2 Bright n-soliton solutions of multicomponent CNLS system with focusing nonlinearity

2.1 Introduction

In the last few decades considerable efforts have been made to understand the formation of solitons, and their dynamics, in nonlinear media. The many-faceted applications of solitons make them to be considered as very useful entities in a variety of areas in science. To be specific, soliton type pulse propagation in multimode optical fibres [43] and in fibre arrays [23] is governed by a set of N-CNLS equations which is non integrable, in general. However, it becomes integrable for specific choices of parameters [75, 93]. In this chapter, we consider the integrable N-CNLS system with focusing nonlinearity (1.42a). The focusing type nonlinearity corresponds to the choice \( \sigma_l = 1, \ l = 1, 2, \ldots, N \), in Eq. (1.42a). Due to the vanishing boundary conditions, \( q_i \to 0, \ t \to \pm \infty, \ i = 1, 2, \ldots, N \), this system admits only bright soliton solutions. For the \( N = 2 \) case, with \( \sigma_1 = \sigma_2 = 1 \), system (1.42a) reduces to the celebrated Manakov model [52] describing the interaction between two polarization components of an optical pulse in a birefringent fibre. Detailed description of focusing N-CNLS system has been given earlier in the introductory chapter.

We now briefly review the already existing results of focusing CNLS equa-
tions. For the focusing 2-CNLS system, the exact soliton solutions have been derived with different procedures [52, 68, 94, 95]. Kaup and Malomed [94] have briefly discussed the role of focusing 2-CNLS equations in nonlinear optics using its one-soliton solution obtained from the inverse scattering method. They have pointed out that the system, besides the birefringence property, covers many other physical phenomena such as soliton trapping and daughter wave (shadow) formation in optical fibres. By considering the analytic solution of the system, conditions have been established for soliton switching and energy coupling among the two modes in a nonlinear fibre [96]. The most general bright two-soliton solution have been given by Radhakrishnan, Lakshmanan and Hi- etarinta [62] using the Hirota bilinearization method and they revealed that the solitons exhibit certain novel shape changing collision properties. However this one- and two- soliton solutions alone are not enough in the light of increasing technological, computational and communicational advancements. For this purpose, in Ref. [66], the more general one-, two-, three- and four-soliton solutions of the integrable 2-CNLS, 3-CNLS, and N-CNLS equations have been reported by Kanna and Lakshmanan and also the shape changing collision scenario have been discussed. Using three-soliton solution, it has been demonstrated [66] that one can construct logic gates based on soliton collisions. Now it is of interest to obtain the general n-soliton solution of (1.42a) using Hirota’s method. Though expressions for n-soliton solution of the focusing type CNLS system have been obtained through Darboux transformation [97] and inverse scattering transform method [98], in this chapter we present the Gram determinant form for the n-soliton solution of N-CNLS with focusing nonlinearity, for arbitrary N case.

In this regard, first we present the explicit forms of one- two- and three- soliton solution [66] and review the shape changing collision of solitons which has been discussed in Ref. [66]. Then we explicitly write down the determinant form for the n-soliton solution and prove that indeed the Grammian form satisfies
2.2 Hirota bilinearization and soliton solution of the integrable N-CNLS system with focusing nonlinearity

Section 2.2 deals with Hirota bilinearization procedure for the multicomponent N-CNLS equations with focusing nonlinearity and we obtain the one, two and three soliton solutions. Section 2.3 is devoted to a discussion of the shape changing (intensity redistribution) nature of soliton interactions. The procedure to obtain one, two soliton solutions is extended to multisoliton solutions and also for multicomponent system in section 2.4 where the proof is also given.

2.2 Hirota bilinearization and soliton solution of the integrable N-CNLS system with focusing nonlinearity

Following the lines of Ref. [66] and applying the specific bilinearizing transformation,

\[ q_j = \frac{g^{(j)}}{f}, \quad j = 1, 2, ..., N, \]  

(2.1)

to Eq. (1.42a), one can deduce the following set of bilinear equations,

\[ (iD_z + D_t^2)g^{(j)}.f = 0, \quad j = 1, 2, ..., N, \]  

(2.2a)

\[ D_t^2(f.f) = 2\mu \sum_{n=1}^{N} g^{(n)}g^{(n)*}. \]  

(2.2b)

where * denotes the complex conjugate, \( g^{(j)} \)'s are complex functions, while \( f(z, t) \) is a real function and the Hirota’s bilinear operators \( D_z \) and \( D_t \) are defined by

\[ D_z^n D_t^m(a,b) = \left( \frac{\partial}{\partial z} - \frac{\partial}{\partial z'} \right)^n \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^m a(z, t)b(z', t') \left|_{z=z', t=t'} \right.. \]  

(2.2c)
The above set of equations can be solved by introducing the following power series expansions for $g^{(j)}$'s and $f$:

\[
\begin{align*}
    g^{(j)} &= \chi g_1^{(j)} + \chi^2 g_3^{(j)} + \ldots, \ j = 1, 2, \ldots, N, \\
    f &= 1 + \chi^2 f_2 + \chi^4 f_4 + \ldots,
\end{align*}
\]  

(2.3a) (2.3b)

where $\chi$ is the formal expansion parameter. The resulting set of equations, after collecting the terms with the same power in $\chi$, can be solved recursively to obtain the forms of $g^{(j)}$'s and $f$. The explicit forms of bright one-, two-, and three-soliton solutions are presented below [66]:

### 2.2.1 One-soliton solution

The one-soliton solution of Eq. (1.42a) is given by

\[
(q_1, q_2, \ldots, q_N)^T = k_{1R} e^{i\mu} \text{sech} \left( \eta_1 + \frac{R}{2} \right) (A_1, A_2, \ldots, A_N)^T,
\]  

(2.4)

where $\eta_1 = k_1 (t + ik_1 z)$, $A_j = \alpha_1^{(j)} / \Delta$, $\Delta = (\mu (\sum_{s=1}^{N} |\alpha_1^{(s)}|^2))^{1/2}$, $e^R = \Delta^2 / (k_1 + k_1^*)^2$, $\alpha_1^{(j)}$ and $k_1$, $j = 1, 2 \ldots, N$, are $(N + 1)$ arbitrary complex parameters. Further $k_{1R} A_j$ gives the amplitude of the $j$th mode $(j = 1, 2, \ldots, N)$ and $2k_{1I}$ is the soliton velocity in all the $N$ modes.

