CHAPTER - 2

ON LINE SPLITTING
GRAPHS

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ABSTRACT

In this chapter, we obtain some properties of line splitting graphs. We characterize those graphs whose line splitting graphs are connected and unicyclic. We find the girth, connectivity, edge connectivity, chromatic number and covering invariants (i.e. $\alpha_0, \beta_0, \alpha_1, \beta_1, n_0, \gamma$ and $d$) for line splitting graphs. Further, we determine graphs whose line splitting graphs are 2-trees and also characterize graphs whose line splitting graphs are $k$-trees, $k \geq 3$. In addition, we establish necessary and sufficient conditions for line splitting graphs to be eulerian and hamiltonian.
2.1. INTRODUCTION

We require the following definitions.

If $x = uv$ is an edge of a graph $G$ and $w$ is not a vertex of $G$, then $x$ is subdivided when it is replaced by the edges $uw$ and $wv$. If every edge of $G$ is subdivided, the resulting graph is the subdivision graph of $G$.

A vertex-cut in a graph $G$ is a set $S$ of vertices of $G$ such that $G - S$ is disconnected. Similarly, an edge-cut in a graph $G$ is a set $X$ of edges of $G$ such that $G - X$ is disconnected.

A set $S$ of edges of a graph $G$ is a dominating set of edges if every edge of $G$ either belongs to $S$ or is adjacent to an edge of $S$. If $\langle S \rangle$ is circuit $C$, then $C$ is called a dominating circuit of $G$. Equivalently, a circuit $C$ in a graph $G$ is a dominating circuit if every edge of $G$ is incident with a vertex of $C$.

A graph is called triangulated if every cycle of length strictly greater than 3 possesses a chord, i.e., an edge joining two non-consecutive vertices of the cycle. Equivalently, $G$ does not contain an induced subgraph isomorphic to $C_n$, $n > 3$.

The open neighborhood $N(u)$ of a vertex $u$ in $V(G)$ is the set of vertices adjacent to $u$. $N(u) = \{v \mid uv \in E(G)\}$.
For each vertex $u_t$ of $G$, a new vertex $u'_t$ is taken and the resulting set of vertices is denoted by $V_t(G)$.

The splitting graph $S(G)$ of a graph $G$ is defined as the graph having vertex set $V(G) \cup V_t(G)$ with two vertices adjacent if they correspond to adjacent vertices of $G$ or one corresponds to a vertex $u'_t$ of $V_t(G)$ and the other to a vertex $w_j$ of $G$ where $w_j$ is in $N(u_t)$. This concept was introduced by Sampathkumar and Walikar in [10].

The open neighborhood $N(e_i)$ of an edge $e_i$ in $E(G)$ is the set of edges adjacent to $e_i$. $N(e_i) = \{ e_j / e_j, e_j are adjacent in G \}$. For each edge $e_i$ of $G$, a new vertex $e'_i$ is taken and the resulting set of vertices is denoted by $E_t(G)$.

The line splitting graph $L_s(G)$ of a graph $G$ is defined as the graph having vertex set $E(G) \cup E_t(G)$ with two vertices adjacent if they correspond to adjacent edges of $G$ or one corresponds to an element $e'_i$ of $E_t(G)$ and the other to an element $e_j$ of $E(G)$ where $e_j$ is in $N(e_i)$. This concept was introduced by Kulli and Biradar in [6].

In Figure 2.1, a graph $G$, $S(G)$ and $L_s(G)$ are shown.

The graph $G$ is an induced subgraph of $S(G)$. The line graph $L(G)$ is an induced subgraph of $L_s(G)$. 

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Figure 2.1
The following observations are of use.

REMARK 2.1. If $G = L_s(H)$ for some graph $H$, then $G = S(L(H))$.

REMARK 2.2. $G$ is a splitting graph if and only if its vertex set $V$ can be partitioned into two subsets $V_1$ and $V_2$ and there exists a bijective mapping from $V_1$ onto $V_2$ such that $N(v') = N(v) \cap V_1$, for all $v \in V_1$. Since any line splitting graph is a splitting graph, a similar result holds for line splitting graphs.

REMARK 2.3. Let $v \in V(G)$. If the neighbors of $v$ in $G$ are $v_1, v_2, \ldots, v_k$, then the neighbors of $v$ in $S(G)$ are $v_1, v_2, \ldots, v_k, v_1', v_2', \ldots, v_k'$. Hence $\deg_{S(G)} v = 2 \deg_{G} v$. Also $\deg_{S(G)} v' = \deg_{G} v$. Similarly, if $e \in E(G)$, then $\deg_{L_s(G)} e = 2 \deg_{L(G)} e$ and $\deg_{L_s(G)} e' = \deg_{L(G)} e$.

REMARK 2.4 [7]. Every 2-tree is planar.

The following will be useful in the proof of our results.

THEOREM 2.A [9]. A splitting graph $S(G)$ is an unicyclic graph if and only if $G = P_3$.

THEOREM 2.B [2]. If a graph $G$ is $m$-edge connected, $m \geq 2$, then its line graph $L(G)$ is $m$-connected.

THEOREM 2.C [5]. A graph $G$ is $n$-connected if and only if every pairs of vertices are joined by at least $n$ vertex disjoint paths.
THEOREM 2.D [1]. Let $G$ be a graph without isolated vertices. Then $L(G)$ is hamiltonian if and only if $G = K_{1,t}$ for some $t \geq 3$ or $G$ contains a dominating circuit.

