CHAPTER 5
ON THE ENERGY OF TREES WITH EDGE INDEPENDENCE NUMBER FOUR

5.1 INTRODUCTION

We know that characteristic polynomial of tree can be determined easily using matchings of different size as there are no cycles, using Sachs formula or by using the recurrence formula obtained by H.S. Ramane [96]. In particular energy of trees with respect to independence number 2 and 3 is already investigated [57, 82].

In this chapter we extend it for trees with independence no 4.

The energy of a bipartite graph is given by [13, 37, 44]

\[ E(G) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d}{d\lambda} \log \left[ 1 + a_2 \lambda^2 + a_4 \lambda^4 + \ldots + a_{2k} \lambda^{2k} \right] \]  

(5.1)

Where, \( \Phi(G : \lambda) = \lambda^p - a_2 \lambda^{p-2} + a_4 \lambda^{p-4} - \ldots + (-1)^k a_{2k} \lambda^{p-k} \)  

(5.2)

is the characteristic polynomial of G, \( p \) is the number of vertices in G and

\[ k = \begin{cases} 
(p/2), & \text{if } p \text{ is even} \\
(p-1)/2, & \text{if } p \text{ is odd.} 
\end{cases} \]

If the characteristic polynomial is written as in equation (5.2) then

\[ a_{2j} \geq 0 \text{ for all } j = 1, 2, \ldots, k \]
Let $G$ and $H$ be the bipartite graphs with the same number of vertices. Suppose the coefficients $a_{2j}$ of these graphs satisfy the inequalities,

$$a_{2j}(G) \geq a_{2j}(H)$$

for all $j = 1, 2, \ldots, k$ then from equation (5.1) it follows that,

$$E(G) \geq E(H)$$

Further if the characteristic polynomials of $G$ and $H$ are not same then the relation (5.1) implies that

$$E(G) > E(H)$$

An independent set of edges of $G$ has no two of its edges incident to a common vertex and the maximum cardinality among all such sets, is the edge independence number of $G$ denoted by $\beta_1(G)$

### 5.2 EXISTING RESULTS

Let $T$ be the tree with the vertex set $V(T) = \{v_1, v_2, \ldots, v_p\}$ and edge set $E(T) = \{e_1, e_2, \ldots, e_{p-1}\}$. Let $M_k = \{e_1, e_2, \ldots, e_k\}$ be the set of independent edges of $T$ where $1 < k \leq \lfloor p/2 \rfloor$.

Define a set $Q_n(M_k) = \{S \mid S \subseteq M_k \text{ and } |S| = n\}$, for $1 \leq n \leq k$. In addition $Q_0(M_k)$ is an empty set. Let $T_k$ be the graph obtained from $T$ by removing $k$ independent edges of $T$, that is, $T_k = T - e_1 - e_2 - \ldots - e_k = T - \sum_{i=1}^{k} e_i$

Let $T$ be the tree with $p$ vertices and $M_k = \sum_{n=0}^{k} (-1)^n \sum_{S \in Q_n(M_k)} \Phi(T_k - S : \lambda)$

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\[ \phi(K_0 : \lambda) = 1 \] where \( K_0 \) is a graph without vertices and edges.

**Theorem 5.2.1 [96] (Recurrence Relation):**

If \( e_1, e_2, \ldots, e_k \) be the \( k \)-independent edges of a tree \( T \), \( 1 \leq k \leq \lfloor p/2 \rfloor \), \( p \) is the number of vertices of \( T \) then,

\[
\phi(T : \lambda) = \sum_{n=0}^{k} \left[ (-1)^n \sum_{S \in Q_n(M_k)} \phi(T_k - S : \lambda) \right]
\]

Trees with edge independence number 2 as shown in Figure 5.1-5.2 were investigated by H. S. Ramane [96] and with edge independence number 3 were investigated by P.R. Hampiholi [57].

