CHAPTER 2

UNIQUENESS AND VALUE-SHARING OF MEROMORPHIC FUNCTIONS
CHAPTER 2
UNIQUENESS AND VALUE-SHARING OF MEROMORPHIC FUNCTIONS *

In this chapter, we prove two theorems on the uniqueness of nonlinear differential polynomials. As a particular case, we get a result of C.C.Yang and Xinhou [43] and one of which considerably improves a result of Indrajit Lahiri and Nintu Mandal [24]

2.1 INTRODUCTION

Let \( f(z) \) be a non constant transcendental meromorphic function in the whole complex plane. Let \( 'a' \) be a finite complex number and \( k \) a positive integer.

We denote by \( N_{k}\left(r,\frac{1}{f-a}\right) \) the counting function for zeros of \( f(z)-a \) with multiplicity \( \leq k \), and by \( \overline{N}_{k}\left(r,\frac{1}{f-a}\right) \) the corresponding one for which each zero of \( f(z)-a \) with multiplicity \( \leq k \), is counted once. Let \( N_{k}\left(r,\frac{1}{f-a}\right) \) be the counting function for zeros of \( f(z)-a \) with multiplicity at least \( k \) and \( \overline{N}_{k}\left(r,\frac{1}{f-a}\right) \) the corresponding one for which each zero of \( f(z)-a \) with multiplicity at least \( k \) is counted once.

* The results in this chapter have been accepted for publication in “Computers and mathematics with Applications”.
“Elsevier Publication”
We define

\[ \delta_k(a, f) = 1 - \lim_{r \to \infty} \frac{N_k\left( r, \frac{1}{f-a} \right)}{T(r, f)}. \]

Hayman [19] and Clunie [9] proved the following result

**Theorem 2.1.A**: Let \( f(z) \) be a transcendental entire function, \( n \geq 1 \) be a positive integer, then \( f^n f' = 1 \) has infinitely many solutions.

Fang and Hua [12] obtained a unicity theorem corresponding to the above result.

**Theorem 2.1.B**: Let \( f(z) \) and \( g(z) \) be two non-constant entire functions, \( n \geq 6 \) a positive integer. If \( f^n(z)f'(z) \) and \( g^n(z)g'(z) \) share 1 CM, then either \( f(z) = c_1e^{cz}, g(z) = c_2e^{-cz} \), where \( c_1, c_2 \) and \( c \) are three constants satisfying \( (c_1c_2)^{n+1}c^2 = -1 \) or \( f(z) = t g(z) \) for a constant \( t \) such that \( t^{n+1} = 1 \).

Hennekemper [20], Chen [7] and Wang [39, 40] extended Theorem 2.1.A by proving the following theorem.

**Theorem 2.1.C**: Let \( f(z) \) be a transcendental entire function, \( n, k \) two positive integers with \( n \geq k + 1 \). Then \( \left( f^n(z) \right)^{(k)} = 1 \) has infinitely many solutions.

Ming-Liang Fang [11] obtained a unicity theorem corresponding to Theorem 2.1.C.

**Theorem 2.1.D**: Let \( f(z) \) and \( g(z) \) be two non-constant entire functions, and let \( n, k \) be two positive integers with \( n > 2k + 4 \). If \( \left[ f^n(z) \right]^k \) and \( \left[ g^n(z) \right]^k \) share 1 CM, then either \( f(z) = c_1e^{cz}, g(z) = c_2e^{-cz} \), where
\(c_1, c_2 \text{ and } c \text{ are three constants satisfying } (-1)^k (c_1 c_2)^n (nc)^{2k} = 1 \text{ or } f(z) = t g(z) \text{ for a constant } t \text{ such that } t^n = 1.\)

Indrajit Lahiri and Nintu Mandal [24] proved the following result.

**Theorem 2.1.E**: Let \(f\) and \(g\) be two transcendental meromorphic functions such that \(\Theta(\infty, f) + \Theta(\infty, g) > 4/(n+1)\) and let \(n \geq 17\) be an integer. If \(E_{2k} (f; fn(f-1)f') = E_{2k} (g; gn(g-1)g')\), then \(f \equiv g\). Here we denote by \(E_k (a; f)\) the set of all \(a\)-points of \(f\) with multiplicities not exceeding \(k\), where an \(a\)-point is counted according to its multiplicity.

Ming-Liang Fang [11] proved the following theorem

**Theorem 2.1.F**: Let \(f(z)\) be a transcendental entire function, \(n, k\) two positive integers with \(n \geq k + 2\). Then \([f^n(f-1)f'] = 1\) has infinitely many solutions.

Ming-Liang Fang [11] also obtained a corresponding unicity theorem corresponding to Theorem 2.1.F as follows.

**Theorem 2.1.G**: Let \(f(z)\) and \(g(z)\) be two non-constant entire functions, and let \(n, k\) two positive integers with \(n \geq k + 8\). If \([f^n(f-1)f]^k)\) and \([g^n(g-1)f]^k)\) share \(1\) CM, then \(f(z) \equiv g(z)\)

In this chapter, we extend Theorem 2.1.C and Theorem 2.1.D to meromorphic functions by proving

**Theorem 2.1.1**: Let \(f(z)\) be a transcendental meromorphic function and let \(n, k\) be two positive integers with \(n \geq k + 3\). Then \(f^n(z)^{(k)} = 1\) has infinitely many solutions.
In view of Theorem 2.1.1, Theorem 2.1.2 naturally motivates us to the following theorem.

**Theorem 2.1.2**: Let \( f(z) \) and \( g(z) \) be two non-constant meromorphic functions, and let \( n, k \) be two positive integers with \( n > 3k + 8 \). If \( \left[f^n(z)\right]^{(k)} \) and \( \left[g^n(z)\right]^{(k)} \) share 1 CM, then either \( f(z) = c_1 e^{cz}, g(z) = c_2 e^{-cz} \), where \( c_1, c_2 \) and \( c \) are three constants satisfying \((-1)^k (c_1 c_2)^n (nc)^{2k} = 1\) or \( f(z) = t g(z) \) for a constant \( t \) such that \( t^n = 1 \).

