CHAPTER - 4

CHARACTERIZATION OF MASS GRAPHS WITH CROSSING NUMBER ONE
ABSTRACT

In this chapter we deduce a necessary and sufficient condition for graphs whose mass graphs have crossing number 1. We also obtain a necessary and sufficient condition for mass graphs to have crossing number 1 in terms of forbidden subgraphs.
4.1. INTRODUCTION

In 1963, Harary and Hill [2] introduced one of the topological invariants, viz., the crossing number of a graph. In [4], Kulli, Akka and Beineke established a necessary and sufficient condition for line graphs with crossing number 1. In [3], Jendroľ and Klešč obtained characterization of graphs whose line graphs have crossing number one. In [8], Patil and Kulli established a necessary and sufficient condition for graphs whose middle graphs have crossing number k; k=1,2 or 3. In [5], Kulli and Annigeri obtained characterizations of total graphs with crossing number 1. In [7], Patil established a necessary and sufficient condition for total graphs with crossing number 1 in terms of forbidden subgraphs.

The purpose of this chapter is to establish a necessary and sufficient condition for graphs whose mass graphs have crossing number 1.

We now need a few basic terms and results.

In a drawing, the vertices of a graph G are mapped into vertices on the plane, and the edges of G into Jordan arcs of the plane, no three having a vertex in common, unless it be an endvertex of an edge.

A crossing is said to occur (in a graph) if the Jordan arcs of the plane corresponding to two edges of G intersect at a vertex which corresponds to no vertex of the plane.
An optimal drawing in a given surface is one which exhibits the least possible number of crossings. The least number is called the crossing number of a graph. In other words, the crossing number of a graph $G$, denoted by $\text{cr}(G)$, is the least number of intersections of pairs of edges in any embedding of $G$ in the plane. Obviously, $G$ is planar if and only if $\text{cr}(G) = 0$. A graph $G$ has crossing number 1 if $\text{cr}(G) = 1$. It is implicit that the edges in a drawing are Jordan arcs (hence, non-selfintersecting), and it is easy to see that a drawing with the minimum number of crossings (an optimal drawing) must be a good drawing, that is, each two edges have at most one vertex in common, which is either a common endvertex or a crossing.

In [6], Kuratowski proved the following interesting theorem.

**THEOREM 4.A.** A graph $G$ is planar if and only if it has no subgraph homeomorphic to $K_5$ or $K_{3,3}$.

We may revise Theorem 4.A to read.

**THEOREM 4.B.** A graph $G$ has crossing number zero if and only if it has no subgraph homeomorphic to $K_5$ or $K_{3,3}$.

The following will be useful in the proof of our results.

**THEOREM 4.C.** [4]. The line graph of any nonplanar graph has crossing number at least 3.
THEOREM 4.D. [1]. A graph $G$ has a planar line graph if and only if it has no subgraph homeomorphic to $K_{3,3}$, $K_{1,5}$, $P_4 + K_1$ or $K_2 + K_3$.

THEOREM 4.E. [3]. Let $G$ be a nonplanar graph. Then $\text{cr}(L(G)) = 1$ if and only if the following conditions hold:

1. $\text{cr}(G) = 1$
2. $\Delta(G) \leq 4$, and every vertex of degree 4 is a cut-vertex of $G$;
3. There exists a drawing of $G$ in the plane with exactly one crossing in which each crossed edge is incident with a vertex of degree 2.

THEOREM 4.F. [3]. The line graph of a planar graph $G$ has crossing number one if and only if (1) or (2) holds:

1. $\Delta(G) = 4$ and there is a unique non-cut-vertex of degree 4
2. $\Delta(G) = 5$, every vertex of degree 4 is a cut-vertex, there is a unique vertex of degree 5, and it has atmost 3 incident edges in any block.

4.2. MAIN RESULTS

The following theorem supports the main theorem.

THEOREM 4.1 Let $x$ be any edge of $K_4$. If $G$ is homeomorphic to $K_4 - x$, then $\text{cr}(M'(G)) = 1$.

