CHAPTER 3

UNIQUENESS THEOREM FOR ENTIRE FUNCTIONS CONCERNING DIFFERENTIAL POLYNOMIALS SHARING SMALL FUNCTIONS
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CONCERNING DIFFERENTIAL POLYNOMIALS
SHARING SMALL FUNCTIONS

In this chapter, we continue the study of meromorphic functions that share small functions and prove several results. As particular cases, we get results due to Fang and Hua [12] and results due to M. L. Fang and W. Hong [13] and Milloux inequality.

3.1 INTRODUCTION AND MAIN RESULTS

Here, we need the following definition

Let \( f(z) \) and \( g(z) \) be non-constant transcendental entire functions. If \( f(z) - P(z) \) and \( g(z) - P(z) \) assume the same zeroes with the same multiplicities then we call that \( f(z) \) and \( g(z) \) share the function \( P(z) \) CM, where \( P(z) \) is an entire small function.

M.L. Fang and X.H. Hua [12] obtained the following unicity theorem.

**Theorem 3.1.A:** Let \( f(z) \) and \( g(z) \) be two transcendental entire functions. 

\( n \geq 6 \) be a positive integer. If \( f^n f' \) and \( g^n g' \) share \( 1 \) CM, then either \( f^n f' g^n g' = 1 \) or \( f = cg \) for a constant \( c \) with \( c^{n+1} = 1 \).

Above theorem is improved by Chung-Chun Yang and Xinhou Hua [43] by proving the unicity theorem 1.1.A.(stated in chapter 1)

Ming–Liang Fang and Wei-Hong [13] also obtained another unicity theorem.
Theorem 3.1.B : Let \( f(z) \) and \( g(z) \) be two transcendental entire functions. \( n \geq 11 \) be a positive integer. If \( f^n(z)[f(z)-1]f'(z) \) and \( g^n(z)[g(z)-1]g'(z) \) share \( 1 \) CM, then \( f(z) = g(z) \).

In this chapter, we obtain significant improvements of the above theorems.

Theorem 3.1.1 : Let \( f(z) \) and \( g(z) \) be two transcendental entire functions. \( n \) be a positive integer. If \( f^n f' \) and \( g^n g' \) share an entire small function \( P(z) \) CM and \( f - q \) and \( g - q \) have no zeroes, where
\[
F = \frac{f^{n+1}}{n+1}, \quad G = \frac{g^{n+1}}{n+1}
\]
and \( Q'(z) = P(z) - c, \quad c \neq 0 \). Then either \( F - Q = c_1 e^{kz} \) and \( G - Q = c_2 e^{-kz} \) or \( f = g \), where \( c_1, c_2 \) and \( k \) are constants such that \(-c_1 c_2 k^2 = c^2\).

Remark 3.1.1 : If \( Q = 0 \) and \( c = a \), then \( p(z) = a \) and the above theorem reduces to Theorem 2.1.B, a result of Fang and Hau [12].

Theorem 3.1.2 : Let \( f(z) \) and \( g(z) \) be two transcendental entire functions. \( n \) is a positive integer. If \( f^n(f-1)f' \) and \( g^n(g-1)g' \) share an entire small function \( P(z) \) CM and \( f - q \) and \( g - q \) have no zeroes, where
\[
F = \frac{f^{n+2}}{n+2} - \frac{f^{n+1}}{n+1}, \quad G = \frac{g^{n+2}}{n+2} - \frac{g^{n+1}}{n+1}
\]
and \( Q'(z) = P(z) - c, \quad c \neq 0 \), then either \( F - Q = c_1 e^{kz} \) and \( G - Q = c_2 e^{-kz} \) or \( f = g \), where \( c_1,c_2 \) and \( k \) are being constants such that \(-c_1 c_2 k^2 = c^2\).

Remark 3.1.2 : If \( Q = 0 \) and \( c = 1 \), then \( P(z) = 1 \) and the above result reduces to Theorem 3.1.B, a result of Ming-Liang Fang and Wei Hong [13].
3.2 LEMMAS

We require the following lemmas for the proof of Theorem 3.1.1 and Theorem 3.1.2.