**Remark**: Let \( k = 1 \), Then by Theorem 2.1.2, we get Theorem 1.1.1A, a result of C.C. Yang and Xinhou Hua [43].

We extend theorem 2.1.3 for transcendental meromorphic functions as follows.

**Theorem 2.1.3**: Let \( f(z) \) be a transcendental meromorphic function, \( n, k \) two positive integers with \( n > k + 3 \). Then \( \left[f^n (f - 1)\right]^{(k)} = 1 \) has infinitely many solutions.

Theorem 2.1.3, naturally motivates us to think of the following unicity theorem.

**Theorem 2.1.4**: Let \( f(z) \) and \( g(z) \) be two non-constant meromorphic functions satisfying \( \Theta(\infty, f) > \frac{3}{n + 1} \) and let \( n, k \) be two positive integers with \( n \geq 3k + 13 \). If \( \left[f^n (f(z) - 1)\right]^{(k)} \) and \( \left[g^n (g(z) - 1)\right]^{(k)} \) share 1 CM, then \( f(z) \equiv g(z) \).

**Remark**: Let \( k = 1 \) in the theorem 2.1.4, we get a result which is an improvement of theorem 2.1.5.
2.2 USEFUL LEMMAS

For the proof of our results we need the following lemmas.

**Lemma 2.2.1:** Let \( f(z) \) be a non constant meromorphic function, \( k \) a positive integer, and let \( c \) be a nonzero finite complex number. Then

\[
T(r, f) \leq \tilde{N}(r, f) + N_{k+1} \left( r, \frac{1}{f} \right) + \tilde{N} \left( r, \frac{1}{f^{(k)} - c} \right) - N_{0} \left( r, \frac{1}{f^{(k+1)}} \right) + S(r, f),
\]

where \( N_{0} \left( r, \frac{1}{f^{(k+1)}} \right) \) is the counting function which only counts those points such that \( f^{(k+1)} = 0 \) but \( f^{(k)}(z) - c \neq 0 \).

**Proof:** By Lemma 1.2.1 (Stated in the Chapter 1) and by the definition of \( N_{k+1} \left( r, \frac{1}{f} \right) \), easily we get the above inequality.

**Lemma 2.2.2** [16]: Let \( f(z) \) be a non-constant entire function, and let \( k \geq 2 \) be a positive integer. If \( f(z)f^{(k)}(z) \neq 0 \), then \( f = e^{az+b} \), where \( a \neq 0, b \) are constants.

We now prove the following lemma which plays a cardinal role in the proof of Theorem 2.1.2 and Theorem 2.1.4.

**Lemma 2.2.3:** Let \( f(z) \) and \( g(z) \) be two meromorphic functions, and let \( k \) be a positive integer. If \( f^{(k)} \) and \( g^{(k)} \) share the value \( 1 \) CM and

\[
\Delta = [(k+2) \Theta(\infty, f) + 2 \Theta(\infty, g) + \Theta(0, f) + \Theta(0, g) + \delta_{+1}(0, f) + \delta_{+1}(0, g)] > k + 7,
\]

then either \( f^{(k)} = g^{(k)} \equiv 1 \) or \( f \equiv g \).

**Proof:** Let \( \Phi(z) = \frac{f^{(k+2)}}{f^{(k+1)}} - 2 \frac{f^{(k+1)}}{f(k)} - \frac{g^{(k+2)}}{g^{(k+1)}} + 2 \frac{g^{(k+1)}}{g(k)} \).

\[
\Phi(z) = \frac{f^{(k+2)}}{f^{(k+1)}} - 2 \frac{f^{(k+1)}}{f(k)} - \frac{g^{(k+2)}}{g^{(k+1)}} + 2 \frac{g^{(k+1)}}{g(k)}.
\]
Suppose \( \Phi(z) \neq 0 \).

If \( z_0 \) is a common simple 1-point of \( f^{(k)}(z) \) and \( g^{(k)}(z) \), substituting their Taylor series at \( z_0 \) into (2.2.2), we see that \( z_0 \) is a zero of \( \Phi(z) \).

Thus, we have
\[
N_1 \left( r, \frac{1}{f^{(k)}(z) - 1} \right) = N_1 \left( r, \frac{1}{g^{(k)}(z) - 1} \right) \leq \bar{N} \left( r, \frac{1}{\Phi} \right) \leq T(r, \Phi) + O(1) \leq N(r, \Phi) + S(r, f) + S(r, g) \quad \ldots(2.2.3)
\]

here \( N_1 \left( r, \frac{1}{f^{(k)}(z) - 1} \right) \) is the counting function which only counts those points such that \( f^{(k)}(z) - 1 = 0 \) but \( f^{(k+1)}(z) \neq 0 \).

By our assumptions, \( \Phi(z) \) has simple poles only at zeros of \( f^{(k+1)}(z) \) and \( g^{(k+1)}(z) \) and poles of \( f \) and \( g \). Thus, we deduce from (2.2.2) that
\[
N(r, \Phi) \leq \bar{N}(r, f) + \bar{N}(r, g) + \bar{N} \left( r, \frac{1}{f} \right) + \bar{N} \left( r, \frac{1}{g} \right) + N_0 \left( r, \frac{1}{f^{(k+1)}} \right) + N_0 \left( r, \frac{1}{g^{(k+1)}} \right) \quad \ldots(2.2.4)
\]

here \( N_0 \left( r, \frac{1}{f^{(k+1)}} \right) \) has the same meaning as in Lemma 2.2.1. Obviously,
\[
\bar{N} \left( r, \frac{1}{f^{(k)}(z) - 1} \right) + \bar{N} \left( r, \frac{1}{g^{(k)}(z) - 1} \right) = 2 \bar{N} \left( r, \frac{1}{f^{(k)}(z) - 1} \right) \leq N_1 \left( r, \frac{1}{f^{(k)}(z) - 1} \right) + N \left( r, \frac{1}{f^{(k)}(z) - 1} \right) . \quad \ldots(2.2.5)
\]