### 2.2.2 Two-soliton solution

The two-soliton solution of Eq. (1.42a) can be written as

\[
q_j = \frac{\alpha_1^{(j)} e^{\eta_1^2 + \eta_2^2 + \delta_{1j}} + \alpha_2^{(j)} e^{\eta_1^2 + \eta_2^2 + \delta_{2j}}}{D}, \ j = 1, 2, \ldots, N,
\]  

(2.5a)

where

\[
D = 1 + e^{\eta_1^2 + \eta_2^2 + \delta_{11}} + e^{\eta_1^2 + \eta_2^2 + \delta_{0}} + e^{\eta_1^2 + \eta_2^2 + \delta_{10}} + e^{\eta_1^2 + \eta_2^2 + \delta_{10} + \delta_{12}} + e^{\eta_1^2 + \eta_2^2 + \delta_{10} + \delta_{12} + \delta_{22}}.
\]  

(2.5b)
In Eqs. (2.5), the various quantities are defined as follows,

\[ \eta_i = k_i(t + i k^*_i z), \quad e^{\delta_0} = \frac{k_{12}}{k_1 + k^*_2}, \quad e^{R_1} = \frac{k_{11}}{k_1 + k^*_1}, \quad e^{R_2} = \frac{k_{22}}{k_2 + k^*_2}, \]
\[ e^{\delta_{12}} = \frac{(k_1 - k_2)(\alpha_1^{(j)} \alpha_{21}^{(j)} - \alpha_2^{(j)} \alpha_{12}^{(j)})}{(k_1 + k^*_1)(k^*_1 + k_2)}, \quad e^{\delta_{2j}} = \frac{(k_2 - k_1)(\alpha_2^{(j)} \kappa_{12}^{(j)} - \alpha_1^{(j)} \kappa_{22}^{(j)})}{(k_2 + k^*_2)(k_1 + k^*_2)}, \]
\[ e^{R_3} = \frac{|k_1 - k_2|^2}{(k_1 + k^*_1)(k_2 + k^*_2)|k_1 + k^*_2|^2}(\kappa_{11} \kappa_{22} - \kappa_{12} \kappa_{21}), \quad (2.6a) \]

and

\[ \kappa_{il} = \mu \sum_{s=1}^{N} \frac{\alpha_{1s}^{(s)} \alpha_{ls}^{(s)}}{(k_i + k^*_l)^s}, \quad i, l = 1, 2. \quad (2.6b) \]

One may also note that the above two-soliton solution depends on \(2(N + 1)\) arbitrary complex parameters \(\alpha_1^{(j)}, \alpha_2^{(j)}, k_1, \) and \(k_2, j = 1, 2, ..., N.\)

### 2.2.3 Three-soliton solution

We can write down the three-soliton solution as

\[ q_j = \frac{\alpha_1^{(j)} e^{\eta_1} + \alpha_2^{(j)} e^{\eta_2} + \alpha_3^{(j)} e^{\eta_3} + e^{\eta_1 + \eta_2 + \eta_3 + \delta_{12}} + e^{\eta_1 + \eta_2 + \eta_3 + \delta_{2j}} + e^{\eta_2 + \eta_3 + \eta_1 + \delta_{3j}} + e^{\eta_1 + \eta_2 + \eta_3 + \delta_{13}} + e^{\eta_1 + \eta_2 + \eta_3 + \delta_{23}} + e^{\eta_1 + \eta_2 + \eta_3 + \delta_{12}}}{D} \]
\[ + \frac{e^{\eta_2 + \eta_3 + \eta_1 + \delta_{13}} + e^{\eta_3 + \eta_1 + \eta_2 + \delta_{23}} + e^{\eta_3 + \eta_1 + \eta_2 + \delta_{13}} + e^{\eta_3 + \eta_1 + \eta_2 + \delta_{23}} + e^{\eta_1 + \eta_2 + \eta_3 + \delta_{13}} + e^{\eta_1 + \eta_2 + \eta_3 + \delta_{13}} + e^{\eta_1 + \eta_2 + \eta_3 + \delta_{13}} + e^{\eta_1 + \eta_2 + \eta_3 + \delta_{13}}}{D} \]
\[ + \frac{e^{\eta_1 + \eta_2 + \eta_3 + \delta_{12}} + e^{\eta_1 + \eta_2 + \eta_3 + \delta_{12}}}{D} + \frac{e^{\eta_1 + \eta_2 + \eta_3 + \delta_{12}} + e^{\eta_1 + \eta_2 + \eta_3 + \delta_{12}}}{D} + \frac{e^{\eta_1 + \eta_2 + \eta_3 + \delta_{12}} + e^{\eta_1 + \eta_2 + \eta_3 + \delta_{12}}}{D} + \frac{e^{\eta_1 + \eta_2 + \eta_3 + \delta_{12}} + e^{\eta_1 + \eta_2 + \eta_3 + \delta_{12}}}{D}, \quad j = 1, 2, ..., N. \quad (2.7a) \]