Lemma 3.2.1: Suppose that $f(z)$ is meromorphic and non-constant in the plane. Then

$$T(r, f) < N(r, f) + (k + 1)N\left(r, \frac{1}{f - Q(z)}\right) - N\left(r, \frac{1}{f^{(k)} - P(z)}\right) + S(r, f),$$

where $P(z)$ and $Q(z)$ are small entire functions such that

$$(Q(z))^{(k)} = P(z) - c, \quad c \neq 0 \quad \text{and} \quad N_0\left(r, \frac{1}{f^{(k+1)} - P'(z)}\right)$$

is the function which only counts those points such that $f^{(k+1)} - P'(z) = 0$ and $f - Q \neq 0$.

Further, if $f(z)$ is an entire function, then

$$T(r, f) < (k + 1)N\left(r, \frac{1}{f - Q(z)}\right) - N\left(r, \frac{1}{f^{(k+1)} - P'(z)}\right) + S(r, f).$$

Proof: Let $\Psi = f^{(k)} - P(z) + c = f^{(k)} - Q^{(k)}$

Using Second Fundamental Theorem (q = 3), we have

$$m(r, \Psi) + m(r, \frac{1}{\Psi}) + m(r, \frac{1}{\Psi - c}) < 2T(r, \Psi) - N(t, \Psi) + S(r, f), \quad \ldots(3.2.1)$$

where

$$N_1(r, \Psi) = N(r, \frac{1}{\Psi}) + 2N(r, \Psi') - N(r, \Psi')$$

and $c \neq 0$.

Now

$$2T(r, \Psi) - N_1(r, \Psi) = m(r, \Psi) + m(r, \frac{1}{\Psi - c}) + N(r, \Psi) + N(r, \frac{1}{\Psi - c}) - N(r, \frac{1}{\Psi})$$

$$- 2N(r, \Psi') + N(r, \Psi') + O(1)$$

$$\ldots(3.2.2)$$

Again at a pole of $\Psi(z)$ of order $k$, $\Psi'(z)$ has a pole of order $k + 1$, and such poles of $\Psi(z)$ occur only at poles of $f(z)$. 
Hence
\[ N(r, \Psi') - N(r, \Psi) = \overline{N}(r, \Psi) \leq \overline{N}(r, f) + S(r, f). \]

Again, we have
\[ S(r, \Psi) = O[T(r, \Psi)] = O[T(r, f)], \]
outside a set of finite linear measure, so that \( S(r, \Psi) = S(r, f). \)

Thus equations (3.2.1) and (3.2.2) yield
\[ m(r, \frac{1}{\Psi}) < \overline{N}(r, f) + N(r, \frac{1}{\Psi - c}) - N(r, \frac{1}{\Psi}) + S(r, f). \quad \ldots (3.2.3) \]

Again we have
\[ T[r, f - Q(z)] = m(r, \frac{1}{f - Q(z)}) + N(r, \frac{1}{f - Q(z)}) + S(r, f) \]
\[ \leq m\left(r, \frac{1}{f(k) - Q(k)(z)}\right) + m\left(r, \frac{f(k) - Q(k)(z)}{f - Q(z)}\right) + N\left(r, \frac{1}{f - Q(z)}\right) + S(r, f) \]
\[ \leq m\left(r, \frac{1}{f(k) - Q(k)(z)}\right) + m\left(r, \frac{f(k) - Q(k)(z)}{f - Q(z)}\right) + N\left(r, \frac{1}{f - Q(z)}\right) + S(r, f) \]

Or,
\[ T(r, f) \leq m\left(r, \frac{1}{\Psi}\right) + N\left(r, \frac{1}{f - Q(z)}\right) + S(r, f). \]

From (3.2.3), we get
\[ T(r, f) < \overline{N}(r, f) + N\left(r, \frac{1}{f - Q(z)}\right) + N\left(r, \frac{1}{\Psi - c}\right) - N\left(r, \frac{1}{\Psi}\right) + S(r, f). \]
\[ \leq \overline{N}(r, f) - (k + 1) \overline{N}\left(r, \frac{1}{f - Q(z)}\right) + \overline{N}\left(r, \frac{1}{f(k) - P(z)}\right) - N_o\left(r, \frac{1}{f(k+1) - P'(z)}\right) + S(r, f) \]