From Lemma 2.1.1, we have
\[
T(r, f) \leq \bar{N}(r, f) + N_{k+1} \left( r, \frac{1}{f} \right) + \bar{N} \left( r, \frac{1}{f^{(k)}(z) - c} \right) - N_0 \left( r, \frac{1}{f^{(k+1)}} \right) + S(r, f) , \quad \ldots(2.2.6)
\]
\[
T(r, g) \leq \bar{N}(r, g) + N_{k+1} \left( r, \frac{1}{g} \right) + \bar{N} \left( r, \frac{1}{g^{(k)}(z) - c} \right) - N_0 \left( r, \frac{1}{g^{(k+1)}} \right) + S(r, g) . \quad \ldots(2.2.7)
\]

Thus, we deduce from (2.2.3)-(2.2.7) that
\[
T(r, f) + T(r, g) \leq 2 \bar{N}(r, f) + 2 \bar{N}(r, g) + N_{k+1} \left( r, \frac{1}{f} \right) + N_{k+1} \left( r, \frac{1}{g} \right) + \bar{N} \left( r, \frac{1}{f} \right) + \bar{N} \left( r, \frac{1}{g} \right) + S(r, f) + S(r, g) . \quad \ldots(2.2.8)
\]
Since
\[ N\left(r, \frac{1}{f^{(k)}}\right) \leq T\left(r, f^{(k)}\right) = m\left(r, f^{(k)}\right) + N\left(r, f^{(k)}\right) \]
\[ \leq m\left(r, f\right) + m\left(r, \frac{f^{(k)}}{f}\right) + N\left(r, f\right) + k \overline{N}\left(r, f\right) \]
\[ \leq T\left(r, f\right) + k \overline{N}\left(r, f\right) + S\left(r, f\right). \]

We obtain from (2.2.8) that
\[ T\left(r, f\right) \leq (k + 2)\overline{N}\left(r, f\right) + 2\overline{N}\left(r, g\right) + N_{\kappa}\left(r, \frac{1}{f}\right) + N_{\tilde{\kappa}}\left(r, \frac{1}{g}\right) \]
\[ \leq T\left(r, g\right) + k \overline{N}\left(r, f\right) + S\left(r, f\right) + S\left(r, g\right). \]

Without loss of generality, we suppose that there exists a set \( I \) with infinite measure such that
\[ T\left(r, f\right) < T\left(r, g\right) \text{ for } r \in I. \]
Hence
\[ T\left(r, g\right) < \left(1 - \delta_{\kappa}\right)T\left(r, f\right) + 2 \overline{N}\left(r, f\right) + \left(1 - \Theta\left(0, f\right)\right) + \left(1 - \Theta\left(0, g\right)\right) + \varepsilon \]
\[ \leq T\left(r, g\right) + S\left(r, g\right), \]
\[ T\left(r, g\right) \leq \left\{ \delta_{\kappa}, \Theta\left(0, f\right) + \left(1 - \Theta\left(0, g\right)\right) + \varepsilon \right\} T\left(r, g\right) + S\left(r, g\right), \]
\[ \text{for } r \in I \text{ and } 0 < \varepsilon < \Delta - (k + 7), \]
\[ \text{i.e. } \Delta - (k + 7) \leq 0 \]
\[ \text{i.e. } \Delta \leq k + 7, \]
which is a contradiction to our hypothesis \( \Delta > k + 7 \) from (2.2.1).

Hence, we get \( \Phi(z) = 0 \); that is,
\[ \frac{f^{(k+2)}}{f^{(k+1)}} - 2 \frac{f^{(k+1)}}{f^{(k)} - 1} = \frac{g^{(k+2)}}{g^{(k+1)}} - 2 \frac{g^{(k+1)}}{g^{(k)} - 1}. \]
\[ \text{(2.2.10)} \]

Integrating this equation, we get
\[ \log f^{(k+1)} - 2 \log (f^{(k)} - 1) = \log g^{(k+1)} - 2 \log (g^{(k)} - 1) + \log a, \]
where \( a \) is constant and \( a \neq 0 \). That is
\[ \log \frac{f^{(k+1)}}{(f^{(k)} - 1)^2} = \log \frac{ag^{(k+1)}}{(g^{(k)} - 1)^2}. \]
\[ \Rightarrow \frac{f^{(k+1)}}{(f^{(k)} - 1)^2} = \frac{ag^{(k+1)}}{(g^{(k)} - 1)^2}. \]
Again integrating above equation, we get
\[
\frac{1}{f^{(k)}-1} = \frac{a}{g^{(k)}-1} - b,
\]
where \( b \) is a constant.

Solving above equation, we get
\[
\frac{1}{f^{(k)}-1} = \frac{b g^{(k)} + a - b}{g^{(k)}-1}, \quad \text{(2.2.11)}
\]
where \( a, b \) are two constants and \( a \neq 0 \).

Next, we consider three cases

**Case 1:** \( a = b \). From (2.2.11),

(i) If \( b = -1 \), then \( f^{(k)}(z) g^{(k)}(z) \equiv 1 \).

(ii) If \( b \neq -1 \), then

\[
\frac{1}{f^{(k)}-1} = \frac{b g^{(k)}}{g^{(k)}-1} \Rightarrow \frac{1}{f^{(k)}} = \frac{b g^{(k)}}{(1+b)g^{(k)}-1}, \quad \text{...(2.2.12)}
\]

We can write
\[
\overline{N} \left[ r, \left( \frac{1}{g^{(k)} - (1/(1+b))} \right) \right] \leq \overline{N} \left[ r, \left( \frac{g^{(k)}}{g^{(k)} - (1/(1+b))} \right) \right]. \quad \text{...(2.2.13)}
\]

From (2.2.12), we have
\[
\overline{N} \left[ r, \left( \frac{g^{(k)}}{g^{(k)} - (1/(1+b))} \right) \right] = \overline{N} \left[ r, \left( \frac{1}{f^{(k)}} \right) \right]. \quad \text{...(2.2.14)}
\]

From (2.2.13) and (2.2.14), we get
\[
\overline{N} \left[ r, \left( \frac{1}{g^{(k)} - (1/(1+b))} \right) \right] \leq \overline{N} \left[ r, \left( \frac{1}{f^{(k)}} \right) \right]. \quad \text{...(2.2.15)}
\]