Now let us consider the trees with independence number 4. First we prove that if \( \beta_i(T) = 4 \) then \( 4 \leq \text{diam}(T) \leq 8 \) and then, construct all those trees \( T \) with \( \beta_i(T) = 4 \) and \( 4 \leq \text{diam}(T) \leq 8 \).

**Theorem 5.2.2:** If \( \beta_i(T) = 4 \) then \( 4 \leq \text{diam}(T) \leq 8 \) for any tree \( T \).

**Proof:** Suppose \( \beta_i(T) = 4 \) and let \( \text{diam}(T) \leq 4 \) then \( T \) is any of the trees \( K_1, K_2, \) or \( K_{1,p-1} \) or \( A_p(k) \), as shown in Figure 5.1.

\[ A_p(k) \]

Figure 5.1

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In each of these cases $\beta_1(T) \leq 3$ which contradicts to the assumption

$\beta_1(T) = 4$

Hence $\text{diam (T)} \geq 4$

Next to prove $\text{diam (T)} \leq 8$. Assume that $\text{diam}(T) = d \geq 9$. Let $u$ and $v$ be the end vertices of the longest path $uu_1u_2u_3u_4\ldots u_{d-1}v$ in $T$. Then the edges $e_1 = uu_1$, $e_2 = u_2u_3$, $e_3 = u_4u_5$, $e_{d/2} = u_{d-2}u_{d-1}$ are independent edges in $T$ if $d$ is even, so that

$\beta_1(T) = \left(\frac{d+1}{2}\right) > 4$ as $d \geq 9$ contradiction to $\beta_1(T) = 4$

If $d$ is odd $e_1 = uu_1$, $e_2 = u_2u_3$, $e_3 = u_4u_5$, $e_{d+1/2} = u_{d-1}v$ are independent edges in $T$, so that $\beta_1(T) = \left(\frac{d+1}{2}\right) > 4$ again contradiction to $\beta_1(T) = 4$

Hence $\text{diam (T)} \leq 8$

**Proposition 5.2.2** If $\beta_1(T) = 4$ then $T$ is one of the following trees shown in Figure 5.2-5. For convenience we denote $T_{d,b}$ as $b$'th tree with diameter 'd' and $\beta_1(T) = 4$
In each of these cases $\beta_1(T) \leq 3$ which contradicts to the assumption $\beta_1(T) = 4$

Hence $\text{diam}(T) \geq 5$

Next to prove $\text{diam}(T) \leq 8$ Assume that $\text{diam}(T) = d \geq 9$ Let $u$ and $v$ be the end vertices of the longest path $uu_1u_2u_3u_4 v$ in $T$ Then the edges $e_1=uu_1, e_2 = u_2u_3, e_3 = u_4u_5, e_{d-2} = u_{d-2}u_{d-1}$ are independent edges in $T$ if $d$ is even, so that $\beta_1(T) = \frac{d+1}{2} > 4$ as $d \geq 9$ contradiction to $\beta_1(T) = 4$
If \( d \) is odd, \( e_1 = u_1, e_2 = u_2u_3, e_3 = u_4u_5, \ldots, e_{d+1/2} = u_{d+1}v \) are independent edges in \( T \), so that \( \beta_1(T) = \binom{d+1}{2} > 4 \) again contradiction to \( \beta_1(T) = 4 \).

Hence \( \text{diam}(T) \leq 8 \).

**Proposition 5.2.2** If \( \beta_1(T) = 4 \) then \( T \) is one of the following trees shown in Figure 5.9.

For convenience we denote \( T_{d,b} \) as \( b \)th tree with diameter \( 'd' \) and \( \beta_1(T) = 4 \).

\[ \begin{align*}
T_{5,1} & \quad \text{Figure 5.9} \\
T_{5,2} & \quad \text{Figure 5.10} \\
T_{5,3} & \quad \text{Figure 5.11} \\
\end{align*} \]
Figure 5.13

Figure 5.14

Figure 5.15

Figure 5.16
Proof: By the proposition 5.2.2, if \( \beta(T) = 4 \) then \( 4 \leq \text{diam}(T) \leq 8 \). We consider here \( 5 \) cases for \( \text{diam}(T) = 5, 6, 7, \) and \( 8 \).