where

\[ D = 1 + e^{\eta_1 + \eta_2 + R_1} + e^{\eta_2 + \eta_3 + R_2} + e^{\eta_3 + \eta_1 + R_3} + e^{\eta_1 + \eta_2 + \delta_{10}} + e^{\eta_1 + \eta_2 + \delta_{10}} \]
\[ + e^{\eta_1 + \eta_2 + \delta_{20}} + e^{\eta_1 + \eta_3 + \delta_{20}} + e^{\eta_2 + \eta_3 + \delta_{30}} + e^{\eta_1 + \eta_2 + \delta_{30}} + e^{\eta_2 + \eta_3 + \delta_{02}} + e^{\eta_1 + \eta_2 + \delta_{02}} + e^{\eta_1 + \eta_2 + \delta_{02}} + e^{\eta_1 + \eta_2 + \delta_{02}} \]
\[ + e^{\eta_1 + \eta_2 + \eta_3 + R_5} + e^{\eta_2 + \eta_3 + \eta_1 + R_6} + e^{\eta_1 + \eta_2 + \eta_3 + \tau_{10}} + e^{\eta_1 + \eta_2 + \eta_3 + \tau_{10}} \]
\[ + e^{\eta_2 + \eta_3 + \eta_1 + \tau_{20}} + e^{\eta_2 + \eta_3 + \eta_1 + \tau_{20}} + e^{\eta_3 + \eta_1 + \eta_2 + \tau_{30}} + e^{\eta_3 + \eta_1 + \eta_2 + \tau_{30}} \]
\[ + e^{\eta_1 + \eta_2 + \eta_3 + \eta_1 + R_7}. \quad (2.7b) \]
Here

\( \eta_i = k_i(t + ik_i z), \ i = 1, 2, 3, \)  

\[ e^{\delta_{1j}} = \frac{(k_1 - k_2)(\alpha_j^{(j)} \kappa_{21} - \alpha^j_{(j)} \kappa_{11})}{(k_1 + k_1^*)(k_1^* + k_2^*)}, \quad e^{\delta_{2j}} = \frac{(k_1 - k_3)(\alpha_j^{(j)} \kappa_{31} - \alpha^j_{(j)} \kappa_{11})}{(k_1 + k_1^*)(k_1^* + k_3^*)}, \]

\[ e^{\delta_{3j}} = \frac{(k_1 - k_2)(\alpha_j^{(j)} \kappa_{22} - \alpha^j_{(j)} \kappa_{12})}{(k_1 + k_2^*)(k_2^* + k_2^*)}, \quad e^{\delta_{4j}} = \frac{(k_2 - k_3)(\alpha_j^{(j)} \kappa_{32} - \alpha^j_{(j)} \kappa_{22})}{(k_2 + k_2^*)(k_2^* + k_3^*)}, \]

\[ e^{\delta_{5j}} = \frac{(k_1 - k_3)(\alpha_j^{(j)} \kappa_{33} - \alpha^j_{(j)} \kappa_{13})}{(k_3 + k_3^*)(k_3^* + k_1^*)}, \]

\[ e^{\delta_{6j}} = \frac{(k_2 - k_3)(\alpha_j^{(j)} \kappa_{33} - \alpha^j_{(j)} \kappa_{23})}{(k_3^* + k_2^*)(k_3^* + k_3^*)}, \]

\[ e^{\delta_{7j}} = \frac{(k_1 - k_3)(\alpha_j^{(j)} \kappa_{33} - \alpha^j_{(j)} \kappa_{23})}{(k_1 + k_3^*)(k_1^* + k_3^*)}, \quad e^{\delta_{8j}} = \frac{(k_1 - k_3)(\alpha_j^{(j)} \kappa_{32} - \alpha^j_{(j)} \kappa_{12})}{(k_1 + k_2^*)(k_2^* + k_3^*)}, \]

\[ e^{\tau_{1j}} = \frac{(k_2 - k_1)(k_3 - k_1)(k_3 - k_2)(k_2^* - k_1^*)}{(k_1^* + k_1)(k_1^* + k_2)(k_1^* + k_3)(k_2^* + k_1)(k_2^* + k_2)(k_2^* + k_3)} \times \left[ \alpha_j^{(j)}(\kappa_{21}\kappa_{32} - \kappa_{22}\kappa_{31}) + \alpha_j^{(j)}(\kappa_{12}\kappa_{31} - \kappa_{13}\kappa_{21}) + \alpha_j^{(j)}(\kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21}) \right], \]

\[ e^{\tau_{2j}} = \frac{(k_2 - k_1)(k_3 - k_1)(k_3 - k_2)(k_2^* - k_1^*)}{(k_1^* + k_1)(k_1^* + k_2)(k_1^* + k_3)(k_2^* + k_1)(k_2^* + k_2)(k_2^* + k_3)} \times \left[ \alpha_j^{(j)}(\kappa_{33}\kappa_{21} - \kappa_{32}\kappa_{23}) + \alpha_j^{(j)}(\kappa_{31}\kappa_{13} - \kappa_{33}\kappa_{11}) + \alpha_j^{(j)}(\kappa_{23}\kappa_{11} - \kappa_{13}\kappa_{21}) \right], \]

\[ e^{\tau_{3j}} = \frac{(k_2 - k_1)(k_3 - k_1)(k_3 - k_2)(k_2^* - k_1^*)}{(k_2^* + k_1)(k_2^* + k_2)(k_2^* + k_3)(k_3^* + k_1)(k_3^* + k_2)(k_3^* + k_3)} \times \left[ \alpha_j^{(j)}(\kappa_{22}\kappa_{33} - \kappa_{23}\kappa_{32}) + \alpha_j^{(j)}(\kappa_{13}\kappa_{32} - \kappa_{33}\kappa_{12}) + \alpha_j^{(j)}(\kappa_{12}\kappa_{32} - \kappa_{22}\kappa_{13}) \right], \]