If \( f(z) \) is an entire function, then
\[ T(r, f) < (k + 1) \overline{N}\left(r, \frac{1}{f - Q(z)}\right) + \overline{N}\left(r, \frac{1}{f(k) - P(z)}\right) - N_o\left(r, \frac{1}{f(k+1) - P'(z)}\right) + S(r, f) \]
...(3.2.4) where \( P(z) = Q^{(k)} + c, \quad c \neq 0. \)

**Remark 3.2.1:** If \( Q(z) = 0 \), then above theorem reduces to Lemma 1.2.1 (Milloux inequality).

**Lemma 3.2.2:** Let \( f(z) \) and \( g(z) \) be two transcendental entire functions. \( k \) be a positive integer. If \( f^{(k)} \) and \( g^{(k)} \) share the small entire function \( P(z) \) CM and

\[ \Theta(Q.f) + \Theta(Q.g) > \frac{2k+3}{k+2}, \]  

...(3.2.5)

then either
\[ \left( f^{(k)} - P(z) + c \right) \left( g^{(k)} - P(z) + c \right) = c^2 \quad \text{or} \quad f \equiv g \]

where \( P(z) = Q^{(k)} + c \) and \( c \neq 0. \)

**Remark 3.2.2:** If \( c = 1 \) and \( Q = 0 \), then \( P(z) = 1 \), then we get a result of Ming-Liang Fang and Wei-Hong [13, Lemma 4].

**Proof of lemma 3.2.2:** Let
\[
\Phi(z) = \frac{f^{(k+2)} - P^*}{f^{(k+1)} - P^*} - \frac{2(f^{(k+1)} - P^*)}{f^{(k)} - P} - \frac{g^{(k+2)} - P^*}{g^{(k+1)} - P^*} + \frac{2(g^{(k+1)} - P^*)}{g^{(k)} - P} \quad \text{...(3.2.6)}
\]

Clearly \( m(r, \Phi) = S(r, f) + S(r, g) \).

Suppose \( \Phi(z) \equiv 0. \)

If \( z_0 \) is a common simple zero of \( f^{(k)} - P \) and \( g^{(k)} - P \), then \( z_0 \) is a zero of \( \Phi(z) \). Thus, we have
\[ N_1\left(r, \frac{1}{f^{(k)} - P}\right) = N_1\left(r, \frac{1}{g^{(k)} - P}\right) \leq N\left(r, \frac{1}{\Phi}\right) \]

\[ \leq T(r, \Phi) + O(1) \leq N(r, \Phi) + S(r, f) + S(r, g). \]

...(3.2.7)

Here \( N_1\left(r, \frac{1}{f^{(k)} - P}\right) \) is the counting function of the simple zeros of \( f^{(k)} - P \) in \( \{z : |z| \leq r\} \).

By our assumptions \( \Phi(z) \) has poles only at zeros of \( f^{(k+1)} - P' \) and \( g^{(k+1)} - P' \) and from (3.2.6), we deduce that

\[ N(r, \Phi) \leq N\left(r, \frac{1}{f - Q}\right) + N\left(r, \frac{1}{g - Q}\right) + N_0\left(r, \frac{1}{f^{(k+1)} - P'}\right) + N_0\left(r, \frac{1}{g^{(k+1)} - P'}\right), \]

...(3.2.8)

where \( N_0\left(r, \frac{1}{f^{(k+1)} - P'(z)}\right) \) is the counting function which only counts those zeros of \( f^{(k+1)} - P'(z) \) but not the zeros of \( f - Q \), where \( Q^{(k)} = P(z) - c, \text{ and } c \neq 0. \)

Since \( f^{(k)} \) and \( g^{(k)} \) share \( P(z) \), we have

\[ N\left(r, \frac{1}{f^{(k)} - P(z)}\right) + N\left(r, \frac{1}{g^{(k)} - P(z)}\right) = 2N\left(r, \frac{1}{f^{(k)} - P(z)}\right) \]

\[ \leq N_1\left(r, \frac{1}{f^{(k)} - P(z)}\right) + N\left(r, \frac{1}{f^{(k)} - P(z)}\right), \]

...(3.2.9)

By Lemma 3.2.1, we have.