By First fundamental theorem and Lemma of logarithmic derivative, we obtain the following inequality
\[ N\left( r, \frac{1}{f(k)} \right) \leq N\left( r, \frac{f}{f(k)} \right) + \overline{N}\left( r, \frac{1}{f} \right) \leq T\left( r, \frac{f}{f(k)} \right) + \overline{N}\left( r, \frac{1}{f} \right) \]

\[ \leq T\left( r, \frac{f}{f(k)} \right) + \overline{N}\left( r, \frac{1}{f} \right) + S(r, f) \]

\[ \leq N\left( r, \frac{f}{f} \right) + m\left( r, \frac{f}{f(k)} \right) + \overline{N}\left( r, \frac{1}{f} \right) + S(r, f) \]

\[ \leq k \overline{N}(r, f) + \overline{N}\left( r, \frac{1}{f} \right) + S(r, f) \]

\[ \leq (k+2)\overline{N}(r, f) + \overline{N}\left( r, \frac{1}{f} \right) + S(r, f). \]

Therefore \[ \overline{N}\left( r, \frac{1}{f(k)} \right) \leq (k+2)\overline{N}(r, f) + \overline{N}\left( r, \frac{1}{f} \right) + S(r, f). \quad \ldots(2.2.16) \]

From (2.2.15) and (2.2.16), we get

\[ \overline{N}\left[ r, \left( \frac{1}{g^{(k)} - (1/(1+b))} \right) \right] \leq \overline{N}\left( r, \frac{1}{f(k)} \right) \]

\[ \leq (k+2)\overline{N}(r, f) + \overline{N}\left( r, \frac{1}{f} \right) + S(r, f). \]

From Lemma 2.1.1, we have

\[ T(r, g) \leq \overline{N}(r, g) + N_{r\mapsto g}\left( \frac{1}{g} \right) + \overline{N}\left( r, \frac{1}{g} - c \right) - N_{r\mapsto g}\left( \frac{1}{g^{(k)}} \right) \]

\[ \leq \overline{N}(r, g) + N_{r\mapsto g}\left( \frac{1}{g} \right) + \overline{N}\left( r, \frac{1}{g^{(k)}} - (1/(1+b)) \right) + S(r, g) \quad \text{since } c = 1/(1+b) \neq 0 \]

\[ \leq \overline{N}(r, g) + N_{r\mapsto g}\left( \frac{1}{g} \right) + (k+2)\overline{N}(r, f) + \overline{N}\left( r, \frac{1}{f} \right) + S(r, f) + S(r, g) \]

\[ \leq (k+2)\overline{N}(r, f) + 2\overline{N}(r, g) + N_{k+1}\left( \frac{1}{f} \right) + N_{k+1}\left( \frac{1}{g} \right) + \overline{N}\left( r, \frac{1}{f} \right) + \overline{N}\left( r, \frac{1}{g} \right) + S(r, f) + S(r, g). \]
That is \( T(r, g) \leq (k + 8 - \Delta)T(r, g) + S(r, g) \),
for \( r \in I \) and \( r \) sufficiently large.

That is \( (\Delta - k - 7)T(r, g) \leq S(r, g) \).

Hence, by (2.2.1), we deduce that \( T(r, g) \leq S(r, g) \), a contradiction.

**Case 2:** \( b \neq 0 \) and \( a \neq b \). Then from (2.2.11),

(i) If \( b = -1 \), we obtain \( f^{(k)} = \frac{a}{-g^{(k)} + a + 1} \).

Therefore \( \bar{N}\left(r, \frac{1}{g^{(k)} - (1 + a)}\right) = \bar{N}(r, f^{(k)}) = \bar{N}(r, f) \).

From Lemma 2.1.1, we have

\[
T(r, g) \leq \bar{N}(r, g) + N_{k+1} \left( r, \frac{1}{g} \right) + \bar{N} \left( r, \frac{1}{g^{(k)} - (1 + a)} \right) + S(r, g)
\]

\[
\leq \bar{N}(r, g) + N_{k+1} \left( r, \frac{1}{g} \right) + \bar{N}(r, f) + S(r, g)
\]

\[
\leq (k + 2)\bar{N}(r, f) + 2\bar{N}(r, g) + N_{k+1} \left( r, \frac{1}{f} \right) + N_{k+1} \left( r, \frac{1}{g} \right) + N_{k+1} \left( r, \frac{1}{f} \right) + \bar{N} \left( r, \frac{1}{g} \right) + S(r, g).
\]

Using the argument as in case 1, we get a contradiction.

(ii) If \( b \neq -1 \), we obtain that \( f^{(k)} - (1 + 1/b) = \frac{-a}{b^2 \left[ g^{(k)} + a - b \right]} \).

Therefore \( \bar{N} \left( r, \left( \frac{1}{g^{(k)} + a - b} \right) \right) = \bar{N} \left( r, f^{(k)} - (1 + 1/b) \right) = \bar{N}(r, f^{(k)}) = \bar{N}(r, f) \).
By Lemma 2.1.1, we have

\[
T(r, g) \leq \overline{N}(r, g) + N_{k+1}\left(\frac{r}{g} + \frac{1}{g(k + a-b)}\right) - N_0\left(\frac{r}{g(k+1)}\right) + S(r, g).
\]

\[
\leq \overline{N}(r, g) + N_{k+1}\left(\frac{r}{g}\right) + \overline{N}(r, f) + S(r, g).
\]

\[
\leq (k + 2)\overline{N}(r, f) + 2\overline{N}(r, g) + N_{k+1}\left(\frac{r}{f}\right) + N_{k+1}\left(\frac{r}{g}\right) + \overline{N}(r, f) + S(r, f) + S(r, g).
\]

Using the argument as in case 1, we get a contradiction.