THEOREM 2.E [6]. A connected graph $G$ is a cycle if and only if the graphs $L_0(G)$ and $S(G)$ are isomorphic.

THEOREM 2.F [5]. If $G$ is a $(p, q)$ graph whose vertices have degree $d_i$, then $L(G)$ has $q$ vertices and $q_L$ edges, where $q_L = -q + \frac{1}{2} \sum d_i^2$.

THEOREM 2.G [4]. If $L(G)$ is the line graph of a nontrivial connected graph $G$, then

(i) $\alpha_t(L(G)) = \left\lfloor \frac{q}{2} \right\rfloor$

(ii) $\beta_t(L(G)) = \left\lfloor \frac{q}{2} \right\rfloor$

(iii) $\alpha_0(L(G)) = q - \beta_t(G)$

(iv) $\beta_0(L(G)) = \beta_t(G)$.

THEOREM 2.H [6]. The line splitting graph $L_s(G)$ of a graph $G$ is planar if and only if $G$ is planar and (i) or (ii) holds:

(i) $G$ is either $K_{1,4}$ or $C_{2n}$, $n \geq 2$.

(ii) $\Delta(G) \leq 3$ and $G$ has no subgraph homeomorphic from the subdivision graph of $K_{1,3}$ and also every block of $G$ is either a $K_2$ or a triangle such that each triangle has at most one cut-vertex.
THEOREM 2.1 [8]. Let G be a graph of order p and let k < p. Then the following assertions are equivalent:

(i) G is a k-tree.

(ii) G is k-connected, triangulated and $K_{k+2}$-free.

(iii) G is k-connected, triangulated of size $kp - \binom{k+1}{2}$

2.2. BASIC PROPERTIES OF LINE SPLITTING GRAPHS

2.2.1. We start with a few preliminary results.

THEOREM 2.1. Let G be a nontrivial connected (p,q) graph.

$L_{s}(G) = L(G) \cup K_1$ if and only if $G = K_2.$

PROOF. Suppose $G = K_2.$ By definition of $L_{s}(G)$, it is totally disconnected graph of order 2. Also by the definition of $L(G)$, it has exactly one vertex. Clearly $L_{s}(G) = L(G) \cup K_1.$

Conversely, suppose $L_{s}(G) = L(G) \cup K_1.$ Assume G is not $K_2.$ Then G has at least two edges. By definition of $L(G)$ and $L_{s}(G)$, $L(G) \cup K_1$ has $q+1$ vertices whereas $L_{s}(G)$ has $2q$ vertices, where $q > 1.$ Thus $L(G) \cup K_1$ has less number of vertices than $L_{s}(G).$ Clearly $L_{s}(G) \neq L(G) \cup K_1,$ a contradiction. Hence $G = K_2.$ This completes the proof.

In the following Theorem, we establish a relation between the edge degree of an edge in G and degree of the corresponding vertex in $L_{s}(G).$
**THEOREM 2.2.** For any edge in a graph $G$, with edge degree $n$, the degree of the corresponding vertex in $L_n(G)$ is $2(n-2)$.

**PROOF.** If an edge $e$ in $G$ is of edge degree $n$. Then $e$ is adjacent to $(n-2)$ edges, say, $e_1, e_2, \ldots, e_{n-2}$. Since the edge itself contributes one degree to each edge adjacent to it in $G$, $(n-2)$ degrees are contributed for the corresponding vertex in $L_n(G)$. Also, by definition of $L_n(G)$, the newly introduced vertices $e'_1, e'_2, \ldots, e'_{n-2}$ corresponding to the edges $e_1, e_2, \ldots, e_{n-2}$ of $G$ are adjacent to the vertex corresponding to the edge $e$ and hence contribute $(n-2)$ more degrees to the corresponding vertex in $L_n(G)$. Hence the degree of the vertex in $L_n(G)$ corresponding to an edge in $G$ with edge degree $n$ is $2(n-2)$. This completes the proof.

In the following theorem, we characterize graphs whose line splitting graphs are connected.

**THEOREM 2.3.** Let $G$ be a $(p, q)$ graph. Then $L_n(G)$ is connected if and only if $G$ is a connected graph with $p \geq 3$.

**PROOF.** Let $G$ be a connected graph with $p \geq 3$ vertices. Let $V(L_n(G)) = V_1 \cup V_2$ where $V_1 = L(G)$ and $V_2$ is the set of all newly introduced vertices, such that $v_1 \to v_2$ is a bijective map from $V_1$ onto $V_2$ satisfying $N(v_2) = N(v_1) \cap V_1$ for all $v_1 \in V_1$. Let $a, b \in V(L_n(G))$. We consider the following cases.
Case 1. a, b ∈ V₁. Since G is a connected graph with p ≥ 3, L(G) is a nontrivial connected graph. Since L(G) is an induced subgraph of Lₐ(G), there exists an a – b path in Lₐ(G).

Case 2. a ∈ V₁ and b ∈ V₂. Let v ∈ V₁ be such that N(b) = N(v) ∩ V₁. Choose w ∈ N(b). Since a and w ∈ V₁, as in Case 1, a and w are joined by a path in Lₐ(G). Hence a and b are connected by a path in Lₐ(G).