\[ T(r, g) \leq (k + 1) N\left(r, \frac{1}{g - Q}\right) + N\left(r, \frac{1}{g^{(k)} - P(z)}\right) - N_0\left(r, \frac{1}{g^{(k+1)} - P'}\right) + S(r, g) \]

...(3.2.10)
Thus, we deduce, from (3.2.7) to (3.2.10), that
\[
T(r,f) + T(r,g) \leq (k + 2) N \left( r - \frac{1}{f - Q} \right) + (k + 2) N \left( r - \frac{1}{g - Q} \right) + S(r,f) + S(r,g) * 
\]
...(3.2.11)

Without loss of generality we suppose there exists a set I with infinite measure such that \( T(r,f) \leq T(r,g) \) and \( r \in I \).

Since
\[
N \left( r, \frac{1}{f^{(k)} - P(z)} \right) \leq T(r,f^{(k)}) \leq m(r,f^{(k)}) + N(r,f^{(k)}) 
\]
\[
\leq m \left( r, \frac{f^{(k)}}{f} \right) + m(r,f) + O(1) \leq T(r,f) + S(r,f) 
\]

We obtain from (3.2.11), that
\[
T(r,g) \leq (k + 2) N \left( r - \frac{1}{f - Q} \right) + (k + 2) N \left( r - \frac{1}{g - Q} \right) + S(r,g) 
\]
\[
\leq (k + 2)[2 - \Theta(Q,f) - \Theta(Q,g) + \varepsilon]T(r,g) + S(r,g) 
\]
...(3.2.12)

for \( r \in I \).

Thus, we obtain from (3.2.12) and given condition (3.2.5) that \( T(r,g) \leq S(r,g) \) for \( r \in I \), a contradiction.

Hence, we have \( \Phi(z) = 0 \)

That is
\[
\frac{f^{(k+2)} - P'}{f^{(k+1)} - P'} - \frac{2(f^{(k+1)} - P')}{f^{(k)} - P} = \frac{g^{(k+2)} - P'}{g^{(k+1)} - P'} - \frac{2(g^{(k+1)} - P')}{g^{(k)} - P} 
\]
...(3.2.13)

By solving this, we obtain
\[
\frac{1}{f^{(k)} - P(z)} = \frac{bg^{(k)} + a - bP(z)}{g^{(k)} - P(z)} 
\]
...(3.2.14)
Or,
\[ \frac{1}{f^{(k)} - Q^{(k)} - c} = \frac{b}{g^{(k)} - Q^{(k)} + a - bc} \frac{g^{(k)} - Q^{(k)} - c}{g^{(k)} - Q^{(k)} - c}, \]

where \( a \) and \( b \) are two constants and \( a \neq 0 \).

Next, we consider three cases.

**Case 1**: \( b \neq 0 \) and \( a = bc \)

From (3.2.14), we obtain
\[ g^{(k)} - Q^{(k)} \neq 0 \]

Thus there exists an entire function \( h(z) \) such that
\[ g^{(k)} - Q^{(k)} = e^{h(z)} \quad \text{and} \quad f^{(k)} = Q^{(k)} + c + \frac{1}{b} \frac{e^{-h}}{bc} \quad \ldots(3.2.15) \]

If \( bc = -1 \), then
\[ f^{(k)} - Q^{(k)} = c + \frac{1}{b} - \frac{c}{bc h} = \frac{bc + 1}{b} - \frac{c}{bc h} = -\frac{c}{bc h} \]
\[ f^{(k)} - Q^{(k)} = -\frac{c}{be^h} = \frac{-c^2}{b e^h} = \frac{c^2}{e^h} \]

Therefore,
\[ [f^{(k)} - Q^{(k)}] e^h = c^2 \]
\[ \Rightarrow [f^{(k)} - Q^{(k)}] [g^{(k)} - Q^{(k)}] = c^2 \]
\[ \Rightarrow [f^{(k)} - P(z) + c] [g^{(k)} - P(z) + c] = c^2 \]

If \( bc \neq -1 \), from (3.2.15), we get
\[ f^{(k)} - Q^{(k)} - \left( c + \frac{1}{b} \right) = \frac{-c}{b} e^{-h} \neq 0 \]

