CHAPTER - 5

ON PATHOS PLICK GRAPH
OF A TREE*

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ABSTRACT

In this chapter we introduce the concept of pathos plick graph of a graph. We concentrate our study only on trees. We obtain some properties of these graphs. We present characterizations of those graphs whose pathos plick graphs are planar, outerplanar, minimally nonouterplanar and maximal outerplanar. Further we investigate eulerian and hamiltonian properties of pathos plick graphs. Also, the necessary and sufficient conditions for pathos plick graphs to have crossing number one are established.
5.1. INTRODUCTION

The concept of Pathos of a graph $G$ was introduced by Harary [2] as a collection of minimum number of edge disjoint open paths whose union is $G$. The path number of a graph $G$ is the number of paths in a pathos. Stanton [8] and Harary [4] have calculated the path number for certain class of graphs like trees and complete graphs.

For a graph $G$, let $E(G)$ and $b(G)$ denote its edge set and block set respectively. If two distinct blocks $B_i$ and $B_j$ of a graph $G$ are incident with a common cut-vertex then they are called adjacent blocks of $G$. The pluck graph $P(G)$ of a graph $G$ is defined as the graph having vertex set $E(G) \cup b(G)$, with two vertices adjacent if they correspond to adjacent edges of $G$ or one corresponds to an edge $e_i$ of $G$ and the other corresponds to a block $B_j$ of $G$ and $e_i$ lies in $B_j$ [5].

We now define the pathos pluck graph $PP(T)$ of a tree $T$ as the graph whose vertex set is the union of the set of edges, set of paths of pathos and the set of blocks of $T$, in which two vertices are adjacent if and only if the corresponding edges of $T$ are adjacent or the edge lies on the path of pathos or one corresponds to an edge $e_i$ of $G$ and the other corresponds to a block $B_j$ of $G$ and $e_i$ lies in $B_j$. Since the system of paths of pathos for a tree $T$ is not unique, the corresponding pathos pluck graph is also not unique. In Figure 5.1, a tree $T$ and its different pathos pluck graphs $PP(T)$ are shown.
Figure 5.1
The edge degree of an edge uv of a tree T is the sum of the degrees of u and v. The pathos length is the number of edges which lie on a particular path P_i of pathos in T. A pendant pathos is a path P_i of pathos having unit length which corresponds to a pendant edge in T. A pathos vertex is a vertex in PP(T) corresponding to the path P_i of pathos in T.

A set of vertices of a planar graph G is called an inner vertex set S of G if G can be drawn on the plane in such a way that each vertex of S lies only on the interior region and S contains the minimum possible number of vertices of G. The number of vertices of S is said to be the inner vertex number of G and is denoted by i(G). A graph is said to be k-minimally nonouterplanar if i(G) = k, k ≥ 1. An 1-minimally nonouterplanar graph is called minimally nonouterplanar. These concepts are introduced by Kulli [6] in 1975. A graph G is outerplanar if and only if i(G) = 0. This concept was introduced by Chartrand and Harary in 1967.

We need the following theorems for the proof of our further results.

**THEOREM 5.A** [5]. If G is nontrivial connected (p,q) graph whose vertices have degree d_i and if b_i is the number of blocks to which vertex v_i belongs in G, then the plick graph P(G) has q - p + 1 + \( \sum_{i=1}^{p} b_i \) vertices and \( \frac{1}{2} \sum_{i=1}^{p} d_i^2 \) edges.
THEOREM 5.B [1]. If G is a connected graph with p vertices and \( b_i \) is the number of blocks to which vertex \( v_i \) belongs in G, then it has
\[
\sum_{i=1}^{p} (b_i - 1) + 1 \text{ blocks.}
\]

THEOREM 5.C [7]. The plick graph \( P(G) \) of a graph G is planar if and only if G satisfies the following conditions:

1. \( \Delta(G) \leq 4 \) and
2. every block of G is either a cycle or a \( K_2 \).

THEOREM 5.D [7]. The plick graph \( P(G) \) of a connected graph G is outerplanar if and only if G satisfies the following conditions:

1. \( \Delta(G) \leq 3 \) and
2. G is a tree.

THEOREM 5.E [3]. A graph G is outerplanar if and only if it has no subgraph homeomorphic to \( K_4 \) or \( K_{2,3} \).

THEOREM 5.F [3]. A connected graph G is eulerian if and only if every vertex of G has even degree.

5.2. PATHOS PLICK GRAPHS

We start with a few preliminary results.

REMARK 5.1. For any tree T, \( P(T) \) is an induced subgraph of \( PP(T) \).
REMARK 5.2. The degree of a vertex in PP(T) corresponding to an edge of T is equal to the edge degree in T.

REMARK 5.3. The degree of a vertex in PP(T) corresponding to a block of T is equal to 1.

REMARK 5.4. The degree of the pathos vertex in PP(T) is equal to the pathos length of the corresponding path P_i of pathos in T.

REMARK 5.5. The number of cut-vertices in PP(T) is equal to the number of edges of T.

REMARK 5.6. The number of pendant vertices in PP(T) is greater than or equal to the number of edges of T.

