CHAPTER 4
EQUIENERGETIC GRAPHS

4.1 INTRODUCTION

The concept of graph energy was introduced by I. Gutman in 1978 [45]. Recently this concept started to attract considerable attention of mathematicians involved in the study of spectral graph theory. For recent mathematical work on the energy of graphs, one can refer [2, 67-70, 36, 49, 50, 52]. The chemical aspects are outlined in the book [82].

Let G be a graph on p vertices. The eigenvalues of adjacency matrix of G denoted by $\lambda_1, \lambda_2, \ldots, \lambda_p$ are eigen values of G and they form the spectrum of G [22].

The energy of a graph is then defined as $\sum_{i=1}^{p} |\lambda_i|$

If for two graphs $G_1$ and $G_2$, $E(G_1) = E(G_2)$ then $G_1$ and $G_2$ are said to be equienergetic. Recall that two nonisomorphic graphs are co spectral if they have the same spectrum. The simplest example of connected co spectral graphs is as shown in Figure 4.1

![Figure 4.1](image)
They share a common spectrum \( \begin{pmatrix} 2.709 & .194 & 1 & -1 & -1.903 \\ 1 & 1 & 1 & 2 & 1 \end{pmatrix} \)

The elements in first row stand for the eigen values and in the second, for the corresponding multiplicity. Another pair of co spectral non-connected pair is as in Figure 4.2,

Figure 4.2

They share the spectrum \( \begin{pmatrix} 0 & 2 & -2 \\ 3 & 1 & 1 \end{pmatrix} \)

For obvious reasons co spectral graphs are equienergetic. The less trivial cases are of pairs of non-co spectral equienergetic graphs on same number of vertices. They can also be found easily because the union of any graph and an arbitrary number of isolated vertices has the same energy as that of the graph itself.

Figure 4.3
We are not interested in such examples, but looking for a pair of connected non co spectral equienergetic graphs on the same number of vertices. Since on different vertices one can find graphs such as,

\[ E(G) = E(G) \cup K_k \]

Here although \( E(P_4) = E(K_{1,5}) \),

Figure 4.4

\[ E(C_3) = E(C_4) = 4 \] with all integer eigen values.

Or

Figure 4.5
\[ \text{Sp}(P_4) = \begin{pmatrix} \frac{\sqrt{3} + \sqrt{5}}{2} & \frac{-3 + \sqrt{5}}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad \text{where as } \text{Sp}(K_{1,5}) = \begin{pmatrix} \sqrt{5} & 0 & -\sqrt{5} \\ 1 & 3 & 1 \end{pmatrix} \]

The smallest pair of non co spectral connected equienergetic graphs on same number of vertices are, the hexagon (\(C_6\)) and trigonal prism (\(C_3 \times K_2\)) as shown in Figure 4.6 with energy 8 and respective eigen values \(\pm 2, \pm 1, \pm 1 & 3, 0, 0, -2, -2\).

With this background we now deal with the problem of finding connected non-co spectral equienergetic graphs.

### 4.2 LINE GRAPHS OF REGULAR GRAPHS

Let \(G\) be a graph and \(L^1(G) = L(G)\) it's line graph [59]. Let further \(L^2(G) = L(L^1(G))\), \(L^3(G) = L(L^2(G))\), ............be the iterated line graphs of \(G\).

It's well known that the line graph of regular graph is also regular. More precisely if \(G\) is 'r' regular on 'p' vertices, then \(L(G)\) is '2r-2 = r_1'.

Figure 4.6
regular on \( pr/2 \) vertices, \( L \) is \( r_2 = 2r_1 - 2 = 4r - 6 \) regular on \( pr(2r - 2)/2 \) vertices and so on. Consequently all iterated line graphs \( L_k(G) \) of a regular graph \( G \), are regular graphs.