(2.7d)
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\[ e^{\tau_{10}} = \frac{(k_2 - k_1)(k_3^* - k_1^*)}{(k_1 + k_2)(k_3^* + k_1)(k_3^* + k_2)} \left[ \kappa_{11}\kappa_{23} - \kappa_{21}\kappa_{13} \right], \]

\[ e^{\tau_{20}} = \frac{k_1 - k_2}{(k_2 + k_1)(k_3^* + k_1)(k_3^* + k_2)} \left[ \kappa_{22}\kappa_{13} - \kappa_{12}\kappa_{23} \right], \]

\[ e^{\tau_{30}} = \frac{k_3 - k_1}{(k_3 + k_1)(k_3^* + k_1)(k_3^* + k_2)} \left[ \kappa_{33}\kappa_{12} - \kappa_{13}\kappa_{32} \right], \]

\[ e^{R_T} = \frac{|k_1 - k_2|^2|k_2 - k_3|^2|k_3 - k_1|^2}{(k_1 + k_1^*)(k_2 + k_2^*)(k_3 + k_3^*)|k_1 + k_2^2|^2|k_2 + k_3^2|^2|k_3 + k_1^2|^2} \times \left[ (\kappa_{11}\kappa_{22}\kappa_{33} - \kappa_{11}\kappa_{23}\kappa_{32}) + (\kappa_{12}\kappa_{23}\kappa_{31} - \kappa_{12}\kappa_{21}\kappa_{33}) \right] \]

\[ + (\kappa_{21}\kappa_{13}\kappa_{32} - \kappa_{22}\kappa_{13}\kappa_{31}) \], \quad (2.7e) \]

and

\[ \kappa_{il} = \mu \sum_{n=1}^{2} \alpha_i^{(n)} \alpha_l^{(n)^*}, \quad i, l = 1, 2, 3. \] \quad (2.7f) \]

It can be observed from the above expression that as the number of solitons increases the complexity also increases and the present three-soliton solution is characterized by \( 3(N + 1) \) complex parameters \( \alpha_1^{(j)}, \alpha_2^{(j)}, \alpha_3^{(j)}, \quad j = 1, 2, ..., N, \ k_1, \ k_2 \) and \( k_3 \).

### 2.3 Shape changing nature of soliton interactions and intensity redistributions

The remarkable fact about the above bright multisoliton solutions of the integrable CNLS system is that they exhibit a fascinating shape changing (intensity redistribution / energy exchange) collision as shown in Refs. [62,66] and briefly presented below. As the N-CNLS equations arise in diverse areas of physics as mentioned in the introductory chapter, it is of interest to analyze the interaction properties of the soliton solutions of 2-, 3-, and N-CNLS equations. This can be done by plotting the multicomponent two and three soliton solutions obtained in earlier subsections. In fact, the two soliton solution represents interaction
of two one solitons and the corresponding collision dynamics is discussed in the following subsections. The understanding of this collision process can be facilitated by making an asymptotic analysis of the two soliton solution for the Manakov case [62, 63, 66].

The analysis can be performed for the choice \( k_{1R}, k_{2R} > 0 \) and \( k_{1I} > k_{2I} \). For any other choice the analysis is similar. The study shows that due to collision, the amplitudes of the colliding solitons \( S_1 \) and \( S_2 \) change from 
\[
(\alpha_{1}^{(1)} - k_{1R}, \alpha_{1}^{(2)} - k_{1R})
\]
and 
\[
(\alpha_{2}^{(1)} - k_{2R}, \alpha_{2}^{(2)} - k_{2R})
\]
to 
\[
(\alpha_{1}^{(1)} + k_{1R}, \alpha_{1}^{(2)} + k_{1R})
\]
and 
\[
(\alpha_{2}^{(1)} + k_{2R}, \alpha_{2}^{(2)} + k_{2R})
\], respectively. Here the superscripts in \( \alpha_i \)'s denote the solitons (number(1,2)), the subscripts represent the components (number(1,2)) and '±' signs stand for '\( z \rightarrow \pm \infty \)'. They are defined as [66]

\[
\begin{align*}
\begin{pmatrix} A_1^- \ A_2^- \end{pmatrix} &= \begin{pmatrix} \alpha_{1}^{(1)} \\ \alpha_{1}^{(2)} \end{pmatrix} \frac{e^{-R_{1}/2}}{(k_{1} + k_{1}^{*})}, \quad \text{(2.8a)} \\
\begin{pmatrix} A_1^+ \ A_2^+ \end{pmatrix} &= \begin{pmatrix} \alpha_{2}^{(1)} \\ \alpha_{2}^{(2)} \end{pmatrix} \frac{e^{-R_{2}/2}}{(k_{2} + k_{2}^{*})}, \quad \text{(2.8d)} \\
\end{align*}
\]

All the quantities in the above expressions are given in Eq. (2.6a). The next natural step is to extend this analysis to N-CNLS focusing system. To get the asymptotic forms of 2-soliton solution of the N-CNLS case, as may be checked by a careful asymptotic analysis along the lines of the \( N = 2 \) case, one can simply increase the number of components in the \( A^\pm \) vectors above up to \( N \) (\( A^\pm = (A_1^\pm, A_2^\pm, ..., A_N^\pm)^T \)) by adding two more complex parameters \( \alpha^{(i)}_1, \alpha^{(i)}_2, i = 3, 4, ..., N \), to each of the components so that the forms of the quantities \( e^{R_{1}}, e^{R_{2}}, e^{R_{3}}, e^{\delta_{11}}, e^{\delta_{12}}, e^{\delta_{21}}, e^{\delta_{22}} \) in Eq. (2.6a) remain the same as above except for the
replacement of the range of the summation in $\kappa_{il}$ (Eq. (2.6b)) from $n = 1, 2$ to $n = 1, 2, \ldots, N$.

- **Intensity redistribution**

  The above analysis clearly shows that due to the interaction between two copropagating solitons $S_1$ and $S_2$ in an N-CNLS system, their amplitudes change from $A_j^1 - k_1 R$ and $A_j^2 - k_2 R$ to $A_j^1 + k_1 R$ and $A_j^2 + k_2 R$, $j = 1, 2, \ldots, N$, respectively. However, during the interaction process the total energy of each of the solitons is conserved, that is

  $$\sum_{j=1}^{N} |A_j^{1\pm}|^2 = \sum_{j=1}^{N} |A_j^{2\pm}|^2 = \frac{1}{\mu}.$$  \hspace{1cm} (2.9)

  Note that this is a consequence of the conservation of $L^2$ norm. Another noticeable observation of this interaction process is that one can observe from the equation of motion (1.42a) itself that the intensity of each of the modes is separately conserved, that is

  $$\int_{-\infty}^{\infty} |q_j|^2 dt = \text{constant}, \quad j = 1, 2, \ldots, N.$$  \hspace{1cm} (2.10)