Case 1: If \( \text{diam}(T) = 5 \) then there is a longest path say \( uu_1u_2u_3u_4v \) in \( T \). If \( (k_1-1) \) vertices from the remaining \( (p-6) \) vertices are adjacent to \( u_1 \), \( (k_2-1) \) vertices from the remaining \( (p-k_1-5) \) vertices are adjacent to \( u_2 \), \( (k_3-1) \) vertices from the remaining \( (p-k_1-k_2-5) \) vertices are adjacent to \( u_3 \), and \( (k_4-1) \) vertices of the remaining \( (p-k_1-k_2-k_3-5) \) are adjacent to \( u_4 \), then resulting tree \( T_{5,1} \) is as shown in Figure 5.4. Similarly we have trees \( T_{5,2} \) and \( T_{5,3} \) in Figure 5.5.
Case 2 If diam (T) = 6 then as in case 1 we can show that T is either \( T_{6,1} \) or \( T_{6,2} \) or \( T_{6,3} \) or \( T_{6,4} \) or \( T_{6,5} \) as shown in Figures 5.1-5.14.

Case 3 If diam (T) = 7 then as in previous cases, we can prove that T is one of \( T_{7,1} \), or \( T_{7,2} \) as in Figure 5.12-5.13.

Case 4 If diam (T) = 8 then we can have only tree \( T_{8,1} \) as shown in Figure 5.14.

5.3 ENERGY OF TREES WITH \( \beta_1(T) = 4 \):

To find energy of trees with \( \beta_1(T) = 4 \), we need to compute the characteristic polynomial of these trees from \( T_{5,1} \) to \( T_{8,1} \). They are given by proposition 5.3.1 to 5.3.11 and computed by using theorem 5.2.1.

**Proposition 5.3.1**

\[
\Phi(T) = \lambda^p - (p-1)\lambda^{p-2} + (k_1k_2 + k_1k_3 + k_1k_4 + k_2k_3 + k_2k_4 + k_3k_4 + k_2 + k_3 + 2k_4 + 1) \lambda^{p-4} + (k_1k_2k_3 + k_1k_3k_4 + k_2k_3k_4 + k_1k_2k_4 \text{ where } k_1 + k_2 + k_3 + k_4 = p - 4
\]

**Proposition 5.3.2**

\[
\Phi(T) = \lambda^p - (p-1)\lambda^{p-2} + (k_1k_2 + k_1k_3 + k_1k_4 + k_2k_3 + k_3k_4 + k_2k_4 + 3k_1 + 2k_3 + 2k_4 + 2k_2 + 2k_1k_2 + 2k_1k_3 + 2k_1k_4 + 2k_2k_3 + 2k_2k_4 + 2k_3k_4 + 2k_1 + 2k_2 + 2k_3 + 2k_4 \text{ where } k_1 + k_2 + k_3 + k_4 = p - 4
\]
Proposition 5.3.3

\( \Phi(T_{5,3} \lambda) = \lambda^{p-1} \lambda^{p-2} + (k_1k_2 + k_1k_3 + k_1k_4 + (k_2-1)k_3 + k_2k_4 + k_3k_4 + \\
3k_1k_2 + 3k_3k_4) \lambda^p \frac{1}{(k_1k_2 + k_1k_4 + (k_2-1)k_3k_4 + k_2k_3 + k_2k_4 + k_3k_4 + \\
2k_3k_4 + (k_2-1)k_3 + 2k_1k_3 + k_3)} \lambda^{p-6} + (k_1(k_2-1)k_3k_4 + k_1(k_2-1)k_3) \lambda^{p-8} \)

\( \text{where } k_1 + k_2 + k_3 + k_4 = p - 4 \)

