.

**THEOREM 4.14.9:** Every $\alpha$-closed set is $\alpha \hat{\alpha}$-closed set in $X$.

**PROOF:** Trivial.

Converse is not true.
EXAMPLE 4.14.10: Let \( X = \{a, b, c, d\} \) and \( \tau = \{ \emptyset, X, \{a\}, \{b\}, \{a, b\}\} \)
Then the sets \( \{a, d\} \) and \( \{b, d\} \), \( \{a, b, d\} \) are \( \alpha \tilde{g} \alpha \)-closed set but not \( \alpha \)-closed sets in \( X \).

THEOREM 4.14.11: Every \( \alpha \tilde{g} \alpha \)-closed set is \( \alpha \)-*g- closed set in \( X \).

PROOF: Let \( A \) be an \( \alpha \tilde{g} \alpha \)-closed set in \( X \). Let \( A \subseteq U \) and \( U \) be \( \omega (= \tilde{g} \) open set in \( X \). Every \( \omega \)-open set in \( X \) is \( \alpha g s \) - open in \( X \). Hence \( \alpha Cl(A) \subseteq U \). So \( A \) is \( \alpha \)-*g- closed set in \( X \).

Converse is not true.

EXAMPLE 4.14.12: Let \( X = \{a, b, c\} \) and \( \tau = \{ \emptyset, \{a\}, X \} \). Then the sets \( \{a, b\} \) and \( \{a, c\} \) are \( * g \)-closed sets but not \( \alpha \tilde{g} \alpha \)-closed sets in \( X \).

THEOREM 4.14.13: Every \( \alpha \tilde{g} \alpha \)-closed set is \( g \alpha \)-closed set in \( X \).

PROOF: Let \( A \) be an \( \alpha \tilde{g} \alpha \)-closed set in \( X \). Let \( A \subseteq U \) and \( U \) be an \( \alpha \)-open set in \( X \), Since every \( \alpha \)-open set in \( X \) is \( \alpha g s \)-open in \( X \) \( \), \( U \) is \( \alpha g s \)-open in \( X \). Hence \( \alpha Cl(A) \subseteq U \). So \( A \) is \( g \alpha \)-closed set in \( X \).

Converse is not true.

EXAMPLE 4.14.14: Let \( X = \{a, b, c, d\} \) and \( \tau = \{ \emptyset, \{a\}, \{b\}, \{a, b\}, X \} \)
Then \( \{c\} \) is \( g \alpha \)- closed but not \( \alpha \tilde{g} \alpha \)-closed sets in \( X \).

THEOREM 4.14.16: If \( A \) is an on \( \alpha \tilde{g} \alpha \)-closed set in \( X \), then \( \alpha Cl (A) – A \) contains no non - empty closed set in \( X \).
**PROOF:** Suppose that A is an $\alpha g\alpha$-closed set in X. Let $F$ be a closed subset of $\alpha Cl(A) - A$, then $A \subseteq F^c$. Since A is an $\alpha g\alpha$-closed set, $\alpha Cl(A) \subseteq F^c$. Therefore we have $F \subseteq \alpha Cl(A)$. Thus $F \subseteq \alpha Cl(A) \cap (\alpha Cl(A))^c = \emptyset$ and $F$ is empty.

Converse of the above Theorem is not true as seen from the following example.

**EXAMPLE 4.14:** Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, X\}$ and let $A = \{a, b\}$ and $B = \{a, c\}$. Then $\alpha cl(A) - (A)$, and $\alpha cl(B)$ - B contains no non empty closed set in X but A and B are not $\alpha g\alpha$-closed set in X.

**THEOREM 4.14.17:** A subset $A$ is an $\alpha g\alpha$-closed set if and if $\alpha Cl(A) - A$ contains no non empty $\alpha g\alpha$s-closed set in X.

**PROOF:** Suppose $A$ is an $\alpha g\alpha$-closed set. Let $S$ be an $\alpha g\alpha$s-open subset of $\alpha Cl(A) - A$. Then $A \subseteq S^c$. Since $A$ is an $\alpha g\alpha$-closed set, we have $\alpha Cl(A) \subseteq S^c$. Consequently $S \subseteq (\alpha Cl(A))^c$, hence $\alpha Cl(A) \cap (\alpha Cl(A))^c = \emptyset$. Therefore, $S$ is empty.

Conversely, suppose that $\alpha Cl(A) - A$. contains no non-empty $\alpha g\alpha$s-closed set. Let $A \subseteq G$ and let $G$ be $\alpha g\alpha$s-open. If $\alpha Cl(A) \not\subseteq G$, then $\alpha Cl(A) \cap G^c$ is non empty $\alpha g\alpha$s-closed subset of $\alpha Cl(A) - A$. Therefore $A$ is an on $\alpha g\alpha$-closed set in X.

**THEOREM 4.14.18:** If $A$ is an $\alpha g\alpha$-closed set and $A \subseteq B \subseteq \alpha Cl(A)$, then $B$ is an $\alpha g\alpha$-closed set in X.
PROOF: Let \( B \subseteq U \) where \( U \) is \( \alpha \text{gs-open in } X \). Since \( A \) is an \( \alpha \tilde\alpha \)-closed set and \( A \subseteq U \), it follows that \( \alpha \text{Cl}(A) \subseteq U \). By hypothesis \( B \subseteq \alpha \text{Cl}(A) \) and hence \( \alpha \text{Cl}(B) \subseteq \alpha \text{Cl}(A) \). Consequently, \( \alpha \text{Cl}(B) \subseteq U \) and so, \( B \) becomes an \( \alpha \tilde\alpha \)-closed in \( X \).