PROOF. We prove the theorem first for $G = K_4 - x$. One can see that the graph $M'(K_4 - x)$ has 6 vertices and 13 edges. But a planar graph with 6 vertices has atmost 12 edges. This shows that $M'(K_4 - x)$ has crossing number atleast 1. Figure 4.1, being drawing of $M'(K_4 - x)$ concludes that
Figure 4.1
cr(M*(K_4-x))=1. Suppose now G is the graph as in the statement. Referring to Figure 4.1, it is immediate to see that cr(M*(G))=1. This completes the proof.

The following theorem gives a necessary and sufficient condition for graphs whose mass graphs have crossing number 1.

**THEOREM 4.2.** The mass graph M*(G) of a graph G has crossing number 1 if and only if G is planar and one of the following holds:

(1) \( \Delta(G) = 3 \), G has exactly two non-cut-vertices of degree 3 and they are adjacent.

(2) \( \Delta(G) = 4 \), G has unique cut-vertex of degree 4 and every block of G is either a cycle or a K_2.

**PROOF.** Suppose M*(G) has crossing number one. Then by Remark 3.1, and Theorem 4.E, G is planar. By Theorem 3.6, if \( \Delta(G) \leq 3 \), then atleast one vertex of degree 3 is a non-cut-vertex or \( \Delta(G) \geq 4 \).

Suppose G has a vertex v of degree 5. We have the following cases.

**Case 1.** Assume v is a non-cut-vertex. Then by Theorem 4.F, L(G) has crossing number greater than one. By Remark 3.1, cr(M*(G)) > 1, a contradiction.
Case 2. Assume $v$ is a cut-vertex. Then the cut-vertex $v$ together with its five incident edges form $K_{1,5}$ as subgraph of $G$. By Remark 3.2, $K_6$ is a subgraph of $M'(G)$ and hence $\text{cr}(M'(G)) > 1$, a contradiction. This implies that $\Delta(G) \leq 4$.

Suppose $\Delta(G) \leq 2$. Then by Theorem 3.6, $M'(G)$ is planar, a contradiction. Thus $\Delta(G) = 3$ or $4$.

We now consider the following cases:

Case 1. Suppose $\Delta(G) = 3$. By Theorem 3.6 and since $\text{cr}(M'(G)) = 1$, $G$ has a non-cut-vertex of degree 3. Clearly $G$ contains a subgraph homeomorphic to $K_4 - x$, where $x$ is any edge of $K_4$, so that there exist at least two non-cut-vertices of degree 3. More precisely, there is an even number, say $2n$, of non-cut-vertices of degree 3. Now suppose $G$ has at least two diagonal edges. Then there are two subcases to consider depending on whether 2 diagonal edges exist in one cycle or in two different edge disjoint cycles.

Subcase 1.1. If two diagonal edges exist in one cycle of $G$. Then $G$ has a subgraph homeomorphic to $K_4$. The graph $M'(K_4)$ has 7 vertices and 18 edges. It is known that a planar graph with 7 vertices has at most 15 edges. This shows that $M'(K_4)$ must have crossing number exceeding 1 and hence $M'(G)$ has crossing number greater than 1, a contradiction.
**Subcase 1.2.** If two diagonal edges exist in two different edge-disjoint cycles of $G$. Then by Theorem 4.1, we see that for every subgraph of $G$ homeomorphic to $K_{4-x}$, there corresponds at least one crossing of $G$. Hence $M^*(G)$ has at least 2 crossings a contradiction.

Hence $G$ has exactly two non-cut-vertices of degree 3 and every other vertex of degree 3 is a cut-vertex.

Suppose a graph $G$ has two non-cut-vertices of degree 3 and they are not adjacent. Then $G$ contains a subgraph homeomorphic to $K_{2,3}$. On drawing $M^*(K_{2,3})$ in a plane one can see that $cr(M^*(K_{2,3}))=2$. Since $M^*(K_{2,3})$ is a subgraph of $M^*(G)$, $M^*(G)$ has crossing number exceeding 1, a contradiction (see Figure 4.2). Therefore, we conclude that $G$ contains exactly two non-cut-vertices of degree 3 and these are adjacent. This proves (1).