Case 3. a, b ∈ V₂. As in Case 2, there exist w₁ and w₂ in V₁ such that w₁ ∈ N(a) and w₂ ∈ N(b). Consequently, w₁a, w₂b ∈ E(Lₐ(G)). Also w₁ and w₂ are joined by a path in Lₐ(G). Hence a and b are connected by a path in Lₐ(G).

In all the cases, a and b are connected by a path in Lₐ(G). Thus Lₐ(G) is connected.

Conversely, if G is disconnected or G = K₂, then obviously Lₐ(G) is disconnected. This completes the proof.

We now establish characterization of graphs whose line splitting graphs are unicyclic.

**THEOREM 2.4.** The line splitting graph Lₐ(G) of a graph G is an unicyclic graph if and only if G = P₄.

**PROOF.** Suppose Lₐ(G) is unicyclic. Then S(L(G)) is unicyclic. By Theorem 2.A, L(G) is P₃ and hence G = P₄.
Conversely, if $G = P_4$, then $L_s(G)$ is the unicyclic graph given in Figure 2.2.

In the following theorem, we obtain the girth of line splitting graphs.

**THEOREM 2.5.** For any connected graph $G$ of order $\geq 4$,

$$g(L_s(G)) = \begin{cases} 3 & \text{if } K_3 \text{ or } K_{1,3} \subseteq G \\ 4 & \text{otherwise} \end{cases}$$

**PROOF.** Let $G$ be a connected graph of order $\geq 4$. If either $K_3$ or $K_{1,3}$ is a subgraph of $G$, then $L_s(G)$ contains a triangle and hence its girth is 3. Otherwise $G$ contains $P_4$ as an induced subgraph and the graph given in Figure 2.2(b) is a subgraph of $L_s(G)$. Hence, $g(L_s(G)) = 4$. This completes the proof.

### 2.2.2. LINE SPLITTING GRAPHS AND $k$-TREES

In the following theorem, we determine all graphs whose line splitting graphs are 2-trees.

**THEOREM 2.6.** There are only two graphs whose line splitting graphs are 2-trees. These graphs are $K_{1,3}$ and $C_3$.

**PROOF.** Suppose the line splitting graph $L_s(G)$ of a graph $G$ is a 2-tree. Clearly $G$ is connected. By Remark 2.4, $L_s(G)$ is planar and hence it follows from Theorem 2.1, that $G$ is planar and is either $K_{1,4}$ or $C_{2n}$, $n \geq 2$ or $\Delta(G) \leq 3$ and has no subgraph homeomorphic from the subdivision...
Figure 2.2

G:

(a)

L_5(G):

(b)
graph of $K_{1,3}$ and also every block of $G$ is either a $K_2$ or a triangle such that each triangle has at most one cut-vertex.

We consider the following cases depending on the magnitude of $\Delta(G)$.

**Case 1.** $\Delta(G) = 1$. Then $G = K_2$. Clearly $L_s(G)$ is disconnected, a contradiction.

**Case 2.** $\Delta(G) = 2$. Then $G$ is either a path or a cycle. Let $G$ be a graph of order $p$ and size $q$. We have the following subcases in this case.

**Subcase 2.1.** $G$ is a path $P_p$, $p \geq 3$.

For $p = 3$, $G = K_{1,2}$. But $L_s(K_{1,2})$ is not triangulated and hence by Theorem 2.1, $L_s(G)$ is not a 2-tree, a contradiction.

For $p \geq 4$, $L_s(G)$ contains a cycle of length $n = 4$ without chords and therefore, $L_s(G)$ is not triangulated, a contradiction.

**Subcase 2.2.** $G$ is a cycle $C_p$, $p \geq 3$. Then $L_s(G)$ has $2q$ vertices and $3q$ edges. Since a 2-tree with $2q$ vertices contains $4q - 3$ edges, it follows that $3q < 4q - 3$ for all $q > 3$ and $3q = 4q - 3$ for $q = 3$. Hence $G = C_3$.

**Case 3.** $\Delta(G) = 3$. We consider the following subcases.

**Subcase 3.1.** $G$ is not a tree. Then $G$ has at least one cycle. So $G$ is a cycle together with a path of length $\geq 1$, adjoined at some vertex. Then $L_s(G)$
contains a cycle of length \( n = 4 \) without chords and therefore, \( L_4(G) \) is not triangulated, a contradiction.

**Subcase 3.2.** \( G \) is a tree other than \( K_{1,3} \). Then \( L_4(G) \) contains a cut-vertex and by Theorem 2.1, \( L_4(G) \) is not a 2-tree, a contradiction. Hence \( G = K_{1,3} \).

**Case 4.** \( \Delta(G) > 3 \). Then \( K_{1,4} \) is a subgraph of \( G \). One can see that \( L_4(K_{1,4}) \) contains a subgraph isomorphic to \( K_4 \) and therefore \( L_4(G) \) is not a 2-tree, a contradiction.

From all the above cases, it follows that \( G = K_{1,3} \) or \( C_3 \). This completes the proof.

In the next theorem, we characterize graphs whose line splitting graphs are \( k \)-trees, \( k \geq 3 \).

**THEOREM 2.7.** A line splitting graph \( L_a(G) \) of order \( 2(k + 1) \), \( k \geq 3 \), is a \( k \)-tree if and only if \( G = K_{1,k+1} \).