Proposition 5.3.4

\( \Phi(T_{6,1} \lambda) = \lambda^{p-1} \lambda^{p-2} + (k_1k_3 + k_1k_3 + k_1k_4 + k_2k_3 + k_2k_4 + k_3k_4 + 3k_1 + \\
2k_2 + 2k_3 + 3k_4 + 3) \lambda^{p-6} + (k_1k_2k_3 + k_1k_2k_4 + k_2k_3k_4 + k_1k_2k_4 + 2k_1k_2 + 2k_1k_4 + \\
2k_2k_3 + k_3k_4 + k_4) \lambda^{p-6} + (k_1k_2k_3 + k_1k_2k_4 + k_1k_3k_4 + k_2k_3k_4) \lambda^{p-8} \)

\( \text{where } k_1 + k_2 + k_3 + k_4 = p - 6 \)

Proposition 5.3.5

\( \Phi(T_{6,2} \lambda) = \lambda^{p-1} \lambda^{p-2} + (k_1k_2 + k_1k_3 + k_1k_4 + k_2k_3 + k_2k_4 + k_3k_4 + 3k_1 + 2k_2 + \\
2k_3 + 3k_4 + 3) \lambda^{p-6} + (k_1k_2k_3 + k_1k_3k_4 + k_2k_3k_4 + k_1k_2k_4 + 2k_1k_2 + 2k_1k_4 + 2k_3k_4 + \\
k_1k_3 + k_2k_4 + k_4) \lambda^{p-6} + (k_1k_2k_3k_4 + k_1k_2k_4) \lambda^{p-8} \)

\( \text{where } k_1 + k_2 + k_3 + k_4 = p - 8 \)

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Proposition 5.3.6

\[ \Phi(T_{6,3} \lambda) = \lambda p^2 \cdot (p-1) \lambda p^2 + \{k_1 k_2 + k_1 k_3 + k_1 k_4 + (k_2 - 1) k_3 + k_2 k_4 + 3 k_4 + 2 k_3 + 4 k_2 + 3 k_4 + 3 \} \lambda^p + \{k_1 (k_2 - 1) k_3 + k_1 k_3 k_4 + (k_2 - 1) k_3 k_4 + k_1 k_3 k_4 + k_1 k_2 \}
\]

+ 2 k_1 k_4 + 3 k_3 k_4 + 3 k_1 k_3 + k_3 k_4 + 3 k_2 - 1) k_3 + k_1 + k_4 + k_2 + 3 k_3 \} \lambda^p + \{k_1 (k_2 - 1) k_3 k_4 + 2 k_1 k_3 k_4 + k_1 (k_2 - 1) k_3 k_4 + (k_2 - 1) k_3 + k_1 k_3 + k_3 k_4 \} \lambda^p^8

where \( k_1 + k_2 + k_3 + k_4 = p^- 8 \)

Proposition 5.3.7

\[ \Phi(T_{6,4} \lambda) = \lambda p^2 \cdot (p-1) \lambda p^2 + \{k_1 k_2 + k_1 k_3 + k_1 k_4 + k_2 k_3 + k_2 k_4 + k_3 k_4 + 3 k_1 + 3 k_2 + 3 k_3 - 3 \} \lambda^p + \{k_1 k_2 k_3 + k_2 k_3 k_4 + k_1 k_3 k_4 + 4 k_3 k_4 + 4 k_2 + 4 k_3 + 4 k_4 + 1\} \lambda^p + \{k_1 k_2 k_3 k_4 + (k_4 - 1) k_2 k_3 + k_1 k_3 k_4 + (k_4 - 1) k_2 k_3 \}
\]

\[ \lambda^p^8 \text{ where } k_1 + k_2 + k_3 + k_4 = p^- 4 \]

Proposition 5.3.8

\[ \Phi(T_{6,5} \lambda) = \lambda p^2 \cdot (p-1) \lambda p^2 + \{k_1 k_2 + k_2 k_3 + k_1 k_4 + 4 k_2 + 4 k_3 + 5\} \lambda^p + \{k_1 k_2 k_3 + 3 k_1 k_2 + 3 k_1 k_3 + 3 k_1 k_4 + 2 k_2 + 3 k_3 \} \lambda^p + \{k_1 k_3 k_4 + k_1 k_2 k_3 + k_1 k_2 k_4 \}
\]

\[ + k_1 k_4 \lambda^p \text{ where } k_1 + k_2 + k_3 + k_4 = p^- 8 \]