**Case 3:** \(b = 0\). From (2.2.11), we obtain

\[
f = \frac{1}{a}g + p(z), \quad \ldots(2.2.17)
\]

where \(p(z)\) is a polynomial. If \(p(z) \neq 0\), then by Second fundamental theorem for three small functions, we have

\[
T(r, f) \leq \overline{N}(r, f) + \overline{N}\left(\frac{r}{f}\right) + \overline{N}\left(\frac{r}{f-p}\right) + S(r, f)
\]

\[
\leq \overline{N}(r, f) + \overline{N}\left(\frac{r}{f}\right) + \overline{N}\left(\frac{r}{g}\right) + S(r, f). \quad \ldots(2.2.18)
\]

From (2.2.17), we obtain

\[
T(r, f) = T(r, g) + S(r, f).
\]

Hence, substituting this into (2.2.18), we get

\[
T(r, f) \leq \left\{3 - [\Theta(\infty, f) + \Theta(0, f) + \Theta(0, g)] + 0.5T(r, f) + S(r, f)\right\}, \quad \ldots(2.2.19)
\]

where \(0 < \varepsilon < 1 - \delta_{\epsilon_1}(0, f) + 1 - \delta_{\epsilon_1}(0, g) + 2[1 - \Theta(\infty, g)] + (k + 1)[1 - \Theta(\infty, f)]\).

Therefore

\[
T(r, f) \leq \left\{k + 8 - \Delta\right\}T(r, f) + S(r, f). \quad \text{That is}
\]

\[
[\Delta - k - 7] T(r, f) \leq S(r, f).
\]
Hence, by (2.2.1), we deduce that $T(r, f) \leq S(r, f)$, a contradiction.

Therefore, we deduce that $p(z) = 0$, that is $f = \frac{1}{a} g$. \hfill \ldots (2.2.20)

If $a \neq 1$, then $f^{(k)}$ and $g^{(k)}$ sharing the value 1 CM, we deduce from (2.2.20) that $g^{(k)} \neq 1$. That is $\overline{N} \left( r, \frac{1}{g^{(k)} - 1} \right) = 0$.

Next, we can deduce a contradiction as in case 3. Thus, we get that $a = 1$, that is, $f \equiv g$. Thus proof of Lemma 2.2.3 is completed.

**Lemma 2.2.4** [18]: Suppose that $f_1(z), f_2(z), \ldots, f_n(z)$ ($n \geq 2$) are meromorphic functions and $g_1(z), g_2(z), \ldots, g_n(z)$ are entire functions satisfying the following conditions
1. $f_j(z) e^{g_j(z)} = 0 \quad (i)$
2. $g_j(z) - g_k(z)$ are not constants for $1 \leq j < k \leq n \quad (ii)$
3. For $1 \leq j \leq n, 1 \leq h < k \leq n, \ T(r, f_j) = o(T(r, e^{g_h - g_k}))$ $(r \to \infty, r \notin E)$. \hfill (iii)

Then $f_j(z) = 0 \quad (j = 1, 2, \ldots, n)$

**Remark 2.2.1**: In fact, we only need to assume that the growth condition of Lemma 2.2.4 holds on a set of values of infinite linear measure.

### 2.3 PROOF OF THEOREM 2.1.1

From Theorem 0.2.9 and Lemma 1.2.1, we have

\[
T(r, f) = T(r, f^n(z)) \\
\leq N \left( r, \frac{1}{f^n(z)} \right) + N \left( r, \frac{1}{f^n(z)} \right) + N \left( r, f^n(z) \right) - N \left( r, \left( f^n(z) \right)^{k+i} \right) + S(r, f)
\]
\[ \leq \overline{N}(r,f) + n N\left(\frac{1}{f}\right) - (n-k-1) N\left(\frac{1}{f}\right) + N\left(\frac{1}{f^n(z)} - 1\right) + S(r,f) \]

\[ \leq \overline{N}(r,f) + (k+1) N\left(\frac{1}{f}\right) + N\left(\frac{1}{f^n(z)} - 1\right) + S(r,f) \]

\[ \leq (k+2) T(r,f) + N\left(\frac{1}{f^n(z)} - 1\right) + S(r,f). \]

Therefore
\[ (n-k-2) T(r,f) \leq N\left(\frac{1}{f^n(z)} - 1\right) + S(r,f). \] ... (2.3.1)

Hence, we deduce from (2.3.1) and \( n \geq k + 3 \) that \( f^n(z)\) has infinitely many solutions.

### 2.4 PROOF OF THEOREM 2.1.2

Consider \( F(z) = f^n(z) \) and \( G(z) = g^n(z) \). We have

\[ \Delta = [(k+2)\Theta(\infty, F) + 2\Theta(\infty, G) + \Theta(0, F) + \Theta(0, G) + \delta_{n+1}(0, F) + \delta_{n+1}(0, G)]. \] ... (2.4.1)

Consider \( \Theta(0, F) = 1 - \lim_{r \to \infty} \frac{\overline{N}\left(\frac{1}{f}\right)}{T(r,f)} = 1 - \lim_{r \to \infty} \frac{\overline{N}\left(\frac{1}{f^n}\right)}{n T(r,f)}. \)

\[ = 1 - \lim_{r \to \infty} \frac{\overline{N}\left(\frac{1}{f^n}\right)}{n T(r,f)} \geq 1 - \lim_{r \to \infty} \frac{T(r,f)}{n T(r,f)}. \]

i.e
\[ \Theta(0, F) \geq \frac{n-1}{n}. \] ... (2.4.2)

Similarly
\[ \Theta(0, G) \geq \frac{n-1}{n}. \] ... (2.4.3)

Consider \( \Theta(\infty, F) = 1 - \lim_{r \to \infty} \frac{\overline{N}(r,F)}{T(r,F)} = 1 - \lim_{r \to \infty} \frac{\overline{N}(r,F^n)}{n T(r,f)} \)

\[ = 1 - \lim_{r \to \infty} \frac{\overline{N}(r,F^n)}{n T(r,f)} \geq 1 - \lim_{r \to \infty} \frac{T(r,f)}{n T(r,f)}. \]
\[ \Theta(\infty, F) \geq \frac{n-1}{n} \quad \text{(2.4.4)} \]

Similarly
\[ \Theta(\infty, G) \geq \frac{n-1}{n} \quad \text{(2.4.5)} \]