In the following theorem we determine the number of vertices and edges in a pathos plick graph of a graph.

THEOREM 5.1. If G is nontrivial connected (p, q) graph whose vertices have degree d_i and if b_i is the number of blocks to which vertex v_i belongs in G, then the pathos plick graph PP(G) has \( q - p + k + 1 + \sum_{i=1}^{p} b_i \) vertices and \( q + \frac{1}{2} \sum_{i=1}^{p} d_i^2 \) edges, where k is the path number.

PROOF. By Theorem 5.1 and definition of PP(G), the number of vertices of PP(G) is

\[
= q + \sum_{i=1}^{p} (b_i - 1) + 1 + k
\]
The number of edges of \( PP(G) \) is the sum of the number of edges in \( P(G) \) and the number of edges which lie on the paths \( P_i \) of pathos of \( G \) which is \( q \). Hence by Theorem 5.A, the number of edges in \( PP(G) \) is

\[
= q + \sum_{j=1}^{p} d_j^2.
\]

This completes the proof.

The following lemma gives the number of vertices and edges of pathos plick graph of a tree.

**Lemma 5.1.** If \( T \) is a tree with \( p \) vertices and \( q \) edges, the number of vertices of \( PP(T) \) is \( 2q + k \) and the number of edges of \( PP(T) \) is \( q + \frac{1}{2} \sum_{i=1}^{p} d_i^2 \), where \( k \) is the path number.

**Proof.** The number of blocks in a tree is equal to the number of edges in the tree. By Theorem 5.B, \( \sum_{i=1}^{p} (b_i - 1) + 1 = q \). Hence the number of vertices of \( PP(T) \) is \( 2q + 1 \). Also by Theorem 5.1, the number of edges of \( PP(T) \) is \( q + \frac{1}{2} \sum_{i=1}^{p} d_i^2 \).

The following lemma gives the number of regions in pathos plick graph.
LEMMA 5.2. For any \((p, q)\) graph \(T\), the number of regions in pathos plick graph \(PP(T)\) is \(q\).

Further we obtain a relation between the edge degree in \(T\) and degree of vertex in \(PP(T)\) in the following lemma.

LEMMA 5.3. For any edge in a tree \(T\) with edge degree \(n\), the degree of the corresponding vertex in \(PP(T)\) is also \(n\).

PROOF. If an edge in a tree is of edge degree \(n\), then it is adjacent to \((n-2)\) edges. Since the edge itself contributes one degree to each end vertices in \(T\), \((n-2)\) degrees are contributed for the corresponding vertex in \(PP(T)\). Also by definition of \(PP(T)\), every edge in \(T\) lies on only one path of pathos. This contributes one more degree to the corresponding vertex in \(PP(T)\). Since every edge in \(T\) is a block, a vertex in \(PP(T)\) corresponding to a block of \(T\) contributes one more degree to the corresponding vertex in \(PP(T)\). Hence, the degree of the vertex in \(PP(T)\) corresponding to an edge in \(T\) with edge degree \(n\) is \(n - 2 + 1 + 1 = n\). This completes the proof.

5.3. PLANAR PATHOS PLICK GRAPHS

In the following theorem, we obtain the condition for planarity of pathos plick graph.
THEOREM 5.2. The pathos plick graph $PP(T)$ of a tree $T$ is planar if and only if $\Delta(T) \leq 4$.

PROOF : Suppose $PP(T)$ is planar. Assume $\Delta(T) \geq 5$. If there exists a vertex $v$ of degree 5 in $T$, then by Theorem 5.C, $P(T)$ is nonplanar. By Remark 5.1, $PP(T)$ is also nonplanar, a contradiction. Hence, $\Delta(T) \leq 4$.

Conversely, suppose $\Delta(T) \leq 4$, then every vertex of $T$ lies on at most four edges. Since $T$ is acyclic and every block of $T$ is a $K_2$, by Theorem 5.C, $P(T)$ is planar. Each block of $P(T)$ is a complete graph of order 2 or 3 or 4. The edges joining these blocks from the pathos vertices are incident with at most two vertices of each block of $P(T)$. Hence $PP(T)$ is planar. This completes the proof.

We now present a characterization of trees whose pathos plick graphs are outerplanar.

THEOREM 5.3. The pathos plick graph $PP(T)$ of a tree $T$ is outerplanar if and only if $\Delta(T) \leq 3$.

PROOF. Suppose $PP(T)$ is outerplanar. Assume that $T$ has a vertex $v$ of degree 4. Then the edges incident to $v$ form $K_4$ as an induced subgraph of $PP(T)$. Thus $PP(T)$ is nonouterplanar, a contradiction. Hence $\Delta(T) \leq 3$. 

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Conversely, suppose $\Delta(T) \leq 3$, then every vertex of $T$ lies on at most three edges. So, every block of $P(T)$ is either $K_3$ or $K_2$ and by Theorem 5.D, $P(T)$ is outerplanar. The edges joining to $P(T)$ from the corresponding pathos vertices give $PP(T)$ in which each region is a triangle or $K_2$. By Theorem 5.E, $PP(T)$ is outerplanar. This completes the proof.