4.3 RESULTS

Theorem 4.3.1: Let \( G_1 \) and \( G_2 \) be the regular graphs on \( 'p' \) vertices, each with regularity \( r \geq 3 \) Then \( L^2(G_1) \) and \( L^2(G_2) \) are equienergetic.

Proof: We show that the energy of the second iterated line graph of a regular graph of degree \( r \geq 3 \) depends only on the number of vertices and on the regularity \( 'r' \).

Let \( G \) be a regular graph on \( 'p' \) vertices of degree \( r \geq 3 \). Let it’s eigen values be \( 
\lambda_i, i = 1, 2, \ldots, p \).

Then according to a long known result by Sachs [101], the eigen values of \( L(G) \) are

\[
\lambda_i + r - 2 \quad i = 1, 2, \ldots, p
\]

\[
-2 \quad \frac{p(r - 2)}{2} \text{ times}
\]

(4.3.1)

Bearing in mind the structure of \( L^2(G) \) and by a two fold application of equation (4.3.1) we conclude that eigen values of \( L^2(G) \) are,
If $d_{\text{max}}$ is the greatest vertex degree of a graph then all its eigen values belong to the interval $[-d_{\text{max}}, d_{\text{max}}]$ [5].

In particular the eigen values of a regular graph of degree 'r' satisfy,

$$-r \leq \lambda_i \leq r \text{ for all } i = 1, 2, \ldots, p.$$ 

If $r \geq 3$ then $3r-6 \geq 0$ and because $\lambda_i \geq 0$ we get $\lambda_i + 3r-6 \geq 0$

If $r \geq 3$ then $2r-6 \geq 0$. We thus see that all negative valued eigen values of $L^2(G)$ are $-2$ and that they occur in the spectrum of $L^2(G)$ $\frac{pr(r-2)}{2}$ times.

Bearing this in mind we see that second iterated line graphs of all regular graphs having the same number of vertices as well as same degree are mutually equienergetic.

This is tantamount to the theorem 4.3.1.

**Corollary 4.3.2:** Let $G_1$ and $G_2$ be two regular graphs, both on 'p' vertices degree $r \geq 3$ then for any $k \geq 2$, $L^k(G_1)$ and $L^k(G_2)$ are equienergetic.
Proof: The statement of the corollary 4.3.2 for \( k = 2 \) coincides with theorem 4.3.1, therefore we can assume \( k \geq 3 \).

The fact that \( L^{k-2}(G_1) \) and \( L^{k-2}(G_2) \) have equal number of vertices, follow from the repeated application of equations (4.3.1) and (4.3.2). Because \( L^{k-2}(G_1) \) and \( L^{k-2}(G_2) \) are the regular graphs of the same degree, possessing equal number of vertices by Theorem 4.3.1.

Therefore \( L^k(G_1) = L^2(L^{k-2}(G_1)) \) and \( L^k(G_2) = L^2(L^{k-2}(G_2)) \) are equienergetic.

Corollary 4.3.3: Let \( G_1 \) and \( G_2 \) be two connected and non-co spectral regular graphs, both on \( 'p' \) vertices of degree \( r \geq 3 \) respectively. Then for any \( k \geq 2 \) both \( L^k(G_1) \) and \( L^k(G_2) \) are connected non co spectral equienergetic. Furthermore both possess same number of vertices and edges.

Proof: The line graph of a graph without isolated vertices is connected if and only if the graph itself is connected [59]. Hence if \( G \) is connected then \( L^k(G) \) is connected \( k \geq 1 \). According to (4.3.1) if \( G_1 \) and \( G_2 \) are not co spectral, then \( L^1(G_1) \) and \( L^1(G_2) \) are also not co spectral. Thus \( L^k(G_1) \) and \( L^k(G_2) \) are not co spectral for any \( k \geq 1 \).

From the proof of the corollary 4.3.2 it's known that \( L^k(G_1) \) and \( L^k(G_2) \) have the same number of vertices from the fact that the number of edges in \( L^k(G) \) is
equal to number of vertices in $L^{k-1}(G)$ it follows that $L^k(G_1)$ and $L^k(G_2)$ have equal number of edges.