Next, we have (see page 27)
\[ N_k\left( r, \frac{1}{f-a} \right) = \overline{N}\left( r, \frac{1}{f-a} \right) + \overline{N}\left( r, \frac{1}{f-a} \right) + \ldots + \overline{N}\left( r, \frac{1}{f-a} \right). \]
\[ \delta_{k+1}(a, f) = 1 - \lim_{r \to \infty} \frac{N_k\left( r, \frac{1}{f-a} \right)}{T(r, f)} \geq 1 - \lim_{r \to \infty} \frac{(k+1)\overline{N}\left( r, \frac{1}{f-a} \right)}{T(r, f)}. \]
\[ \delta_{k+1}(0, F) = 1 - \lim_{r \to \infty} \frac{(k+1)\overline{N}\left( r, \frac{1}{f} \right)}{nT(r, f)} \geq 1 - \frac{(k+1)}{n} = \frac{n-(k+1)}{n}. \quad \text{(2.4.6)} \]

Similarly, \[ \delta_{k+1}(0, G) \geq \frac{n-(k+1)}{n}. \quad \text{(2.4.7)} \]

From (2.4.1) - (2.4.7), we get
\[ \Delta \geq \frac{n-1}{n} + \frac{n-1}{n} + (k+2) \frac{(n-1)}{n} + 2 \left[ \frac{n-1}{n} \right] + \frac{n-(k+1)}{n} + \frac{n-(k+1)}{n}. \]

Since \( n > 3k + 8 \), we get \( \Delta > k + 7 \).

Considering \( F^{(k)}(z) = [f^n(z)]^{(k)} \) and \( G^{(k)}(z) = [g^n(z)]^{(k)} \), then by condition of theorem 2.1.2, \( F^{(k)}(z) \) and \( G^{(k)}(z) \) share the value ICM and \( F \) and \( G \) satisfies conditions of Lemma 2.2.3, then by Lemma 2.2.3, we deduce that either \( F^{(k)}G^{(k)} \equiv 1 \) or \( F \equiv G \).

Next, we consider two cases.

**Case 1:** \( F^{(k)}(z)G^{(k)}(z) \equiv 1; \) that is \( [f^n(z)]^{(k)} [g^n(z)]^{(k)} = 1. \quad \text{(2.4.8)} \)
We prove that \( f \neq 0, \infty \) and \( g \neq 0, \infty \).

...(2.4.9)

Suppose that \( f(z) \) has a zero \( z_0 \) of order \( p \), then \( z_0 \) is a zero of \( [f^n(z)]^{(k)} \) of order \((3k + k_1)p - k = 3pk + k_1p - k\) and \( z_0 \) is a pole of \( [g^n(z)]^{(k)} \) of order \((3k + k_1)q + k = 3kq + k_1q + k\), where \( k_1 > 8 \). From (2.4.8), we get

\[
3pk + k_1p - k = 3kq + k_1q + k
\]

i.e

\[
3k(p - q) + k_1(p - q) = 2k
\]

i.e

\[
(3k + k_1)(p - q) = 2k
\]

which is impossible since \( p \) and \( q \) are integers and \( k_1 > 8 \).

Therefore \( f \neq 0 \) and \( g \neq 0 \). Similarly \( f \neq \infty \) and \( g \neq \infty \).

Therefore \( f \neq 0, \infty \) and \( g \neq 0, \infty \). \hspace{1cm} ... (2.4.10)

From (2.4.8), (2.4.10), we get \( [f^n(z)]^{(k)} \neq 0 \) and \( [g^n(z)]^{(k)} \neq 0 \). \hspace{1cm} ... (2.4.11)

From (2.4.8), (2.4.9), (2.4.11) and Lemma 2.2.2, we get for \( k \geq 2 \) that

\[
f(z) = c_1e^{cz}, \quad g(z) = c_2e^{-cz},
\]

where \( c_1, c_2 \) and \( c \) are three constants satisfying

\[
(-1)^k (c_1c_2)^n(nc)^{2k} = 1
\]

Next, we consider \( [f^n(z)]^{(k)} [g^n(z)]^{(k)} = 1 \) for the case \( k = 1 \).

That is

\[
n^2 f^{n-1}f' g^{n-1}g' = 1.
\]

...(2.4.12)

We prove that \( f \neq 0, \infty \) and \( g \neq 0, \infty \). \hspace{1cm} ...(2.4.13)

In fact, suppose that \( f \) has a zero \( z_0 \) with order \( p \). Then \( z_0 \) is a pole of \( g(z) \)
(with order \( q \) say), by (2.4.12), we get

\[
(n - 1)p + p - 1 = q(n - 1) + q + 1
\]

\[
n(p - q) = 2,
\]

which is impossible since \( p \) and \( q \) are integers and \( n > 3k + 8 = 11 \).

Therefore \( f \neq 0 \) and \( g \neq 0 \). Similarly \( f \neq \infty \) and \( g \neq \infty \).

Therefore \( f \neq 0, \infty \) and \( g \neq 0, \infty \).
Thus there exist two entire functions $\alpha(z)$ and $\beta(z)$ such that

$$f(z) = e^{\alpha(z)} \quad \text{and} \quad g(z) = e^{\beta(z)} \quad \text{...(2.4.14)}$$

Inserting these in (2.4.12), we get

$$n^2 \alpha' \beta' e^{n(\alpha+\beta)} = 1 \quad \text{...(2.4.15)}$$

Thus $\alpha'$ and $\beta'$ have no zeros and we may set

$$\alpha' = e^{\delta(z)} \quad \text{and} \quad \beta' = e^{\gamma(z)} \quad \text{...(2.4.16)}$$

Then (2.4.15) reduces to

$$n^2 e^{n(\alpha+\beta)+\delta+\gamma} = 1$$

Differentiating this gives

$$n(\alpha'+\beta')+\delta'+\gamma' = 0; \text{ that is } n(e^\delta+e^{\beta'})+\delta'+\gamma' = 0.$$ 

From (2.4.16), we get

$$n(e^{\delta-\gamma}+1)e^{\beta} + \alpha' e^{\delta} + \beta' e^{-\gamma} = 0.$$ 

By Lemma 2.2.4, we get

$$n(\delta-\gamma+1) = 0 \quad \text{i.e} \quad e^{\delta-\gamma} = -1$$

i.e. $\delta - \gamma = (2m+1)\pi i \quad \text{i.e} \quad \delta = \gamma + (2m+1)\pi i$ for some integer $m$.