From Lemma 3.2.1, since \( c + \frac{1}{b} \neq 0 \), we get
\[ T(r, f) \leq (k + 1) N \left( r, \frac{1}{f - Q} \right) + S(r, f) \]
\[ \leq (k + 1) [\Theta(Q, f) + 1 - \Theta(Q, g)] T(r, f) + S(r, f) \]
Or,

\[ \Theta(Q, f) + \Theta(Q, g) \leq \frac{2k + 1}{k + 1} T(r, f) \leq S(r, f) \]

Since by hypothesis \( \Theta(Q, f) + \Theta(Q, g) > \frac{2k + 3}{k + 2} \), we get

\[ T(r, f) \leq S(r, f), \quad \text{a contradiction.} \]

**Case 2: \( b \neq 0 \) and \( a \neq bc \)**

Then from (3.2.14), we have \( g^{(k)} - Q^{(k)} + \frac{a - bc}{b} \neq 0 \).

Since \( \frac{a - bc}{b} \neq 0 \), and using Lemma 3.2.1, we deduce that

\[ T(r, g) \leq (k+1) N \left( r, \frac{1}{g - Q} \right) + S(r, g) \]

As in case 1, we get a contradiction.

**Case 3: \( b = 0 \)**, on integrating (3.2.14) \( k \) times, we get

\[ f - Q = \frac{g - Q}{a} + p(z), \]

where \( p(z) \) is a polynomial of degree \( k \).

If \( p(z) \neq 0 \), then by Second Fundamental Theorem

\[ T(r, g - Q) \leq N \left( r, \frac{1}{g - Q} \right) + N \left( r, \frac{1}{g - Q + ap(z)} \right) + S(r, g) \]

Or

\[ T(r, g) \leq N \left( r, \frac{1}{g - Q} \right) + N \left( r, \frac{1}{f - Q} \right) + S(r, g) \]

\[ \leq \left[ 1 - \Theta(Q, g) + 1 - \Theta(Q, f) \right] T(r, g) + S(r, g) \]

As in case 2 and 3, we get a contradiction.

Therefore

\[ P(z) = 0 \]
That is 

\[ f - Q = \frac{g - Q}{a} \]

If \( a \neq 1 \), then in view of \( f - Q = \frac{g - Q}{a} \), \( f^{(k)} \) and \( g^{(k)} \) do not share the function 

\[ P(z) = Q^{(k)} + c, \] which is a contradiction.

Thus we get 
\[ a = 1 \]

That is 

\[ f - Q \equiv g - Q \]

Implies 
\[ f = g. \]

Thus, either 
\[ \left| f^{(k)} - P(z) + c \right| \left| g^{(k)} - P(z) + c \right| = c^2 \quad \text{or} \quad f = g. \]

This completes the proof of Lemma 3.2.2.

### 3.3 PROOF OF THEOREMS

#### PROOF OF THEOREM 3.3.1

We have 
\[ F = \frac{f^{n+1}}{n+1} \quad \text{and} \quad G = \frac{g^{n+1}}{n+1} \quad \Rightarrow \quad F' = f^n f' \quad \text{and} \quad G' = g^n g' \]

Let \( F' \) and \( G' \) share the function \( P(z) \) **CM**. Since \( F - Q \) and \( G - Q \) have no zeros, we get 
\[ \Theta(Q,F) = \lim_{r \to \infty} \frac{N\left(r, \frac{1}{F' - Q} \right)}{T(r,F)} = 1 \]

Similarly 
\[ \Theta(Q,G) = 1 \]

Therefore 
\[ \Theta(Q,F) + \Theta(Q,G) = 2 > \frac{5}{3} \quad \ldots (3.3.1) \]

Since \( F' \) and \( G' \) share the small function \( P(z) \) **CM**, then by Lemma 3.2.2, we get 

\[ \text{either} \quad \left[ F' - P(z) + c \right] \left[ G' - P(z) + c \right] = c^2, \quad c \neq 0 \quad \text{or} \quad F = G \]
If \([F' - P(z) + c][G' - P(z) + c] = c^2\) holds

That is \([F' - Q'][G' - Q'] = c^2\) ...(3.3.2)

Since \(F - Q\) and \(G - Q\) have no zeros, we get

\[F - Q = e^{h_1(z)} \quad \text{and} \quad G - Q = e^{h_2(z)},\]

where \(h_1(z)\) and \(h_2(z)\) are entire functions.