**Corollary 4.3.4:** From the eigen values of $L^2(G)$ its easy to calculate it’s energy which is,

$$
E(L^2(G)) = \sum_{i=1}^{\nu} (\lambda_i + 3r - 6) + \frac{1}{2} p (r - 2) \times (2r - 6) + \frac{1}{2} pr (r - 2) \times 2
$$

$$
= \sum_{i=1}^{\nu} \lambda_i + 2pr(r-2) \text{ from theorem 4.3.1}
$$

$$
= 2pr(r-2) \text{ as } \sum_{i=1}^{\nu} \lambda_i = 0 \text{ for any graph.}
$$

We also find a family of graphs, which are equienergetic to second line graph of regular graphs.

**Theorem 4.3.5:** $E[L(K_{n_1, n_2, \ldots, n_k})] = E[L^2(G_i)]$ where $K_{n_1, n_2, \ldots, n_k}$ is complete multipartite graph with $n_1=n_2=\ldots=n_k=p/im/n$ and $G_i$ is a regular graph of same order.

**Proof:** The characteristic polynomial of $K_{n_1, n_2, \ldots, n_k}$ is given by,

$$
\Phi(K_{n_1, n_2, \ldots, n_k} : \lambda) = \lambda^{p-k} (\lambda + \frac{p}{k} - p)(\lambda + \frac{p}{k})^{k-1}
$$
\(K_{n_1,n_2,...,n_k}\) is a regular graph of degree \((p - \frac{p}{k})\). Also the line graph of a regular graph being regular, we have the characteristic polynomial of line graph of a regular graph in terms of characteristic polynomial of graph as,

\[
\Phi(L(G): \lambda) = (\lambda + 2)^{\frac{r}{2}} \Phi(\lambda + 2 - r : G) \text{ where 'r' is regularity of } G.
\]

Hence the line graph of \(K_{n_1,n_2,...,n_k}\) denoted by \(L(K_{n_1,n_2,...,n_k})\) has eigen values

- \(2\left(p - \frac{p}{k} - 2\right)\) once
- \(p - \frac{p}{k} - 2\) \(k-1\) times
- \(-2 \frac{p(p - \frac{p}{k})}{2}\) \(p\) times

The energy of \(L(K_{n_1,n_2,...,n_k})\) is thus,

\[
E[L(K_{n_1,n_2,...,n_k})] = 2(p - \frac{p}{k} - 2)(k-1) (p - \frac{2p}{k} - 2) + (p - k)(p - \frac{p}{k} - 2) + 2 \left(\frac{p(p - \frac{p}{k})}{2} - p\right)
\]

\[
= 2p^2 - \frac{2p^2}{k} - 2p + p(k-1) - \frac{2p(k-1)}{k} - 2(k-1)
\]

\[
= 2p^2 - 4p - \frac{2p^2}{k} = 2p\left(p - \frac{p}{k} - 2\right)
\]
\[ = 2p(r - 2) \quad r = \frac{p}{k} \quad \text{(the regularity of } K_{n_1, n_2, \ldots, n_k}) \]

\[ = E(L^2(G)) \quad \text{where } G \text{ is regular of degree 'r'.} \]

CONCLUSION:

Theorem 4.3.1 in particular, its Corollary 4.3.2 provide a general method for constructing families of mutually non-co spectral equienergetic graphs; each member of which is connected and has the same number of vertices. One simply has to find a pertinent collection of mutually non-co spectral regular graphs (of degree >2) and to construct their \(k\)th iterated line graphs for any \(k \geq 2\). Finding the required collection of regular graphs is an easy task because the number of such graphs is usually large and because co spectral mates among them occur only exceptionally [8,20,24]. Within the proof of theorem 4.3.1 we obtained the simple expression for energy of second iterated line graph of a regular graph.