The following theorem gives a nonminimally nonouterplanar pathos plick graph of a tree.

**THEOREM 5.4.** The pathos plick graph $PP(T)$ of a tree $T$ is not minimally nonouterplanar.

**PROOF.** Suppose $PP(T)$ is minimally nonouterplanar. We consider the following cases.

**Case 1.** If $\Delta(T) < 3$, then by Theorem 5.3, $PP(T)$ is outerplanar, a contradiction.

**Case 2.** Suppose $\Delta(T) > 4$. Assume $T$ has a vertex $v$ of degree 4. Then $T$ has subgraph homeomorphic to $K_{1,4}$. Clearly $\langle K_4 \rangle \subset P(T)$. Now, $K_{1,4}$ has two paths of pathos and each of these pathos vertices are adjacent to two vertices of $\langle K_4 \rangle$ of $P(T)$. On embedding $PP(T)$ in any plane, $i(PP(T)) \geq 2$, a contradiction. Hence $PP(T)$ is not minimally nonouterplanar. This completes the proof.
In the next theorem, we prove that pathos plick graph of a tree is not maximal outerplanar.

**THEOREM 5.5.** For any tree $T$, $PP(T)$ is not maximal outerplanar.

**PROOF.** Suppose $PP(T)$ is maximal outerplanar. Clearly, $PP(T)$ is outerplanar. By Theorem 5.3, $\Delta(T) \leq 3$. We consider the following cases.

**Case 1.** Suppose $\Delta(T) < 2$, then $T$ is a path. Clearly, $PP(T)$ has two nonadjacent vertices corresponding to adjacent blocks of $T$ whose join does not alter the outerplanarity of $PP(T)$, a contradiction.

**Case 2.** Suppose $\Delta(T) = 3$. It is possible to join the pathos vertex $p_i$ and the vertex corresponding to an endblock of $T$ which contains the edge $e_i$ of path $p_i$ without losing outerplanarity, a contradiction. Thus $PP(T)$ is not maximal outerplanar. This completes the proof.

We investigate the eulerian property of pathos plick graphs in the following theorem.

**THEOREM 5.6.** For any tree $T$, pathos plick graph $PP(T)$ is not eulerian.

**PROOF.** Suppose $T$ is a tree. Clearly every edge of $T$ is a block. The vertex in $PP(T)$ which corresponds to a block of $T$ is of degree 1. Hence by Theorem 5.6, $PP(T)$ is not eulerian. This completes the proof.

The nonhamiltonian property of pathos plick graph of a tree is given by the following theorem.
THEOREM 5.7. For any tree $T$, pathos plick graph $PP(T)$ is nonhamiltonian.

PROOF. Suppose $T$ is a tree. Then every edge of $T$ is a block. The vertex in $PP(T)$ which corresponds to an edge of $T$ is a cut-vertex. But it is well known fact that every hamiltonian graph is a block. Hence one can easily observe that $PP(T)$ is not hamiltonian. This completes the proof.

The next theorem characterizes pathos plick graphs in terms of crossing number one.

THEOREM 5.8. The pathos plick graph $PP(T)$ of a tree $T$ has crossing number one if and only if $\Delta(T) = 5$ and $T$ has unique vertex of degree 5.

PROOF. Suppose pathos plick graph $PP(T)$ of a tree $T$ has crossing number one. Then by Theorem 5.2, $\Delta(T) \geq 5$. We consider the following cases.

Case 1. Assume $T$ has a vertex $v$ of degree 6. Then the edges incident with this vertex $v$ form $K_6$ as an induced subgraph in $PP(T)$. It is well known that $cr(K_6)=3$. The pathos vertex is adjacent to atmost two adjacent vertices of this $K_6$ in $PP(T)$. This gives $cr(PP(T)) \geq 3$, a contradiction. Hence $\Delta(T) = 5$.

Case 2. Assume $T$ has atleast two vertices of degree 5. Then $T$ contains two $K_{1,5}$ as subgraphs. By definition, $PP(T)$ has two $K_5$ as subgraphs.
Since \( \text{cr}(K_5) = 1 \), we have \( \text{cr}(PP(T)) \geq 2 \), a contradiction. Thus \( T \) has unique vertex of degree 5.

Conversely, suppose \( \Delta(T) = 5 \) and \( T \) has unique vertex \( v \) of degree 5. The edges incident with the vertex \( v \) can be split into two sets of size 2 and 3. Transform \( T \) to \( T' \) as in Figure 5.2. Then \( P(T') \) is again planar and \( P(T) \) can be drawn with one crossing as shown in Figure 5.3. Thus \( \text{cr}(P(T)) = 1 \). Also, in \( PP(T) \), the pathos vertex is adjacent to atmost two adjacent vertices of the \( K_5 \) and hence it adds no more crossings (see Figure 5.4). Hence \( \text{cr}(PP(T)) = 1 \). This completes the proof.
Figure 5.2

T:

\[ \begin{align*}
T' & : \\
\end{align*} \]

\[ \begin{align*}
\text{P}(T') & : \\
\text{P}(T) & : \\
\text{PP}(T) & :
\end{align*} \]

Figure 5.3

Figure 5.4
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