**PROOF.** Let \( G \) be a \((p,q)\) graph. It follows from Theorem 2.F, that \( L(G) \) is \((q, -q + ½ \sum d_i^2)\) graph, where \( d_i \) is degree of each vertex \( v_i \in V(G) \).

Suppose \( L_a(G) \) is a \( k \)-tree, \( k \geq 3 \) of order \( 2(k + 1) \). Then clearly \( p_L = q = k + 1 \).

Also, since \( L_a(G) = S(L(G)) \), \( L_a(G) \) contains \( 3(-q + ½ \sum d_i^2) \) edges. Since \( L_a(G) \) is a \( k \)-tree,
This implies that

\[ 3(-q + \sum d_i^2) = \frac{3k(k+1)}{2}. \]

So, \( q_L = \frac{k(k+1)}{2} \).

Therefore, \( L(G) = K_{k+1}, \ k \geq 3 \) and hence \( G = K_{1, k+1}, \ k \geq 3 \).

Conversely, suppose \( G = K_{1, k+1}, \ k \geq 3 \). Then \( L(G) = K_{k+1} \). It is easy to see that \( L_s(G) = S(L(G)) \) is a \( k \)-tree of order \( 2(k + 1), k \geq 3 \). This completes the proof.

2.3. TRAVERSABILITY OF LINE SPLITTING GRAPHS

Next, we obtain a criterion for line splitting graphs to be eulerian.

THEOREM 2.8. If \( G \) is eulerian then \( L_s(G) \) is eulerian.

PROOF. Suppose \( G \) is eulerian. Then the vertex set of \( L_s(G) \) can be partitioned into two sets \( V_1 \) and \( V_2 \) such that \( V(L_s(G)) = V_1 \cup V_2 \) and \( e \rightarrow e' \) is a bijective mapping from \( V_1 \) onto \( V_2 \) satisfying the condition \( N(e') = N(e) \cap V_1 \). Let \( V_1 = \{e_1, e_2, \ldots, e_q, q \geq 1\} \) and \( V_2 = \{e'_1, e'_2, \ldots, e'_q\} \) be the set of newly introduced vertices under the bijective map \( e \rightarrow e' \).
From Theorem 2.2, deg\(_{L_s(G)}\) \(e_i = 2(n - 2)\), where \(n\) is the edge degree of \(e_i\) in \(G\). Also, since \(G\) is eulerian, every vertex of \(G\) is of even degree. It implies that each edge in \(G\) is adjacent to even number of edges. Thus \(deg_{L_s(G)} e_i'\) is even. Hence \(L_s(G)\) is eulerian.

The converse is not true. For example, for \(G = K_4\), \(L_s(K_4)\) is eulerian, whereas \(K_4\) is not eulerian.

We now characterize graphs whose line splitting graphs are eulerian.

**THEOREM 2.9.** The line splitting graph \(L_s(G)\) of a connected graph \(G\) of order \(p \geq 3\) is eulerian if and only if either of the following conditions holds.

(a) Every vertex of \(G\) is of even degree

(b) Every vertex of \(G\) is of odd degree.

**PROOF.** Suppose \(L_s(G)\) is eulerian. Then every vertex of \(L_s(G)\) is of even degree. Let \(x \in V(L_s(G))\). We consider the following cases.

**Case 1.** If \(x\) corresponds to an edge \(e = uv\) of \(G\). Then by Remark 2.3, \(deg_{L_s(G)} x = 2 deg_{L(G)} e\), which implies that degree of \(e\) in \(L(G)\) is even.

This is possible only when either of (a) or (b) holds.
Case 2. If x corresponds to a newly introduced vertex e' for an edge e = uv of G. Again by Remark 2.3, \( \deg_{L_s(G)} x = \deg_{L(G)} e \), which implies that degree of e in L(G) is even. Hence by Case 1 of this Theorem, it follows that either (a) or (b) holds.

Conversely, suppose either of conditions (a) or (b) holds. Suppose (a) holds. Then G is eulerian. By Theorem 2.8, \( L_s(G) \) is eulerian. Suppose (b) holds. Then edge degree of G is even. Hence, every vertex of L(G) is of even degree. By Remark 2.3, \( L_s(G) \) is eulerian. This completes the proof.

In the following theorem, we obtain necessary and sufficient conditions for line splitting graphs to be hamiltonian.

**THEOREM 2.10.** Let G be a nontrivial connected \((p,q)\) graph. Then \( L_s(G) \) is hamiltonian if and only if either of the following holds.

1) \( G = K_{1,2n+1}, n \geq 1 \) or

2) \( G \) contains a dominating circuit and \( q = 2m + 1, m \geq 1 \).

**PROOF.** If \( G = K_{1,2n+1}, n \geq 1 \), then \( L(G) \) is hamiltonian since \( L(G) = K_{2n+1} \). If \( G \) has a dominating circuit, then by Theorem 2.8, \( L(G) \) is hamiltonian. Thus in both cases \( L(G) \) is a hamiltonian graph with odd number of vertices. Let \( e_1, e_2, e_3, \ldots, e_{2n-1}, e_{2n}, e_{2n+1} \) be a hamiltonian cycle in \( L(G) \). Let \( \{ e'_1, e'_2, e'_3, \ldots, e'_{2n}, e'_{2n+1} \} \) be the set of newly
introduced vertices in $L_3(G)$ for each edge of $G$. Then $(e_1, e'_2, e_3, e'_4, e_5, e'_6, e_7, e'_8, \ldots, e_{2n-1}, e'_{2n}, e_{2n+1}, e'_1, e_2, e'_3, e_4, e'_5, e_6, e'_7, e_8, \ldots, e'_{2n-1}, e_{2n}, e'_{2n+1}, e_1)$ is a hamiltonian cycle in $L_3(G)$. Hence $L_3(G)$ is hamiltonian.