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Proposition 5.3.10

\[ \Phi(T_{7,2}, \lambda) = \lambda^2 - (p-1)\lambda^{p^2} + \{ k_2 + k_1 k_4 + k_1 k_3 + k_2 k_4 + k_3 k_4 + 4 k_1 + 3 k_2 + 3 k_3 + 4 k_4 + 6 \} \lambda^{p^4} - \{ k_1 k_2 k_3 + k_1 k_4 k_4 + k_2 k_3 k_4 + k_1 k_3 k_4 + 3 k_1 k_9 + 2 k_1 k_3 + 3 k_1 k_4 + k_2 k_3 + 2 k_2 k_4 + 3 k_2 + 2 k_3 + 3 k_4 + 1 \} \lambda^{p^6} + (k_1 k_2 k_3 k_4 + k_1 k_2 k_3 + 2 k_1 k_2 k_4 + 2 k_2 k_3 k_4 + 3 k_1 + 2 k_2 + 2 k_3 + 3 k_4 + 1) \lambda^{p^8} \]

where \( k_1 + k_2 + k_3 + k_4 = p \)

Proposition 5.3.11

\[ \Phi(T_{8,1}, \lambda) = \lambda^2 - (p-1)\lambda^{p^2} + \{ k_1 k_2 + k_1 k_4 + k_1 k_3 + k_2 k_3 + k_3 k_4 + 5 k_1 + 4 k_2 + 4 k_3 + 5 k_4 + 1 \} \lambda^{p^4} - \{ k_1 k_2 k_3 + k_1 k_2 k_4 + k_2 k_3 k_4 + k_1 k_3 k_4 + 3 k_1 k_9 + 3 k_2 k_3 + 3 k_2 k_4 + 3 k_3 k_4 + 6 k_1 + 5 k_2 + 5 k_3 + 6 k_4 + 2 \} \lambda^{p^6} + (k_1 k_2 k_3 k_4 + k_2 k_3 k_4 + 2 k_1 k_2 k_4 + 2 k_1 k_3 k_4 + 2 k_1 k_4 + k_2 k_3 + 2 k_2 k_4 + k_1 k_2 + k_3 k_4) \lambda^{p^8} \]

where \( k_1 + k_2 + k_3 + k_4 = p \)
**Proposition 5.3.12:** Let $T_{5,1}$ be the tree as shown in Figure 5, then for any two integers $p$ and $k,$

$$E(T_{5,1}(1)) < E(T_{5,1}(2)) < E(T_{5,1}(3)) < \ldots < E(T_{5,1}(k)), \quad k = \left\lfloor \frac{p-6}{2} \right\rfloor$$

**Proof:** From the characteristic polynomial of $T_{5,1}$ (proposition 5.3.1) by putting $k_1 = k_4 = 1$ and $k_2 = k$ so that $k_3 = p-k-6,$ we have $a_4(T_{5,1}) = pk + 3p-k^2 - 6k-12$ and $a_6(T_{5,1}) = 2p-4k-12$ Both these integer valued functions have maximum value if $k = \left\lfloor \frac{p-6}{2} \right\rfloor,$ with these observations and by the inequalities (5.3) the proof follows.

**Proposition 5.3.13:** Let $T_{5,2}$ be the tree as shown in Figure 5, then for any two integers $p$ and $k,$

$$E(T_{5,2}(1)) < E(T_{5,2}(2)) < E(T_{5,2}(3)) < \ldots < E(T_{5,2}(k)), \quad k = \left\lfloor \frac{p-7}{2} \right\rfloor$$

**Proof:** From the characteristic polynomial of $T_{5,2}$ by putting $k_1 = k_4 = 1$ and $k_2 = k$ so that $k_3 = p-k-6$ we have.

$$a_4(T_{5,2}) = pk + 5p-k^2 - 8k-28 \quad \text{and} \quad a_6(T_{5,2}) = 3kp + 4p-21k-3k^2-25$$

Both these integer valued functions have maximum value if $k = \left\lfloor \frac{p-7}{2} \right\rfloor,$ with these observations and by the inequalities (5.3) the proof follows.