Then, \(F' - Q' = e^{h_1(z)} h_1'(z)\) and \(G' - Q' = e^{h_2(z)} h_2'(z)\).

From (3.3.2), we get

\[e^{h_1(z)} h_1'(z) e^{h_2(z)} h_2'(z) = c^2\] \ ...(3.3.3)

Implies \(h_1'(z) \neq 0\) and \(h_2'(z) \neq 0\).

Therefore \(h_1'(z) = e^{h_3(z)}\) and \(h_2'(z) = e^{h_4(z)},\)

where \(h_3(z)\) and \(h_4(z)\) are entire functions. From (3.3.3),

\[e^{h_1(z)} e^{h_2(z)} e^{h_3(z)} e^{h_4(z)} = c^2\]

\[e^{h_1(z) + h_2(z) + h_3(z) + h_4(z)} = c^2.\]

Differentiating this gives

\[h_1'(z) + h_2'(z) + h_3'(z) + h_4'(z) = 0\]

\[e^{h_3(z)} + e^{h_4(z)} + h_3'(z) + h_4'(z) = 0\]

Using Lemma 2.2.4, we deduce that

\(h_3(z) = h_4(z) = (2m + 1)\pi i\) for some integer \(m.\)

Inserting this in the above equality, we deduce that

\(h_3'(z) = h_4'(z) = 0\)

Therefore \(h_1'(z)\) and \(h_2'(z)\) are constants and hence \(h_1(z)\) and \(h_2(z)\) are linear in \(z.\)
In view of (3.3.2), we get, \( F - Q = c_1 e^{kz} \) and \( G - Q = c_2 e^{-kz} \),
where \(-c_1 c_2 k^2 = c^2\).

Next we consider another case \( F = G \)

\[
\Rightarrow \frac{f^{n+1}}{n+1} = \frac{g^{n+1}}{n+1} \Rightarrow f^{n+1} = g^{n+1}
\]

\( \Rightarrow f = h g \), where \( h^{n+1} = 1 \) i.e. \( h \) is \((n+1)^{th}\) root of unity.

\[
\Rightarrow f^{n+1} = h^{n+1} g^{n+1}
\]

Differentiating both sides w.r.t to \( z \), we get

\[
f^n f' = h^{n+1} g^n g' \quad \forall \ z \in \mathbb{C}, \ \forall \ n \geq 0 \quad \ldots(i)
\]

\[\therefore f^2 f' = h^3 g^2 g' \quad \text{for} \quad n = 2, \ \forall \ z \in \mathbb{C} \quad \ldots(ii)
\]

\[\therefore f^3 f' = h^4 g^3 g' \quad \text{for} \quad n = 3, \ \forall \ z \in \mathbb{C} \quad \ldots(iii)
\]

Suppose \( f^2 f' \) and \( g^2 g' \) share the value \( P(z) \) at \( z_0 \) say, then we have

\[
f^2(z_0) f'(z_0) = g^2(z_0) g'(z_0) = P(z_0) \quad \ldots(iv)
\]

where \( P(z_0) \neq 0 \).

From (ii), we can write \( f^2(z_0) f'(z_0) = h^3 g^2(z_0) g'(z_0) \quad \ldots(v)
\]

From (iv) and (v), we obtain \( h^3 = 1 \quad \ldots(vi)\)

Suppose \( f^3 f' \) and \( g^3 g' \) share the value \( P(z) \) at \( z_1 \) say, then we have

\[
f^3(z_1) f'(z_1) = g^3(z_1) g'(z_1) = P(z_1) , \quad \ldots(vii)
\]

where \( P(z_1) \neq 0 \).

From (iii), we can write as \( f^3(z_1) f'(z_1) = h^4 g^3(z_1) g'(z_1) \quad \ldots(viii)
\]

From (vii) and (viii), we get \( h^4 = 1 \). Implies \( h \cdot h^3 = 1 \).
From (vi) \( h.1 = 1 \). Implies \( h = 1 \).

Therefore \( f = hg \Rightarrow f = g \).

Thus either \( F - Q = c_1 e^{kz} \) and \( G - Q = c_2 e^{-kz} \) or \( f = g \).

This completes the proof of Theorem 3.1.1.