Conversely, suppose $L_3(G)$ is hamiltonian. Consider the hamiltonian cycle $Z$. Let $Z$ be a hamiltonian cycle in $L_3(G)$. Let $e_i, i = 1, \ldots, q$ and $e'_i, i = 1, \ldots, q$ be the vertices of $L_3(G)$, where $e_i$'s are the vertices of $L(G)$ and $e'_i$'s are the newly introduced vertices of $L_3(G)$ for each $e_i$'s. Since $\{e'_i, i = 1, \ldots, q\}$ is an independent set, any hamiltonian cycle of $L_3(G)$ contains $e'_i$'s as alternate vertices. Now let $Z = (e_1, e'_2, e_3, e'_4, e_5, e'_6, e_7, e'_8, \ldots, e'_{q-1}, e_q, e'_1, e_2, e'_3, e_4, e'_5, e_6, e'_7, e_8, \ldots, e_{q-1}, e_q, e_1)$. A hamiltonian cycle of $L(G)$ can be obtained from $Z$ by deleting elements in $Z$ which correspond to the newly introduced vertices of $L_3(G)$, i.e., $Z_1 : (e_1, e_2, e_3, e_4, \ldots, e_{q-1}, e_q, e_1)$ is a hamiltonian cycle of $L(G)$. By Theorem 2.D, it follows that $G = K_{1,q}$, for some $q \geq 3$ or $G$ contains a dominating circuit.

We now show that $q$ is odd. Suppose $q$ is even. Then $L(G)$ has a hamiltonian cycle of even length, i.e., $Z_1 : (e_1, e_2, e_3, e_4, \ldots, e_{2m-1}, e_{2m}, e_1)$. By inserting $e'_i$'s alternately into this cycle we get, a cycle $(e_1, e'_2, e_3, e'_4, e_5, e'_6, \ldots, e'_{2m-1}, e'_{2m}, e_1)$ which is not a hamiltonian cycle of $L_3(G)$. Hence $L_3(G)$ is not hamiltonian. Thus $q = 2m + 1, m \geq 1$. This completes the proof.
2.4. ON CONNECTIVITY OF LINE SPLITTING GRAPHS

In the following, we determine the vertex connectivity for a line splitting graphs.

**Theorem 2.11.** For any graph $G$, $\kappa(L_s(G)) = \min \{2\kappa(L(G)), \delta_e(G) - 2\}$.

**Proof.** By Whitney’s result, $\kappa(L_s(G)) \leq \lambda(L_s(G)) \leq \delta(L_s(G))$. Also, $\kappa(L(G)) \leq \lambda(L(G)) \leq \delta(L(G))$. Since $L(G)$ is an induced subgraph of $L_s(G)$, $\kappa(L_s(G)) \geq \kappa(L(G))$. We have the following cases.

**Case 1.** If $\kappa(L(G)) = 0$, then obviously $\kappa(L_s(G)) = 0$.

**Case 2.** If $\kappa(L(G)) = 1$, then $L(G) = K_2$ or it is connected with a cut-vertex $e_i$.

We consider the following subcases.

**Subcase 2.1.** $L(G) = K_2$, then $L_s(G) = P_4$. Consequently, $\kappa(L_s(G)) = \delta(L(G)) = 1$.

**Subcase 2.2.** $L(G)$ is connected with a cut-vertex $e_i$. Let $e_j$ be a pendant vertex of $L(G)$ which is adjacent to $e_i$. Then $e'_j$ is a pendant vertex of $L_s(G)$ and $e_i$ is also a cut-vertex of $L_s(G)$. Hence $\kappa(L_s(G)) = \delta(L(G))$. If $\delta(L(G)) \geq 2$, then the removal of a cut-vertex $e_i$ of $L(G)$ and its corresponding vertex $e'_i$ from $L_s(G)$ results in a disconnected graph. Hence $\kappa(L_s(G)) = 2\kappa(L(G))$. 

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Now suppose $\kappa(L(G)) = n$. Then $L(G)$ has a minimum vertex-cut 
$\{e_i : 1 \leq i \leq n\}$ whose removal from $L(G)$ results in a disconnected graph.

There are two types of vertex-cuts in $L_s(G)$ depending on the structure of $L(G)$. Among these, one vertex-cut contains exactly $2n$ vertices, $e_i$'s and $e'_i$'s of $L_s(G)$ whose removal increases the components of $L_s(G)$ and the other is $\delta(L(G))$–vertex–cut. Thus we have

$$\kappa(L_s(G)) = \begin{cases} 
2n, & \text{if } n \leq \delta(L(G)) = \frac{\delta_e(G) - 2}{2} \\
\delta(L(G)) = \delta_e(G) - 2, & \text{otherwise.}
\end{cases}$$

Hence,

$$\kappa(L_s(G)) = \min \{ 2 \kappa(L(G)), \delta(L(G)) \}$$

$$= \min \{ 2 \kappa(L(G)), \delta_e(G) - 2 \}.$$

This completes the proof.

In the next theorem, we determine the edge connectivity for a line splitting graph.