**Case 2.** Assume $\Delta(G) = 4$. We first show that $G$ has unique vertex of degree 4. Suppose $G$ has at least two vertices $v_1$ and $v_2$ of degree 4. We consider the following subcases.

**Subcase 2.1.** Suppose $v_1$ and $v_2$ are two non-cut-vertices of degree 4. Then by Theorem 4.1F, $L(G)$ has at least two crossings. By Remark 3.1, $cr(M^*(G)) > 1$, a contradiction.

**Subcase 2.2.** Suppose $v_1$ and $v_2$ are cut-vertices of degree 4. Then the cut-vertices $v_1$ and $v_2$ together with their corresponding incident edges form
Figure 4.2

$K_{2,3}$:

$M'(K_{2,3})$:
form two subgraphs each of which is \( K_{1,4} \) in \( G \). By Remark 3.2, \( M^*(G) \) has two subgraphs each of which is \( K_5 \). Hence \( \text{cr}(M^*(G)) > 1 \), a contradiction.

Subcase 2.3. Suppose \( v_1 \) is a cut-vertex and \( v_2 \) is a non-cut-vertex. Then the cut-vertex \( v_1 \) together with its four incident edges form \( K_{1,4} \) as a subgraph of \( M^*(G) \). By Remark 3.2, \( K_5 \) is a subgraph of \( M^*(G) \). It is known that \( \text{cr}(K_5) = 1 \). Also, since \( v_2 \) is a non-cut-vertex of degree 4, \( G \) has a subgraph homeomorphic to \( K_{4-x} \), where \( x \) is any edge of \( K_4 \). By Theorem 4.1, \( \text{cr}(K_{4-x}) = 1 \). Hence \( \text{cr}(M^*(G)) > 1 \), a contradiction. Hence \( G \) has unique cut-vertex of degree 4.

Now suppose \( v \) is a non-cut-vertex. By Theorem 3.6, removal of any edge at \( v \) leaves a graph \( H \) such that \( M^*(H) \) is nonplanar. It follows that \( \text{cr}(M^*(G)) \geq 2 \), a contradiction. Hence \( v \) is a cut-vertex.

Suppose \( G \) has a vertex \( v \) of degree 4 and at least one block of \( G \) is neither a cycle nor a \( K_2 \). Clearly \( G \) has a block \( B \) which has a subgraph homeomorphic to \( K_{4-x} \). The line graph of \( B \) has a subgraph homeomorphic to \( K_4 \), by Theorem 3.3, \( \text{cr}(M^*(B)) \geq 1 \). Also the cut-vertex \( v \) together with its four incident edges form \( K_{1,4} \) as a subgraph of \( G \). By Remark 3.2, \( K_5 \) is a subgraph of \( M^*(G) \). Hence \( \text{cr}(M^*(G)) \geq 2 \), a contradiction. This proves (2).
Conversely, suppose $G$ is a planar graph satisfying (1) or (2). Then by Theorem 3.6, $M^*(G)$ has crossing number at least 1. We now show that its crossing number is at most 1.

Suppose (1) holds. Then $G$ has exactly one block, say $H$, homeomorphic to $K_4-x$ which contains two adjacent non-cut-vertices of degree 3. By Theorem 4.1, $cr(M^*(H)) = 1$. By Theorem 3.6, all other remaining blocks of $G$ have a planar mass graph. Hence $M^*(G)$ has crossing number 1.

Assume (2) holds. The edges at the vertex $v$ of degree 4 can be split into sets of size 2 such that no edges in different sets are in the same block. Transform $G$ to $G'$ as in Figure 4.3. Then $M^*(G')$ is planar and $M^*(G)$ can be drawn with one crossing as shown in Figure 4.4. This completes the proof.

4.3. FORBIDDEN SUBGRAPHS

By using Theorem 4.2, we now characterize graphs whose mass graphs have crossing number 1 in terms of forbidden subgraphs.