**JUSTIFICATION OF REMARK 3.1.1**

If \( Q = 0 \) and \( c = a \) in the theorem 3.1.1, then we get \( F = c_1 e^{kz} \) and \( G = c_2 e^{-kz} \), where \( c_1, c_2 \) and \( k \) are constants such that \(-c_1c_2k^2 = e^2\).

That is \( F = \frac{f_{n+1}}{n+1} = c_1 e^{kz} \) and \( G = \frac{g_{n+1}}{n+1} = c_2 e^{-kz} \)

\[ \Rightarrow f = [(n+1)c_1]^{(e^{(k/n+1)})z} = d_1 e^{cz}, \quad \text{where} \quad d_1 = [(n+1)c_1]^{kn+1} \] and \( g = [(n+1)c_2]^{(e^{-(k/n+1)})z} = d_2 e^{-cz}, \quad \text{where} \quad d_2 = [(n+1)c_2]^{kn+1} \) and \( c_3 = (k/n+1) \)

and \( (d_1d_2)^n c_3^2 = -1 \) gives \(-c_1c_2k^2 = e^2\).

**PROOF OF THEOREM 3.1.2**

Here

\[ F = \frac{f_{n+2}}{n+2} - \frac{f_{n+1}}{n+1} \quad \text{and} \quad G = \frac{g_{n+2}}{n+2} - \frac{g_{n+1}}{n+1} \]

\[ \Rightarrow F' = f^n (f-1) f' \quad \text{and} \quad G' = g^n (g-1) g' \]

Let \( F' \) and \( G' \) share the function \( P(z) \text{ CM} \). Since \( F - Q \) and \( G - Q \) have no zeroes, we get

\[ \Theta(Q, F) = 1 - \lim_{r \to \infty} N(r, \frac{1}{F-Q}) \]

Similarly

\[ \Theta(Q, G) = 1 \]

Therefore

\[ \Theta(Q, F) + \Theta(Q, G) = 2 > \frac{5}{3} \]
By Lemma 3.2.2, we get either
\[ F' - P(z) + c = G' - P(z) + c = c^2 \]
or
\( F = G \)

If \( [F' - P(z) + c] [G' - P(z) + c] = c^2 \) holds, then equivalently
\[ [F' - Q'] [G' - Q'] = c^2. \] ...(3.3.4)

Since \( F - Q \) and \( G - Q \) have no zeros, we get
\[ F - Q = e^{h_1(z)} \quad \text{and} \quad G - Q = e^{h_2(z)} \]
where \( h_1(z) \) and \( h_2(z) \) are entire functions.

Differentiating, we get \( F' - Q' = e^{h_1(z)} h_1'(z) \) and \( G' - Q' = e^{h_2(z)} h_2'(z) \)

As in proof of theorem 3.1.1, we get \( F - Q = c_1 e^{kz} \) and \( G - Q = c_2 e^{-kz} \)

Next we consider another case \( F = G \)

That is
\[ f^{n+1} \left( \frac{1}{n+2} - \frac{1}{n+1} \right) = g^{n+1} \left( \frac{1}{n+2} g - \frac{1}{n+1} \right) \]

Let \( \frac{f}{g} = h. \) If \( h \neq 1, \) we have \( g = \frac{(n + 2)(1 + h + \ldots + h^n)}{(n + 1)(1 + h + \ldots + h^{n+1})} \)

Thus we deduce by Picard’s theorem that \( h(z) \) is a constant. Hence \( g(z) \) is a constant, a contradiction. Therefore we deduce that \( h(z) = 1, \) that is \( f(z) = g(z). \)

Thus \textbf{either} \( F - Q = c_1 e^{kz} \) and \( G - Q = c_2 e^{-kz} \) \textbf{or} \( f = g, \) where \( c_1, c_2 \) and \( k \) are such that \(-c_1c_2k^2 = c^2.\)

This completes the proof of Theorem 3.1.2.
JUSTIFICATION OF REMARK 3.1.2:

If \( Q = 0 \) and \( c = a \) in the theorem 3.1.2, then from equation (3.3.4), we get

\[
F' G' \equiv c^3 \quad \Rightarrow \quad f''(f-1)f' g'^*(g-1)g' \equiv c^3.
\]

Since \( f \) and \( g \) are entire functions, we deduce that \( f \neq 0,1,\infty \), a contradiction.