**THEOREM 2.12.** For any graph $G$, $\lambda(L_s(G)) = \min \{3\lambda(L(G)), \delta_e(G) - 2\}$.

**PROOF.** Since $\delta(L_s(G)) = \delta(L(G))$, by Whitney’s result $\kappa(L_s(G)) \leq \lambda(L_s(G)) \leq \delta(L(G))$. Since $L(G)$ is an induced subgraph of $L_s(G)$, $\lambda(L_s(G)) \geq \lambda(L(G))$.

We consider the following cases.
Case 1. If $\lambda(L(G)) = 0$, the obviously $\lambda(L_s(G)) = 0$.

Case 2. If $\lambda(L(G)) = 1$, then $L(G) = K_2$ or it is connected with a bridge $x = e_i e_j$.

We have the following subcases of this case.

Subcase 2.1. $L(G) = K_2$, then $L_s(G) = P_4$. Consequently, $\lambda(L_s(G)) = \delta(L(G)) = 1$.

Subcase 2.2. $L(G)$ is connected with a bridge $e_i e_j$. If $e_i$ is a pendant vertex, then $L_s(G)$ is connected with the some pendant vertex $e'_j$. There is only one edge incident with $e'_j$ whose removal disconnects it. Thus $\lambda(L_s(G)) = \delta(L(G)) = 1$. If neither $e_i$ nor $e_j$ is a pendant vertex and $\delta(L(G)) = 2$, then $\delta(L_s(G)) = 2$ and let $e_k$ be a vertex of $L_s(G)$ with $\deg_{L_s(G)} e_k = 2$. In $L_s(G)$, there are only two edges incident with $e_k$ and the removal of these disconnects $L_s(G)$. So $\lambda(L_s(G)) = \delta(L(G))$. If $\delta(L(G)) \geq 3$, then the removal of edges $e_i e_j$, $e'_j e_j$ and $e_i e'_j$ from $L_s(G)$ results in a disconnected graph. Hence $\lambda(L_s(G)) = 3 \lambda(L(G))$.

Now suppose $\lambda(L(G)) = n$. Then $L(G)$ has a minimum edge-cut $\{e_i = u_i v_i: 1 \leq i \leq n\}$ whose removal from $L(G)$ results in a disconnected graph. As above, there are two types of edge-cuts in $L_s(G)$ depending on the structure of $L(G)$. Among these, one edge-cut contains exactly $3n$
edges \{u \_{i} v', u' \_{i} v', u \_{i} v', 1 \leq i \leq n\} whose removal increases the
components of \(L_s(G)\) and the other is \(\delta (L(G))\)-edge-cut. Thus we have

\[
\lambda (L_s(G)) = \begin{cases} 
3n, & \text{if } n \leq \frac{\delta(L(G))}{3} = \frac{\delta_e(G) - 2}{3} \\
\delta(L(G)) = \delta_e(G) - 2 & \text{otherwise.}
\end{cases}
\]

Hence,

\[
\lambda (L_s(G)) = \min \{3\lambda(L(G)), \delta(L(G))\} = \min \{3\lambda(L(G)), \delta_e(G) - 2\}.
\]

This completes the proof.

**THEOREM 2.13.** If a graph \(G\) is \(n\)-edge connected, \(n \geq 2\), then \(L_s(G)\) is
\(n\)-connected.

**PROOF.** Let \(G\) be a \(n\)-edge connected graph, \(n \geq 2\). Then by Theorem
2.B, \(L(G)\) is \(n\)-connected. We show that there exist \(n\)-disjoint paths
between any two vertices of \(L_s(G)\). Let \(x\) and \(y\) be two distinct vertices of
\(L_s(G)\). We consider the following cases.

**Case 1.** Let \(x, y \in E(G)\). Then by Theorem 2.B and Theorem 2.C, \(x\) and \(y\)
are joined by \(n\)-disjoint paths in \(L(G)\). Since \(L(G)\) is an induced subgraph
of \(L_s(G)\), there exist \(n\)-disjoint paths between \(x\) and \(y\) in \(L_s(G)\).

**Case 2.** Let \(x \in E(G)\) and \(y \in E_1(G)\). Since \(\lambda(G) \leq \delta(G) < 2\delta(G) \leq \delta_e(G)\),
there are at least \(n\) edges adjacent to \(x\). Let \(x_i, i = 1, 2, \ldots, n\) be edges of
\(G\), adjacent to \(x\). Then the vertices \(x_i, i = 1, 2, \ldots, n\) are adjacent to the
vertex x in \( L_s(G) \), where \( x_i \in E_i(G) \), \( i = 1, 2, \ldots, n \). It follows from Case 1, that there exist \( n \)-disjoint paths from x to \( x_i \), \( i = 1, 2, \ldots, n \) in \( L_s(G) \). Since \( y \in E_i(G) \), we have \( N(y) = N(w) \cap E \), for some \( w \in E(G) \). Since \( |N(w)| \geq n \), let \( y_1, y_2, \ldots, y_n \in E(G) \) such that \( y_i \in N(w) \), \( i = 1, 2, \ldots, n \). So \( y_i \in N(y) \), \( i = 1, 2, \ldots, n \). Also, since x and \( y_i \in E(G) \), \( i = 1, 2, \ldots, n \), as in Case 1, there exist \( n \)-disjoint paths in \( L_s(G) \) between x and \( y_i \), \( i = 1, 2, \ldots, n \). Hence x and y are joined by \( n \)-disjoint paths in \( L_s(G) \).