**THEOREM 4.3.** Let $G$ be a connected plane graph. Then it has a mass graph with crossing number one if and only if it has no subgraph homeomorphic to any one of the graphs of Figure 4.5.
Figure 4.3

$G : \quad G' :$

$M'(G') :$

$M'(G) :$

Figure 4.4

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PROOF. Suppose G has a mass graph with crossing number one. We now show that all graphs homeomorphic to any one of the graphs of Figure 4.5 have no mass graph with crossing number one. It follows from Theorem 4.2, since graphs homeomorphic to G_1, G_2 or G_3 have more than two non-cut-vertices of degree 3 and graphs homeomorphic to G_4 have two non-cut-vertices of degree 3 which are not adjacent, graphs homeomorphic to G_5 have \( \Delta(G_5) > 4 \), graphs homeomorphic to G_6 have a vertex of degree 4 which is a non-cut-vertex, graphs homeomorphic to G_7 or G_8 have two or more vertices of degree 4, graphs homeomorphic to G_9, G_{10}, or G_{11} have a block which is neither a cycle nor a K_2.

Conversely suppose G has no subgraph homeomorphic to any one of the graphs of Figure 4.5. Since G does not contain a subgraph homeomorphic to G_5 i.e., K_{1,5}, \( \Delta(G) \leq 4 \). Also since \( \Delta(G) \geq 3 \), it follows that \( \Delta(G) = 3 \) or 4.

First we prove condition (1) of Theorem 4.2. Suppose G contains more than two non-cut-vertices of degree 3. Then it is easy to see that G is a planar graph with atleast two diagonal edges. Now we consider two cases depending on whether the two diagonal edges exist in one block or in two different blocks.

Case 1. Suppose two diagonal edges exist in one block of G, then G has a subgraph homeomorphic to G_1 or G_2.
Figure 4.5
Case 2. Suppose two diagonal edges exist in two different blocks of $G$, then $G$ has a subgraph homeomorphic to $G_3$.

In each case we arrive at a contradiction. Hence $G$ has exactly two non-cut-vertices of degree 3.

Suppose these two non-cut-vertices of degree 3 are not adjacent. Then there exist three disjoint paths between these two non-cut-vertices of degree 3. Clearly $G$ contains a subgraph homeomorphic to $G_4$, a contradiction. Thus $G$ has exactly two adjacent non-cut-vertices of degree 3.

Suppose $G$ has a vertex $v$ of degree 4. We prove that $v$ is a cut-vertex. If not, let $a$, $b$, $c$ and $d$ be the vertices of $G$ adjacent to $v$. Then there exist paths between every pair of vertices $a$, $b$, $c$ and $d$ not containing $v$. Then it is proved in Theorem 4.D, that $G$ has a subgraph homeomorphic to $G_6$, this is a contradiction. Thus $v$ is a cut-vertex and every vertex of degree 4 is a cut-vertex.

Suppose $G$ has two cut-vertices $v_1$ and $v_2$ of degree 4. Since $G$ is connected $v_1$ and $v_2$ are connected by a path $P$ and let $(v_1, a_i)$ and $(v_2, b_j)$, $i, j = 1, 2, 3$ be the edges of $G$. We consider the following possibilities.

If $a_i \neq b_j$ for $i, j = 1, 2, 3$ then $G$ contains a subgraph homeomorphic to $G_7$, a contradiction.
If there exists a path between a vertex $a_i$ and a vertex $b_j$, then $G$ has a subgraph homeomorphic to $G_8$, a contradiction.

If $a_i = b_j$, for $i, j = 1, 2$ then clearly $G$ contains a subgraph homeomorphic to $G_9$, a contradiction.

This proves that $G$ has exactly one vertex $v$ of degree 4.

Suppose $G$ has a unique cut-vertex of degree 4 and it lies on blocks, one block of which is neither a cycle nor a $K_2$. Then $G$ contains a subgraph homeomorphic to $G_{10}$ or $G_{11}$.

Thus Theorem 4.2 implies that $G$ has a mass graph with crossing number one. This completes the proof.
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