On similar arguments we can prove the following propositions.

**Proposition 5.3.14:** Let $T_{5,3}$ be the tree as shown in Figure 5, then for any two integers $p$ and $k,$

$$E(T_{5,3}(1)) < E(T_{5,3}(2)) < E(T_{5,3}(3)) < \ldots < E(T_{5,3}(k)), \quad k = \left\lfloor \frac{p-9}{2} \right\rfloor$$
Proposition 5.3.15: Let $T_{6,1}$ be the tree as shown in Figure 5.7, then for any two integers $p$ and $k$,
\[
E(T_{6,1}(1)) < E(T_{6,1}(2)) < E(T_{6,1}(3)) < \ldots < E(T_{6,1}(k)), \quad k = \frac{p - 7}{2}
\]

Proposition 5.3.16: Let $T_{6,2}$ be the tree as shown in Figure 5.8, then for any two integers $p$ and $k$,
\[
E(T_{6,2}(1)) < E(T_{6,2}(2)) < E(T_{6,2}(3)) < \ldots < E(T_{6,2}(k)), \quad k = \frac{p - 5}{2}
\]

Proposition 5.3.17: Let $T_{6,3}$ be the tree as shown in Figure 5.9, then for any two integers $p$ and $k$,
\[
E(T_{6,3}(1)) < E(T_{6,3}(2)) < E(T_{6,3}(3)) < \ldots < E(T_{6,3}(k)), \quad k = \frac{p - 8}{2}
\]

Proposition 5.3.18: Let $T_{6,4}$ be the tree as shown in Figure 5.10, then for any two integers $p$ and $k$,
\[
E(T_{6,4}(1)) < E(T_{6,4}(2)) < E(T_{6,4}(3)) < \ldots < E(T_{6,4}(k)), \quad k = \frac{p - 7}{2}
\]

Proposition 5.3.19: Let $T_{6,5}$ be the tree as shown in Figure 5.11, then for any two integers $p$ and $k$,
\[
E(T_{6,5}(1)) < E(T_{6,5}(2)) < E(T_{6,5}(3)) < \ldots < E(T_{6,5}(k)), \quad k = \frac{p - 8}{2}
\]

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Proposition 5.3.20: Let $T_{7.1}$ be the tree as shown in Figure 5.12, then for any two integers $p$ and $k$,
$$E(T_{7.1}(1)) < E(T_{7.1}(2)) < E(T_{7.1}(3)) \ldots \ldots \ldots < E(T_{7.1}(k))$$
$$k = \left\lfloor \frac{p-11}{2} \right\rfloor$$

Proposition 5.3.21: Let $T_{7.2}$ be the tree as shown in Figure 5.13, then for any two integers $p$ and $k$,
$$E(T_{7.2}(1)) < E(T_{7.2}(2)) < E(T_{7.2}(3)) \ldots \ldots \ldots < E(T_{7.2}(k))$$
$$k = \left\lfloor \frac{p-11}{2} \right\rfloor$$

Proposition 5.3.22: Let $T_{8.1}$ be the tree as shown in Figure 5.14, then for any two integers $p$ and $k$,
$$E(T_{8.1}(1)) < E(T_{8.1}(2)) < E(T_{8.1}(3)) \ldots \ldots \ldots < E(T_{8.1}(k))$$
$$k = \left\lfloor \frac{p-10}{2} \right\rfloor$$

Remark: From the characteristic polynomials of all trees with independence number 4 given by propositions 5.3.1 to 5.3.11, on comparing coefficients we see that tree $T_{8.1}$ of diameter 8 has the highest energy. The inequalities for energy of $T_{7.1}$ & $T_{7.2}$ can be written in the same way.