Inserting this in the above equality, we deduce that $\delta' = \gamma' = 0$, and so $\delta$ and $\gamma$ are constants, i.e. $\alpha'$ and $\beta'$ are constants \text{...(2.4.17)}

From (2.4.12), (2.4.13), (2.4.14) and (2.4.17), we obtain

$$f(z) = c_1 e^{cz}, \quad g(z) = c_2 e^{-cz},$$

where $c_1, c_2$ and $c$ are three constants satisfying $(c_1c_2)^2(n^2) = -1$.

Therefore for the case 1, i.e. $F^{(k)}G^{(k)} = 1$, for all $k \geq 1$, we get

$$f(z) = c_1 e^{cz}, \quad g(z) = c_2 e^{-cz}, \quad \text{where } c_1, c_2 \text{ and } c \text{ are three constants satisfying } (-1)^k (c_1c_2)^n (nc)^{2k} = 1.$$

\textbf{Case 2:} $F(z) = G(z)$; that is $f^n(z) = g^n(z)$.

This implies $f = ig$, where $t^n = 1$ i.e. $t$ is $n^{th}$ root of unity.
2.5 PROOF OF THEOREM 2.1.3

From Theorem 0.2.9 and Lemma 1.2.1, we have
\[(n + 1)T(r, f) = T[r, f^n(f - 1)] + S(r, f)\]
\[\leq N(r, f^n(f - 1)) + N\left(r, \frac{1}{f^n(f - 1)}\right)\]
\[+ N\left(r, \frac{1}{[f^n(f - 1)]^{(a) - 1}}\right) - N\left(r, \frac{1}{[f^n(f - 1)]^{(a) + 1}}\right) + S(r, f)\]
\[\leq N(r, f) + n N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f - 1}\right)\]
\[+ N\left(r, \frac{1}{(f^n(f - 1))^{(a) - 1}}\right) - [n - (k + 1)]N\left(r, \frac{1}{f}\right) + S(r, f)\]
\[\leq (k + 3)T(r, f) + N\left(r, \frac{1}{[f^n(f - 1)]^{(a) - 1}}\right) + S(r, f).\]

Therefore
\[(n - k - 2)T(r, f) \leq N\left(r, \frac{1}{[f^n(f - 1)]^{(a) - 1}}\right) + S(r, f). \quad \ldots(2.5.1)\]

Hence, we deduce by (2.5.1) and \(n \geq k + 3\) that \(f^n(f - 1)^{(a) - 1}\) has infinitely many solutions.

2.6 PROOF OF THEOREM 2.1.4

Let \(F = f^n(f - 1)\) and \(G = g^n(g - 1)\).

Consider \(\Theta(0, F) = 1 - \lim_{r \to \infty} \frac{N\left(r, \frac{1}{F}\right)}{T(r, F)} = 1 - \lim_{r \to \infty} \frac{N\left(r, \frac{1}{f^n(f - 1)}\right)}{(n + 1)T(r, f)}\)
\[= 1 - \lim_{r \to \infty} \frac{N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f - 1}\right)}{(n + 1)T(r, f)} \geq 1 - \lim_{r \to \infty} \frac{2T(r, f)}{(n + 1)T(r, f)}\]
\[ \frac{n-1}{n+1} \leq 1 - \frac{2}{n+1} = \frac{n-1}{n+1}. \]  

...(2.6.1)

Similarly

\[ \Theta(0, G) \geq \frac{n-1}{n+1}. \]  

...(2.6.2)

Consider

\[ \Theta(\infty, F) = 1 - \lim_{r \to \infty} \frac{N(r, F)}{T(r, F)} = 1 - \lim_{r \to \infty} \frac{N(r, f^n(f-1))}{(n+1)T(r, f)} \]

\[ = 1 - \lim_{r \to \infty} \frac{N(r, f^n)}{T(r, f)} \geq 1 - \lim_{r \to \infty} \frac{T(r, f)}{(n+1)T(r, f)} \geq \frac{n}{n+1} \]  

...(2.6.3)

Similarly

\[ \Theta(\infty, G) \geq \frac{n}{n+1}. \]  

...(2.6.4)

Next, we have

\[ N_k\left( r, \frac{1}{F} \right) = N\left( r, \frac{1}{F} \right) + \frac{1}{2} + \cdots + \frac{1}{n+k} \]

\[ \delta_k(0, F) = 1 - \lim_{r \to \infty} \frac{N_k\left( r, \frac{1}{F} \right)}{T(r, F)} = 1 - \lim_{r \to \infty} \frac{N_k\left( r, \frac{1}{f^n(f-1)} \right)}{T(r, F)} \]

\[ \geq 1 - \lim_{r \to \infty} \frac{(k+2)T(r, f)}{(n+1)T(r, f)} \]

i.e

\[ \delta_k(0, F) \geq 1 - \frac{(k+2)}{n+1} = \frac{n-k-1}{n+1}. \]  

...(2.6.5)

Similarly

\[ \delta_k(0, G) \geq \frac{n-k-1}{n+1}. \]  

...(2.6.6)

We have

\[ \Delta = \Theta(0, F) + \Theta(0, G) + (k+2)\Theta(\infty, F) + 2\Theta(\infty, G) + \delta_k(0, F) + \delta_k(0, G). \]

From (2.6.1)-(2.6.6), we get

\[ \Delta \geq 2 \left( \frac{n-1}{n+1} \right) + (k+2) \frac{n}{n+1} + 2 \left( \frac{n}{n+1} \right) + \frac{n-k-1}{n+1} + \frac{n-k-1}{n+1} \]

\[ = 2 \left( \frac{n-1}{n+1} \right) + (k+4) \frac{n}{n+1} + 2 \frac{n-k-1}{n+1} \]

Since \( n > 3k+11 \), we get \( \Delta > k+7 \).
Considering $F^{(k)}(z) = \left[f^{n}(z)[f(z)-1]\right]^{k}$ and $G^{(k)}(z) = \left[g^{n}(z)[g(z)-1]\right]^{k}$, then by the condition of theorem 2.1.4, we obtain that $F^{(k)}$ and $G^{(k)}$ share the value 1 CM and $F$ and $G$ satisfies conditions of Lemma 2.2.3, then by Lemma 2.2.3, we deduce that either $F^{(k)}G^{(k)} = 1$ or $F \equiv G$.