**Case 3.** Let \( x, y \in E_i(G) \). As in Case 2, \( x_i \in N(x) \), \( i = 1, 2, \ldots, n \) and \( y_i \in N(y) \), \( i = 1, 2, \ldots, n \) for some \( x_i, y_i \in E(G) \), \( i = 1, 2, \ldots, n \). Consequently, \( x, y \in E(L_s(G)) \), \( i = 1, 2, \ldots, n \). Also by Case 1, every pair of \( x_i \) and \( y_i \) are joined by \( n \)-disjoint paths in \( L_s(G) \). Hence x and y are joined by \( n \)-disjoint paths in \( L_s(G) \).

Thus it follows from Theorem 2.C, that \( L_s(G) \) is \( n \)-connected. This completes the proof.

However, the converse of the above Theorem is not true. For example, \( L_s(G_1) \) is 2-connected, whereas \( G_1 \) is edge connected (see Figure 2.3).

**COROLLARY 2.13.1.** If a graph G is \( n \)-connected, \( n \geq 2 \), then \( L_s(G) \) is \( n \)-connected.

**PROOF.** We omit the proof.
The converse of above Corollary is not true. For instance, \( L_s(G_2) \) is 2-connected, but \( G_2 \) is connected (see Figure 2.4).

2.5. COVERING INVARIANTS OF LINE SPLITTING GRAPHS

In the following theorem, we determine the vertex covering number and the vertex independence number for line splitting graphs.

**THEOREM 2.14.** For any connected \((p, q)\) graph \( G \) of order \( p \geq 3 \), we have

(i) \( \alpha_0(L_s(G)) = \min \{q, 2 \alpha_0(L(G))\} \)

(ii) \( \beta_0(L_s(G)) = \max \{q, 2 \beta_0(L(G))\} \)

**PROOF.** (i) Let \( G \) be a connected \((p, q)\) graph with \( p \geq 3 \). Since \( L(G) \) is an induced subgraph of \( L_s(G) \), \( \alpha_0(L(G)) < \alpha_0(L_s(G)) \). There are two kinds of vertex covering sets for \( L_s(G) \), depending on the structure of \( G \). The first kind is the set \( E(G) \), which is a trivial edge covering set of \( G \) and also a trivial vertex covering set of \( L(G) \). So, \( \alpha_0(L_s(G)) \leq q \). The second kind of vertex covering set for \( L_s(G) \) is as follows: Let \( N_0 = \{e_i : 1 \leq i \leq n\} \) be a minimum vertex covering set in \( L(G) \). Then \( N'_0 = \{e'_i : 1 \leq i \leq n\} \) is an independent subset of \( V(L_s(G)) \) such that \( N_0 \cup N'_0 \) is a vertex covering set of \( L_s(G) \). By definition, \( \alpha_0(L_s(G)) = \min \{q, 2 \alpha_0(L(G))\} \).
Figure 2.3

$G_1 :$ 

$e_1 e_2$ 

$e_3$ 

$e_4 e_5$ 

$L_4(G_1) :$ 

$e_1 e_2$ 

$e_3$ 

$e_4 e_5 e_6$ 

Figure 2.4

$G_2 :$ 

$e_1 e_2$ 

$e_3$ 

$e_4$ 

$e_5 e_6$ 

$L_5(G_2) :$ 

$e_1 e_2$ 

$e_3$ 

$e_4 e_5 e_6$
(ii) $L_s(G)$ has $2q$ vertices and $\alpha_0(L_s(G)) + \beta_0(L_s(G)) = 2q$. By substituting the value of $\alpha_0(L_s(G))$ from (i), we have, $\beta_0(L_s(G)) = \max \{q, 2\beta_0(L(G))\}$. This completes the proof.

**Corollary 2.14.1.** For any connected $(p, q)$ graph $G$ of order $p \geq 3$, we have

(i) $\alpha_0(L_s(G)) = \min \{q, 2(q - \beta_1(G))\}$

(ii) $\beta_0(L_s(G)) = \max \{q, 2\beta_1(G)\}$

**Proof.** The proof follows from the above Theorem and Theorem 2.5.

In the next theorem, we establish the upper and lower bounds for the edge covering number and edge independence number for line splitting graphs.

**Theorem 2.15.** For any connected $(p, q)$ graph $G$ with $p \geq 3$, we have

$q \leq \alpha_1(L_s(G)) \leq 2\left[\frac{q}{2}\right]$ and $2\left[\frac{q}{2}\right] \leq \beta_1(L_s(G)) \leq q$.

**Proof.** Let $G$ be a connected $(p, q)$ graph $G$ with $p \geq 3$ vertices and $e_1, e_2, \ldots, e_q$ edges. Let $E_i = \{e_i', e_i'', \ldots, e_i^{(s)}\}$ be the set of newly introduced vertices in the construction of $L_s(G)$. For every pair $\{e_i, e_j\}$ of adjacent edges of $G$, we have an edge $e_ie_j$ in $L(G)$. Corresponding to this edge $e_je_i$, there are edges $e_ie_j, e_i'e_j, e_i''e_j$ in $L_s(G)$. Among these, $e_i'e_j$ and $e_i''e_j$
are independent in $L_s(G)$. Thus, each pair of adjacent edges of $G$ gives rise to two independent edges in $L_s(G)$, i.e., each edge of $L(G)$ gives rise to two independent edges in $L_s(G)$. So, $\beta_1(L(G))$ independent edges of $L(G)$ give rise to $2\beta_1(L(G))$ independent edges in $L_s(G)$. Hence $\beta_1(L_s(G)) \geq 2\beta_1(L(G))$. By Theorem 2.3, it follows that $2\left\lfloor \frac{q}{2} \right\rfloor \leq \beta_1(L_s(G))$.