  The above two equations (2.9) and (2.10) ensure that in a two soliton collision process as well as in multisoliton collision processes the total intensity of individual solitons in all the N modes are conserved along with conservation of intensity of individual modes (even while allowing an intensity redistribution). This is a striking feature of the integrable nature of the multicomponent CNLS equations (1.42a). The change in the amplitude of each of the solitons in the $j$th mode can be obtained by introducing the transition matrix $T_j^l$, $j = 1, 2, \ldots, N$, $l = 1, 2$, such that

  $$A_j^{l\pm} = T_j^l A_j^{l\mp}.$$  \hspace{1cm} (2.11a)

  Here the superscripts $l\pm$ represent the solitons designated as $S_1$ and $S_2$. 
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at $z \to \pm \infty$. The form of $T_j^i$'s can be obtained from the asymptotic analysis [66] as

$$T_j^1 = \left( \frac{a_2}{a_2^*} \right) \sqrt{\frac{\kappa_{21}}{\kappa_{12}}} \left[ 1 - \frac{\lambda_2 \left( \frac{a_j^{(j)}}{a_j^{(j)}} \right)}{\sqrt{1 - \lambda_1 \lambda_2}} \right], \quad j = 1, 2, ..., N, \quad (2.11b)$$

where

$$a_2 = (k_2 + k_2^*) \left[ (k_1 - k_2) \sum_{n=1}^{N} \alpha_1^{(n)} \alpha_2^{(n)*} \right]^{1/2}, \quad (2.11c)$$

and

$$T_j^2 = - \left( \frac{a_1}{a_1^*} \right) \sqrt{\frac{\kappa_{21}}{\kappa_{12}}} \left[ \frac{\sqrt{1 - \lambda_1 \lambda_2}}{1 - \lambda_1 \left( \frac{a_j^{(j)}}{a_j^{(j)}} \right)^2} \right], \quad j = 1, 2, ..., N, \quad (2.11d)$$

in which

$$a_1 = (k_1 + k_2^*) \left[ (k_1 - k_2) \sum_{n=1}^{N} \alpha_1^{(n)*} \alpha_2^{(n)} \right]^{1/2}. \quad (2.11e)$$

where $\lambda_1 = \frac{\kappa_{21}}{\kappa_{11}}$ and $\lambda_2 = \frac{\kappa_{12}}{\kappa_{22}}$. Then the intensity exchange in solitons $S_1$ and $S_2$ due to collision can be obtained by taking the absolute square of Eq. (2.11b) and (2.11d), respectively.

The above expressions for the components of the transition matrix implies that in general there is a redistribution of the intensities in the $N$ modes of both the solitons after collision. Only for the special case

$$\frac{\alpha_1^{(1)}}{\alpha_2^{(1)}} = \frac{\alpha_1^{(2)}}{\alpha_2^{(2)}} = ... = \frac{\alpha_1^{(N)}}{\alpha_2^{(N)}}, \quad (2.12)$$

there occurs the standard elastic collision. For all other choices of the
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Figure 2.1: Two distinct possibilities of the shape changing two soliton collision in the integrable 2-CNLS system. The parameters are chosen as (a) $k_1 = 1 + i$, $k_2 = 2 - i$, $\alpha_{1}^{(1)} = \alpha_{1}^{(2)} = \alpha_{2}^{(1)} = 1$, $\alpha_{2}^{(2)} = \frac{39 + 80i}{89}$; (b) $k_1 = 1 + i$, $k_2 = 2 - i$, $\alpha_{1}^{(1)} = 0.02 + 0.1i$, $\alpha_{1}^{(2)} = \alpha_{2}^{(1)} = \alpha_{2}^{(2)} = 1$.

parameters, shape changing (intensity redistribution) collision occurs. For illustrative purpose, we have shown two distinct possibilities of the shape changing two soliton collision in the integrable 2-CNLS system in Fig. 2.1. 

Fig. 2.2 represents the shape changing (intensity redistribution) three soliton collision process for the choice of the parameters, $k_1 = 1 + i$, $k_2 = 1.5 - 0.5i$, $k_3 = 2 - i$, $\alpha_{1}^{(1)} = \frac{39 - 80i}{89}$, $\alpha_{2}^{(1)} = \frac{39 + 80i}{89}$, $\alpha_{3}^{(1)} = 0.3 + 0.2i$, $\alpha_{1}^{(2)} = 0.39$, $\alpha_{2}^{(2)} = \alpha_{3}^{(2)} = 1$. In this figure we have shown the scenario in which the three solitons in the two modes have different amplitudes (intensities) after interaction when compared to the case before interaction. Here $S_1$ is allowed to interact with $S_2$ first and then with $S_3$. Due to this collision, in $q_1$ mode the intensity of $S_1$ is suppressed while that of $S_2$ is enhanced along with suppression of intensity in $S_3$. On the other hand, the reverse scenario occurs in the $q_2$ mode for the three solitons $S_1$, $S_2$, and $S_3$. 


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Figure 2.2: Intensity profiles $|q_1|^2$ and $|q_2|^2$ of the two modes of the three-soliton solution of the 2-CNLS equations.

- **Phase shifts**

  Further, from the asymptotic forms of the solitons $S_1$ and $S_2$, it can be observed that the phases of solitons $S_1$ and $S_2$ also change during a collision process and that the phase shifts are now not only functions of the parameters $k_1$ and $k_2$ but also dependent on $\alpha_i^{(j)}$, $i = 1, 2$, $j = 1, 2, ..., N$. The phase shift suffered by the soliton $S_1$ during collision is

  $$\phi^1 = \frac{(R_3 - R_1 - R_2)}{2} = \left(\frac{1}{2}\right) \ln \left[ \frac{|k_1 - k_2|^2 (\kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21})}{|k_1 + k_2|^2 \kappa_{11}\kappa_{22}} \right], \quad (2.13)$$
Similarly the soliton $S_2$ suffers a phase shift

$$\Phi^2 = -\frac{(R_3 - R_2 - R_1)}{2} = -\Phi^1. \quad (2.14)$$

Then the absolute value of phase shift suffered by the two solitons is

$$|\Phi| = |\Phi^1| = |\Phi^2|. \quad (2.15)$$

Let us consider the case $N = 2$. For a better understanding let us consider the pure elastic collision case ($\alpha^{(1)}_1 : \alpha^{(1)}_2 = \alpha^{(2)}_1 : \alpha^{(2)}_2$) corresponding to parallel modes. Here the absolute phase shift (see Eq. (2.15)) can be obtained as

$$|\Phi| = \left| \ln \left[ \frac{|k_1 - k_2|^2}{|k_1 + k_2^*|^2} \right] \right| = 2 \left| \ln \left[ \frac{|k_1 - k_2|}{|k_1 + k_2^*|} \right] \right|. \quad (2.16)$$