Hence \( F = G \), which gives \( f = g \).

3.4 APPLICATIONS

Solving any non-linear differential equation presents a very difficult problem. As an application of our result, we present the derivations of the meromorphic solutions of the following non-linear differential equation.

\[
y^n y' - P(z) = e^{\alpha(z)} \left[ e^{\beta(z)} - c \right]. \quad n \geq 0, \quad \ldots(3.4.1)
\]

where \( \alpha \) and \( \beta \) are entire functions. If \( e^{\beta(z)} \) can be represented as \( G' - Q' \).

That is

\[
G' - Q' = e^{\beta(z)}, \quad \ldots(3.4.2)
\]

where \( G = \frac{g^{n+1}}{n+1} \) for some entire functions \( g \) and \( Q' = P(z) - c \).

That is

\[
g^n g' - P(z) + c = e^{\beta(z)}. \quad \ldots(3.4.3)
\]

Or

\[
g^n g' - P(z) = e^{\beta(z)} - c. \quad \ldots(3.4.3)
\]

Using (3.4.3) in (3.4.1), we see that \( y^n y' \) and \( g^n g' \) share the polynomial \( P(z) \) CM. We have

\[
G' - Q' = e^{\beta(z)}.
\]

Integrating, we get

\[
G - Q = \int e^{\beta(z)} dz.
\]
If \( G - Q \neq 0 \), then there exists an entire function \( \gamma \) such that
\[
G - Q = \int e^{\beta(z)} dz = e^{\gamma(z)}
\]
Differentiating, we get
\[
e^{\beta(z)} = e^{\gamma(z)} \gamma'(z)
\]
Therefore
\[
\gamma'(z) = e^{\beta(z) - \gamma(z)} \tag{3.4.4}
\]
Therefore \( \gamma'(z) \neq 0 \)

This implies that \( \gamma'(z) \) has no zeroes and thus there exists an entire function \( \delta(z) \) such that
\[
\gamma'(z) = e^{\delta(z)} \tag{3.4.5}
\]

Using (3.4.4) and (3.4.5), we get
\[
\beta(z) = \gamma(z) + \delta(z) + 2k\pi i
\]
Therefore
\[
\beta'(z) = \gamma'(z) + \delta'(z) \tag{3.4.6}
\]
Or,
\[
\beta'(z) = e^{\delta(z)} + \delta'(z) \tag{3.4.7}
\]

Now, by Theorem 3.1.1, the possible solution for the equation (3.4.1) is

**either** \( y = g \) or \( Y - Q = c_1 e^{kz} \) and \( G - Q = c_2 e^{-kz} \), \( \ldots (3.4.8) \)

where \( c_1, c_2 \) and \( k \) are constants and \( Y = \frac{y^{n+1}}{n+1} \)

In the second case if \( \gamma \) is linear then \( \gamma = k_1 z + k_2 \Rightarrow \gamma' = k_1 \).

From (3.4.5), we get \( e^{\gamma(z)} = k_1 \)

Differentiating, we get \( e^{\delta(z)} \delta'(z) = 0 \Rightarrow \delta'(z) = 0 \).

From (3.4.6), we get \( \beta'(z) = k_1 \).

From (3.4.8), we get \( G' - Q' = -c_2 k e^{-kz} \)

Comparing this with (3.4.2), we get \( \beta' = -k \) or \( k = -\beta' \)
Combining the above discussions, we have the following result.

**Theorem 3.4.1:** Suppose that for a non-linear differential equation of (3.4.1) with \( n \geq 0 \), there exists an entire function \( \delta \) such that (3.4.7) holds. Then every solution of (3.4.1) is of the form

\[
y = \left( (n+1) \left[ P(z) + c \right] \right)^{\frac{1}{n+1}}
\]

Or

\[
Y = c_2 e^{(\delta)z} + \left[ P(z) + c \right] dz,
\]

where \( Y = \frac{y^{n+1}}{n+1} \).

**Remark 3.4.1:** If \( \beta \) is polynomial then by (3.4.7), \( \delta \) can only be constant and hence \( \beta' \) is constant. Therefore \( \beta \) is linear.