Next, we consider the case $F^{(k)}G^{(k)} = 1$, that is
$$\left[f^{n}(z)[f(z)-1]\right]^{k} \left[g^{n}(z)[g(z)-1]\right]^{k} = 1.$$ ...(2.6.7)

Let $z_0$ be a zero of $f$ of order $p$. From (2.6.7) we get $z_0$ is a pole of $g$.

Suppose that $z_0$ is a pole of $g$ of order $q$. Again by (2.6.7), we obtain
$$np - k = nq + q + k$$
i.e
$$n(p-q) = q + 2k,$$
which implies that $p \geq q + 1$ and $q + 2k \geq n$. Hence $p \geq n - 2k + 1$. ...(2.6.8)

Let $z_1$ be a zero of $f - 1$ of order $p_1$, then $z_1$ is zero of $\left[f^{n}(f-1)\right]^{k}$ of order $p_1 - k$. Therefore from (2.6.7), we obtain
$$p_1 - k = nq_1 + q_1 + k,$$ since $z_1$ is a pole of $g$ of order $q_1$
i.e
$$p_1 = (n + 1)q_1 + 2k$$
i.e
$$p_1 \geq n + 2k + 1$$ ...(2.6.9)

Let $z_2$ be a zero of $f'$ of order $p_2$ that is not zero of $f(f-1)$, As above, we obtain from (2.6.7)
$$p_2 - (k - 1) = nq_2 + q_2 + k$$
i.e
$$p_2 = (n + 1)q_2 + 2k - 1$$
i.e
$$p_2 \geq n + 2k.$$ ...(2.6.10)
Moreover, in the same manner as above, we have the similar results for the zeros of $\left[g^n(g-1)^k \right]$.

On the other hand, suppose that $z_3$ is a pole of $f$. From (2.6.7), we get $z_3$ is the zero $\left[g^n(z)(g(z)-1)^k \right]$. Thus

$$
\overline{N}(r,f) \leq \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g-1}\right) + \overline{N}\left(r, \frac{1}{g'}\right) 
$$

$$
\leq \frac{1}{n-2k+1}N\left(r, \frac{1}{g}\right) + \frac{1}{n+2k+1}N\left(r, \frac{1}{g-1}\right) + \frac{1}{n+2k}N\left(r, \frac{1}{g'}\right).
$$

Since $n \geq 3k+1$, we get

$$
\overline{N}(r,f) \leq \frac{1}{k+12}N\left(r, \frac{1}{g}\right) + \frac{1}{5k+12}N\left(r, \frac{1}{g-1}\right) + \frac{1}{5k+12}N\left(r, \frac{1}{g'}\right) 
$$

$$
\leq \frac{1}{13}N\left(r, \frac{1}{g}\right) + \frac{1}{17}N\left(r, \frac{1}{g-1}\right) + \frac{2}{16}N\left(r, \frac{1}{g'}\right) 
$$

$$
\leq \left(\frac{1}{13} + \frac{1}{17} + \frac{1}{8}\right)T(r,g) + S(r,g) 
$$

$$
\leq (0.261)T(r,g) + S(r,g). \tag{2.6.11}
$$

From Second fundamental theorem and from (2.6.11), we get

$$
T(r,f) \leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f-1}\right) + \overline{N}(r,f) + S(r,f) 
$$

$$
\leq \frac{1}{13}N\left(r, \frac{1}{f}\right) + \frac{1}{17}N\left(r, \frac{1}{f-1}\right) + (0.261)T(r,g) + S(r,f) + S(r,g) 
$$

$$
\leq (0.1358)T(r,f) + (0.261)T(r,g) + S(r,f) + S(r,g). \tag{2.6.12}
$$

Similarly, we have $T(r,g) \leq (0.1358)T(r,g) + (0.261)T(r,f) + S(r,f) + S(r,g). \tag{2.6.13}$

Adding (2.6.12) and (2.6.13), we obtain

$$
T(r,f) + T(r,g) \leq 0.7936(T(r,f) + T(r,g)) + S(r,f) + S(r,g). 
$$

i.e

$$
(0.2064) \left[T(r,f) + T(r,g)\right] \leq S(r,f) + S(r,g),
$$
which is a contradiction.

**Case 2:** If $F = G$, that is $f^n(f-1) = g^n(g-1)$. ...(2.6.14)

Suppose $f \neq g$, then we consider two cases:

(i) Let $h = \frac{f}{g}$ be a constant. Then from (2.6.14) it follows that $h \neq 1$, $h^n \neq 1$, $h^{n+1} \neq 1$ and $g = \frac{1-h^n}{1-h^{n+1}}$ = constant, which leads to a contradiction.

(ii) Let $h = \frac{f}{g}$ is not a constant. Since $f \neq g$, we have $h \neq 1$ and hence we deduce that $g = \frac{1-h^n}{1-h^{n+1}}$ and $f = \left(\frac{1-h^n}{1-h^{n+1}}\right) h = \frac{(1+h+h^2+..h^{n-1})h}{1+h+h^2+h^n}$, where $h$ is non constant meromorphic function. It follows that

$T(r,f) = T(r,gh) = (n+1)T(r,h) + S(r,f)$.

On the other hand, by the second fundamental theorem, we deduce

$\overline{N}(r,f) = \sum_{j=1}^{n} \left\{ \text{ number of zeros of } \frac{1}{r-h-\alpha_j} \right\} \geq (n-2)T(r,h) + S(r,f)$,

where $\alpha_j (\neq 1) (j=1,2,..n)$ are distinct roots of the algebraic equation $h^{n+1} = 1$.

We have

$\Theta(\infty, f) = 1 - \frac{\overline{N}(r,f)}{T(r,f)} \leq 1 - \frac{(n-2)T(r,h) + S(r,f)}{T(r,f)}$

$\leq 1 - \frac{(n-2)T(r,h) + S(r,f)}{(n+1)T(r,h) + S(r,f)} \leq 1 - \frac{(n-2)}{(n+1)} = \frac{3}{n+1}$

i.e $\Theta(\infty, f) \leq \frac{3}{n+1}$, which contradicts the assumption $\Theta(\infty, f) > \frac{3}{n+1}$.

Thus $f = g$. This completes the proof of Theorem 2.1.4.