Similarly for the case corresponding to orthogonal modes ($\alpha^{(1)}_1 : \alpha^{(1)}_2 = \infty$, $\alpha^{(2)}_1 : \alpha^{(2)}_2 = 0$) the absolute phase shift is found from Eqs. (2.13)-(2.15) to be

$$|\Phi| = \left| \ln \left[ \frac{|k_1 - k_2|}{|k_1 + k_2^*|} \right] \right|. \quad (2.17)$$

The absolute value of the phase shift takes intermediate values for other choices of the parameters $\alpha^{(j)}_i$’s, $i = 1, 2$, $j = 1, 2, ..., N$. Thus phase shifts do vary depending on $\alpha^{(j)}_i$’s (amplitudes) for fixed $k_i$’s.

- **Relative separation distance**

Ultimately the above phase shifts make the relative separation distance $t_{12}^\pm$ between the solitons (that is, the position of $S_2$ (at $z \to \pm \infty$) minus position of $S_1$ (at $z \to \pm \infty$)) also to vary during collision, depending upon the amplitudes of the modes. The change in the relative separation distance
is found to be
\[ \Delta t_{12} = t_{12}^0 - t_{12}^1 = \frac{(k_{1R} + k_{2R})}{k_{1R}k_{2R}} \phi^j. \]  
(2.18)

Thus as a whole the intensity profiles of the two solitons in different modes as well as the phases and hence the relative separation distance are non-trivially dependent on \( \alpha_j^{(j)} \)'s and vary as a result of soliton interaction.

### 2.4 Multisoliton solutions of multicomponent CNLS equations with focusing nonlinearity

By generalizing one-, two- and three-soliton solutions of N-CNLS equations, one can obtain higher order soliton solutions and also show that the \( n \)-soliton solution of N-CNLS system will be dependent on \( n(N + 1) \) arbitrary complex parameters. For this purpose, in this section we will present the \( n \)-soliton solution in a Gram determinant form and prove its validity.

We write down the multisoliton (\( n \)-soliton) solution of the N-CNLS system (1.42a) with \( \sigma_i = 1 \). For this purpose, we define the following \((1 \times n)\) row matrix \( C_s \), \( s = 1, 2, \ldots, N \), \((N \times 1)\) column matrices \( \psi_j \) and \( \phi \), \( j = 1, 2, \ldots, n \), and the \((n \times n)\) identity matrix \( I \):

\[ C_s = - \left( \alpha_1^{(s)}, \alpha_2^{(s)}, \ldots, \alpha_n^{(s)} \right), \quad 0 = (0, 0, \ldots, 0), \]  
\[ (2.19a) \]

\[ \psi_j = \begin{pmatrix} \alpha_j^{(1)} \\ \alpha_j^{(2)} \\ \vdots \\ \alpha_j^{(N)} \end{pmatrix}, \quad \phi = \begin{pmatrix} e^{\eta_1} \\ e^{\eta_2} \\ \vdots \\ e^{\eta_n} \end{pmatrix}, \quad \sigma = I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \]  
\[ (2.19b) \]

Here \( \alpha_j^{(s)} \), \( s = 1, 2, \ldots, N \), \( j = 1, 2, \ldots, n \), are arbitrary complex parameters and \( \eta_i = k_i t + i k_i^2 z, i = 1, 2, \ldots, n \), are the wave variables, \( k_i \)'s are complex parameters.
and \( \sigma_1 = \sigma_2 = \ldots = \sigma_N = 1 \). In Eq. (2.19b) although \( \sigma \) is nothing but the identity matrix \( I \), we retain '\( \sigma \)' in expression (2.21b) and (2.22) as this will be useful in proving the n-soliton solution for the mixed N-CNLS system later on by redefining the \( \sigma \) matrix. Then we can write down the multisoliton solution of the N-CNLS system as below [66, 99]:

\[
q_s = \frac{g^{(s)}}{f}, \quad s = 1, 2, \ldots, N,
\]

where

\[
g^{(s)} = \begin{vmatrix}
A & I & \phi \\
-I & B & 0^T \\
0 & C_s & 0
\end{vmatrix}, \quad f = \begin{vmatrix}
A & I \\
-I & B
\end{vmatrix}. \tag{2.21a}
\]

Here the matrices \( A \) and \( B \) are defined as

\[
A_{ij} = \frac{e^{\eta_i + \eta_j^*}}{(k_i + k_j^*)}, \quad B_{ij} = \kappa_{ji} = \frac{\psi_i^* \sigma \psi_j}{(k_i^* + k_j)}, \quad i, j = 1, 2, \ldots, n. \tag{2.21b}
\]

In equation (2.21b), \( \dagger \) represents the transpose conjugate.

### 2.4.1 Proof of multisoliton solution of N-CNLS system

Using the rational transformation (2.20) to Eq. (1.42a), we obtain the following bilinear equations,

\[
(iD_z + D_t^2) (g^{(s)} \cdot f) = 0, \quad s = 1, 2, \ldots, N, \tag{2.22a}
\]

\[
D_t^2 (f \cdot f) = 2 \sum_{s=1}^{N} \sigma_s g^{(s)} g^{(s)*}, \tag{2.22b}
\]

We now prove that the Gram determinant forms of \( g^{(s)} \) and \( f \) given above indeed satisfy the above bilinear equations. For this purpose we make use of the derivative formula for the determinants [61], and also the properties of bordered
2.4 Multisoliton solutions of multicomponent CNLS equations with focusing nonlinearity

determinants along with standard elementary properties of determinants [61].