Since $L_s(G)$ has $2q$ vertices and $\alpha_1(L_s(G)) + \beta_1(L_s(G)) = 2q$, we have $\alpha_1(L_s(G)) \leq 2\left\lfloor \frac{q}{2} \right\rfloor$.

Now, if $L(G)$ contains a spanning odd cycle $C_q : e_1e_2e_3...e_qe_1; q = 2k + 1, k \geq 0$, then the subset $N$ of $E(L_s(G))$, where $N = \{e_1e'_2, e_2e'_3, ..., e_{q-1}e'_q, e_qe'_1\}$ also forms an edge cover of $L_s(G)$ with $|N| = q$. In this case, $q < 2\alpha_1(L(G))$. Also, since $L_s(G)$ has atleast $q$ independent vertices, it requires atleast $q$ independent edges of $E(L_s(G))$ to cover these independent vertices in $L_s(G)$. Therefore, $\alpha_1(L_s(G))$ cannot be less than $q$. Hence $q \leq \alpha_1(L_s(G))$. Thus, we have $q \leq \alpha_1(L_s(G)) \leq 2\left\lfloor \frac{q}{2} \right\rfloor$.

and by Gallai’s result, $2\left\lfloor \frac{q}{2} \right\rfloor \leq \beta_1(L_s(G)) \leq q$. This completes the proof.

In the following theorem, we determine the neighborhood number $n_0(L_s(G))$ of line splitting graphs.
THEOREM 2.16. For any graph $G$ of order $p \geq 3$, 
\[ n_0(L_s(G)) = \min \{q, 2n_0(L(G))\}. \]

PROOF. Let $G$ be a graph of order $p \geq 3$. Since $L(G)$ is an induced subgraph of $L_s(G)$, $n_0(L_s(G)) > n_0(L(G))$. Now, depending on the structure of $G$, there are only two kinds of neighborhood sets for $L_s(G)$. Trivially, $E(G)$ is the first kind of neighborhood set for $L_s(G)$, since $E(G)$ is a trivial edge neighborhood set in $G$ and also a trivial neighborhood set in $L(G)$. So, in this case $n_0(L_s(G)) \leq q$. The second kind of neighborhood set for $L_s(G)$ is as follows: If $N_0 = \{e_i : 1 \leq i \leq n\}$ is a minimum neighborhood set in $L(G)$, then $N'_0 = \{e'_i : 1 \leq i \leq n\}$ is an independent subset of $L_s(G)$ such that $N_0 \cup N'_0$ forms another neighborhood set of $L_s(G)$. By definition, 
\[ n_0(L_s(G)) = \min \{q, 2n_0(L(G))\}. \] This completes the proof.

We now determine the domination number $\gamma(L_s(G))$ of line splitting graphs.

THEOREM 2.17. For any graph $G$ of order $p \geq 3$, 
\[ \gamma(L_s(G)) = \min \{q, 2\gamma'(G)\}. \]

PROOF. Let $G$ be a graph of order $p \geq 3$. Since $L(G)$ is an induced subgraph of $L_s(G)$, $\gamma(L_s(G)) > \gamma(L(G))$. Also, $\gamma(L(G)) = \gamma'(G)$, it follows that $\gamma(L_s(G)) > \gamma'(G)$. There are only two kinds of dominating sets for $L_s(G)$, depending on the structure of $G$. $E(G)$ is the first kind,
which is a trivial edge dominating set of $G$. So, in this case, $\gamma(L_d(G)) \leq q$.

The second kind of dominating set for $L_d(G)$ is as follows: If $M_0 = \{ e_i : 1 \leq i \leq n \}$ is a minimum edge dominating set in $G$, then $M'_0 = \{ e'_i : 1 \leq i \leq n \}$ is an independent subset of $L_s(G)$ such that $M_0 \cup M'_0$ forms another dominating set for $L_s(G)$. By definition, $\gamma((L_s(G))) = \min \{ q, 2\gamma'(G) \}$.

This completes the proof.

In the following, we establish a relation between domatic number of line splitting graphs and edge domatic number of graphs.

**COROLLORY 2.17.1.** For any graph $G$ of order $p \geq 2$, we have $d(L_s(G)) = d_1(G)$.

The following theorem determines the chromatic number of line splitting graphs.

**THEOREM 2.18.** For any graph $G$, the chromatic number $\chi(L_s(G)) = \chi'(G)$.

**PROOF.** Let $G$ be a graph. For each $e$ in $G$, let $e'$ be the new vertex chosen in the construction of $L_s(G)$. Since $\chi'(G) = \chi(L(G))$, $\chi'(G)$ coloring of $G$ can be extended to $\chi(L(G))$ coloring of $L(G)$. Also, we can assign the same color to $e$ and $e'$ in $L_s(G)$. Hence $\chi(L_s(G)) = \chi(L(G)) = \chi'(G)$. This completes the proof.
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