It is interesting to note that from (2.21), the time derivative of \( f \) can be written as a border determinant of the following form,

\[
f_t = \begin{vmatrix}
A & I & \phi \\
-I & B & 0^T \\
-\phi^\dagger & 0 & 0 
\end{vmatrix}.
\] (2.23a)

The above determinant is obtained by introducing border elements to \( f \) thereby increasing the order of the determinant by one. The form of border elements are obtained by careful inspection. The double derivative of \( f \) \((f_{tt})\) with respect to \( t \) can be written as a sum of two border elements whose order is same as that of \( f_t \) thereby keeping the order of the determinant unchanged.

\[
f_{tt} = \begin{vmatrix}
A & I & \phi_t \\
-I & B & 0^T \\
-\phi^\dagger & 0 & 0 
\end{vmatrix} + \begin{vmatrix}
A & I & \phi \\
-I & B & 0^T \\
-\phi^\dagger_t & 0 & 0 
\end{vmatrix}.
\] (2.23b)

In a similar manner, we write the \( z \) derivative of \( f \) \((f_z)\) as a difference of two border determinants and is given by

\[
f_z = i \begin{vmatrix}
A & I & \phi_z \\
-I & B & 0^T \\
-\phi^\dagger & 0 & 0 
\end{vmatrix} - i \begin{vmatrix}
A & I & \phi \\
-I & B & 0^T \\
-\phi^\dagger_z & 0 & 0 
\end{vmatrix}.
\] (2.23c)

As explained above, the derivatives of \( g \) \((g_t, g_{tt}, g_z)\) can also be written as (see also [100]):
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\[
g^{(s)}_t = \begin{vmatrix}
A & I & \phi & \phi_t \\
-I & B & 0^T & 0^T \\
0 & C_s & 0 & 0 \\
0 & 0 & -1 & 0
\end{vmatrix}, \tag{2.23d}
\]

\[
g^{(s)}_{tt} = \begin{vmatrix}
A & I & \phi & \phi_{tt} \\
-I & B & 0^T & 0^T \\
0 & C_s & 0 & 0 \\
0 & 0 & -1 & 0
\end{vmatrix} + \begin{vmatrix}
A & I & \phi & \phi_t \\
-I & B & 0^T & 0^T \\
0 & C_s & 0 & 0 \\
\phi^\dagger & 0 & 0 & 0
\end{vmatrix}, \tag{2.23e}
\]

\[
g^{(s)}_z = \begin{vmatrix}
A & I & \phi & \phi_{tt} \\
-I & B & 0^T & 0^T \\
0 & C_s & 0 & 0 \\
0 & 0 & -1 & 0
\end{vmatrix} - i \begin{vmatrix}
A & I & \phi & \phi_t \\
-I & B & 0^T & 0^T \\
0 & C_s & 0 & 0 \\
\phi^\dagger & 0 & 0 & 0
\end{vmatrix}, \tag{2.23f}
\]

and the conjugate of \( g^{(s)} \) can be written as

\[
g^{(s)*} = -\begin{vmatrix}
A & I & 0^T \\
-I & B & -C_s^\dagger \\
-\phi^\dagger & 0 & 0
\end{vmatrix}. \tag{2.23g}
\]

In the Gram determinant expression of \( f \) (see Eq. (2.21a)), dividing the \( i \)th row by \( e^{\eta_i} \) and multiplying \((N + i)\)th column by \( e^{\eta_i} \) for \( i = 1, 2, \ldots, n \), and dividing \( j \)th column by \( e^{\eta_j} \) and multiplying \((N + j)\)th row by \( e^{\eta_j^*} \) for \( j = 1, 2, \ldots, n \), we obtain another determinant expression of \( f \) (see [100]).

\[
f = \begin{vmatrix}
A' & I \\
-I & B'
\end{vmatrix}, \tag{2.24a}
\]
where the entries of the modified matrices $A'$ and $B'$ are

$$A'_{ij} = \frac{1}{(k_i + k_j^*)}, \quad B'_{ij} = \kappa_{ji} = \frac{e^{(\eta_i^* + \eta_j)(\psi_i^\dagger \sigma \psi_j)}}{(k_i^* + k_j)}, \quad i, j = 1, 2, \ldots, n. \tag{2.24b}$$

In the above form of $f$, the element $B'_{ij}$ is depending upon $z$ and $t$, and all the other are independent of $t$. By using the derivative formula, one can also be write the $t$-derivative of $f$ as

$$f_t = \sum_{1 \leq i, j \leq N} \frac{\partial}{\partial t} \frac{e^{\eta_i^* + \eta_j} (\psi_i^\dagger \sigma \psi_j)}{k_i^* + k_j} \Delta'_{N+i,N+j}, \tag{2.24c}$$

where $\Delta'_{N+i,N+j}$ is the $(i, j)$ th cofactor of $f$, see(2.24a). Therefore we have

$$f_t = \sum_{s=1}^{N} \begin{vmatrix} A' & I & 0^T \\ -I & B' & -\tilde{C}_s^\dagger \end{vmatrix} = \sum_{s=1}^{N} \begin{vmatrix} A & I & 0^T \\ -I & B & -C_s^\dagger \end{vmatrix}, \tag{2.25a}$$

where $\tilde{C}_s = -\left(\alpha_1^{(s)} e^{\eta_1}, \alpha_2^{(s)} e^{\eta_2}, \ldots, \alpha_n^{(s)} e^{\eta_n}\right)$.

By differentiating the above $f_t$ with respect to $t$ again, we get

$$f_{tt} = \sum_{s=1}^{N} \begin{vmatrix} A & I & \phi & 0^T \\ -I & B & 0^T & -C_s^\dagger \end{vmatrix}. \tag{2.25b}$$

Substituting for $g_z^{(s)}, g_t^{(s)}, g_{tt}^{(s)}, f_z, f_t$, and $f_{tt}$ in equation (2.22a), finally we get
This is nothing but a Jacobian identity and hence \( g^{(s)} \) and \( f \) satisfy the first bilinear equation (2.22a). In a similar way one can also check that the second bilinear equation (2.22b) gives rise to the following Jacobian identity for the Gram determinant forms of \( g^{(s)} \) and \( f \):

\[
\begin{align*}
\sum_{s=1}^{N} \left| \begin{array}{ccc}
A & I & \phi \\
-I & B & 0^T \\
0 & C_s & 0 \\
\phi^\dagger & 0 & 0
\end{array} \right| &= \sum_{s=1}^{N} \left| \begin{array}{ccc}
A & I & 0^T \\
-I & B & -C_s^\dagger \\
0 & C_s & 0 \\
\phi^\dagger & 0 & 0
\end{array} \right| - \sum_{s=1}^{N} \left| \begin{array}{ccc}
A & I & \phi \\
-I & B & 0^T \\
0 & C_s & 0 \\
\phi^\dagger & 0 & 0
\end{array} \right|, \\
\sum_{s=1}^{N} \left| \begin{array}{ccc}
A & I & \phi \\
-I & B & 0^T \\
0 & C_s & 0 \\
\phi^\dagger & 0 & 0
\end{array} \right| &= \sum_{s=1}^{N} \left| \begin{array}{ccc}
A & I & 0^T \\
-I & B & -C_s^\dagger \\
0 & C_s & 0 \\
\phi^\dagger & 0 & 0
\end{array} \right| - \sum_{s=1}^{N} \left| \begin{array}{ccc}
A & I & \phi \\
-I & B & 0^T \\
0 & C_s & 0 \\
\phi^\dagger & 0 & 0
\end{array} \right|.
\end{align*}
\]

Thus equations (2.26a) and (2.26b) clearly show that the given Gram determinants \( g^{(s)} \) and \( f \) satisfy the bilinear equations (2.22), which completes the proof of the Gram determinant form of the n-soliton solution of the focusing type N-CNLS system (1.42a).

One can easily check that the one, two and three soliton solutions obtained earlier in the chapter follow directly from the \( n \)-soliton solution expression (2.21).
• One soliton solution:

Specializing to the case of \( n = 1 \) in Eq. (2.21) so that the Gram determinants take the form

\[
g^{(s)} = \begin{vmatrix} A_{11} & 1 & e^{\eta_1} \\ -1 & B_{11} & 0 \\ 0 & -\alpha^{(s)}_1 & 0 \end{vmatrix}, \quad f = \begin{vmatrix} A_{11} & 1 \\ -1 & B_{11} \end{vmatrix},
\]

where \( A_{11} = \frac{e^{n_1 + \eta_i}}{k_1 + k_i^*} \), and \( B_{11} = \kappa_{11} = \frac{\left( \sum_{s=1}^{N} |\alpha^{(s)}_1|^2 \right)}{k_1 + k_i^*} \).

• Two-soliton solution:

To obtain the two soliton solution, we take \( n = 2 \) in Eq. (2.21) and deduce the Gram determinant forms as

\[
g^{(s)} = \begin{vmatrix} A_{11} & A_{12} & 1 & 0 & e^{\eta_1} \\ A_{21} & A_{22} & 0 & 1 & e^{\eta_2} \\ -1 & 0 & B_{11} & B_{12} & 0 \\ 0 & -1 & B_{21} & B_{22} & 0 \\ 0 & 0 & -\alpha^{(s)}_1 & -\alpha^{(s)}_2 & 0 \end{vmatrix}, \quad f = \begin{vmatrix} A_{11} & A_{12} & 1 & 0 \\ A_{21} & A_{22} & 0 & 1 \\ -1 & 0 & B_{11} & B_{12} \\ 0 & -1 & B_{21} & B_{22} \end{vmatrix},
\]

where \( A_{ij} = \frac{e^{n_i + \eta_j}}{k_i + k_j^*} \), and \( B_{ij} = \kappa_{ij} = \frac{\left( \sum_{s=1}^{N} \alpha^{(s)}_j \alpha^{(s)*}_i \right)}{(k_j + k_i^*)}, \quad i, j = 1, 2. \)

• Three-soliton solution:

Similarly for three soliton solution, we put \( n = 3 \) in Eq. (2.21) and the Gram determinants take the forms as
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\[
g^{(s)} = \begin{pmatrix}
A_{11} & A_{12} & A_{13} & 1 & 0 & 0 & e^{\eta_1} \\
A_{21} & A_{22} & A_{23} & 0 & 1 & 0 & e^{\eta_2} \\
A_{31} & A_{32} & A_{33} & 0 & 0 & 1 & e^{\eta_3} \\
-1 & 0 & 0 & B_{11} & B_{12} & B_{13} & 0 \\
0 & -1 & 0 & B_{21} & B_{22} & B_{23} & 0 \\
0 & 0 & -1 & B_{31} & B_{32} & B_{33} & 0 \\
0 & 0 & 0 & -\alpha_1^{(s)} & -\alpha_2^{(s)} & -\alpha_3^{(s)} & 0 \\
\end{pmatrix}, \quad (2.29a)
\]

\[
f = \begin{pmatrix}
A_{11} & A_{12} & A_{13} & 1 & 0 & 0 \\
A_{21} & A_{22} & A_{23} & 0 & 1 & 0 \\
A_{31} & A_{32} & A_{33} & 0 & 0 & 1 \\
-1 & 0 & 0 & B_{11} & B_{12} & B_{13} \\
0 & -1 & 0 & B_{21} & B_{22} & B_{23} \\
0 & 0 & -1 & B_{31} & B_{32} & B_{32} \\
\end{pmatrix}, \quad (2.29b)
\]

where \( A_{ij} = \frac{e^{\eta_i + \eta_j^*}}{k_i + k_j^*} \), and \( B_{ij} = \kappa_{ji} = \frac{\left( \sum_{s=1}^{N} \alpha_j^{(s)} \alpha_i^{(s)*} \right)}{(k_j + k_i^*)}, \) \( i,j = 1, 2, 3 \).