CHAPTER 1
A UNICITY THEOREM FOR MEROMORPHIC FUNCTIONS
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1.1 INTRODUCTION AND MAIN RESULTS

In this Chapter, by introducing the notion of order of multiplicity, we study meromorphic functions that share only one value CM. The result improves the theorem due to C.C. Yang and Xinhou Hua [43]. In addition, as a consequence of our result several significant new results in the improved form have been obtained.

Here, we need the following definition.

Let \( f(z) \) and \( g(z) \) be meromorphic functions. If \( f(z) - a \) and \( g(z) - a \) have the same zeros with the same multiplicities, then we say that \( f(z) \) and \( g(z) \) share the value \( a \) CM, where \( a \) is a complex number.

In 1997, Chung-Chun Yang and Xinhou Hua [43] proved the following result.

Theorem 1.1.A [1]: Let \( f(z) \) and \( g(z) \) be two non-constant meromorphic functions. \( n \geq 11 \) be an integer and \( a \in \mathbb{C} - \{0\} \). If \( f^n f' \) and \( g^n g' \) share the value \( a \) CM, then either \( f = dg \) for some \((n+1)^{th}\) root of unity \( d \) or \( g(z) = c_1 e^{cz} \) and \( f(z) = c_2 e^{-cz} \), where \( c, c_1, c_2 \) are constants and satisfy \( (c_1 c_2)^{n+1} c^2 = -a^2 \).

* The results in this chapter have been accepted for publication in the Journal of Analysis.
We obtain the following theorem which resembles Theorem 1.1.A to some extent.

**Theorem 1.1.1** : Let \( f(z) \) and \( g(z) \) be two non-constant meromorphic functions whose zeros and poles are of multiplicities at least \( s \), where \( s \) is a positive integer. Let \( n \geq 2 \) be an integer satisfying \((n+1)s > 12\). If \( f^*f' \) and \( g^*g' \) share the value 1 CM, then either \( f = d g \) for some \((n+1)^{\text{th}}\) root of unity \( d \) or \( f(z) = c_1 e^{cz} \) and \( g(z) = c_2 e^{-cz} \), where \( c, c_1, c_2 \) are constants satisfying \((c_1c_2)^n + c_2^2 = -1\).

### 1.2 USEFUL LEMMAS

In order to prove our results, we need the following Lemmas.

**Lemma 1.2.1 (Milloux inequality) [47]**: Let \( f(z) \) be a meromorphic function and \( \alpha \) a non zero finite complex number. Then

\[
T(r,f) \leq N(r,f) + N\left( r, \frac{1}{f} \right) + N\left( r, \frac{1}{f(k+1)} \right) - N\left( r, \frac{1}{f(k) - \alpha} \right) + S(r,f) \\
\leq N(r,f) + (k+1)N\left( r, \frac{1}{f} \right) + N\left( r, \frac{1}{f(k)} \right) - N_0\left( r, \frac{1}{f(k+1)} \right) + S(r,f),
\]

where \( k \) is a positive integer and \( N_0\left( r, \frac{1}{f(k+1)} \right) \) is the counting function for those zeros of \( f^{(k+1)} \) which are not zeros of \( f^{(k)} - \alpha \) and not zeros of \( f \) with multiplicity \( \geq k + 2 \).

In order to prove the Theorem 1.1.1, we shall first prove the following Lemma

**Lemma 1.2.2** : Let \( f(z) \) and \( g(z) \) be two non-constant transcendental meromorphic functions, \( k \) be a positive integer. If \( f^{(k)} \) and \( g^{(k)} \) share the value 1 CM and if \( \Delta = [\Theta(\infty, f) + \Theta(\infty, g) + \Theta(0, f) + \Theta(0, g)] > \frac{4k + 7}{k + 2} \), then either \( f^{(k)} \equiv 1 \) or \( f = g \).
Proof: Let \( \Phi(z) = \frac{f^{(k+2)}}{f^{(k+1)}} - 2 \frac{f^{(k+1)}}{f^{(k)}} - \frac{g^{(k+2)}}{g^{(k+1)}} + 2 \frac{g^{(k+1)}}{g^{(k)}} \). \( \ldots (1.2.1) \)

Clearly \( m(r, \Phi) = S(r, f) + S(r, g). \)

We claim that \( \Phi(z) \equiv 0. \)

Suppose that \( \Phi(z) \neq 0 \), then if \( z_0 \) is a common simple 1-point of \( f^{(k)} \) and \( g^{(k)} \) substituting their Taylor series at \( z_0 \) into (1.2.1), we see that \( z_0 \) is a zero of \( \Phi(z) \). Thus, we have

\[
\overline{N}_1\left( r, \frac{1}{f^{(k)} - 1} \right) = \overline{N}_1\left( r, \frac{1}{g^{(k)} - 1} \right) \leq N\left( r, \frac{1}{\Phi} \right) \leq T(r, \Phi) + O(1).
\]

Thus

\[
\overline{N}_1\left( r, \frac{1}{f^{(k)} - 1} \right) \leq N(r, \Phi) + S(r, f) + S(r, g), \quad \ldots (1.2.2)
\]

where \( \overline{N}_1\left( r, \frac{1}{f^{(k)} - 1} \right) \) is the counting function which only counts the simple zeros of \( f^{(k)} - 1. \)

By our assumptions, \( \Phi(z) \) has only simple poles at zeros of \( f^{(k+1)} \) and \( g^{(k+1)} \) and the multiple poles of \( f^{(k)} \) and \( g^{(k)} \) and the multiple zeros of \( f^{(k)} - 1 \) and \( g^{(k)} - 1 \) are no poles of \( \Phi. \)

From (1.2.1) and (1.2.2) and the above observations, we deduce that

\[
\overline{N}_1\left( r, \frac{1}{f^{(k)} - 1} \right) \leq \overline{N}(r, f) + \overline{N}(r, g) + \overline{N}\left( r, \frac{1}{f} \right) + \overline{N}\left( r, \frac{1}{g} \right)
\]

\[
+ N_o\left( r, \frac{1}{f^{(k+1)}} \right) + N_o\left( r, \frac{1}{g^{(k+1)}} \right),
\]

where \( N_o\left( r, \frac{1}{f^{(k+1)}} \right) \) has the same meaning as in Lemma 1.2.1.
Obviously,
\[
\bar{N}\left( r, -\frac{1}{f(k) - 1} \right) + \bar{N}\left( g, -\frac{1}{g(k) - 1} \right) = 2\bar{N}\left( r, -\frac{1}{f(k) - 1} \right) \\
\leq \bar{N}\left( r, -\frac{1}{f(k) - 1} \right) + \bar{N}\left( r, -\frac{1}{f(k) - 1} \right)
\]
\[\text{...(1.2.4)}\]

By Lemma 1.2.1, we have
\[
T(r, f) \leq \bar{N}(r, f) + (k + 1)\bar{N}\left( r, -\frac{1}{f(k) - 1} \right) - N_0\left( r, -\frac{1}{f(k+1)} \right) + S(r, f),
\]
\[\text{...(1.2.5)}\]
\[
T(r, g) \leq \bar{N}(r, g) + (k + 1)\bar{N}\left( g, -\frac{1}{g(k) - 1} \right) - N_0\left( r, -\frac{1}{g(k+1)} \right) + S(r, g).
\]
\[\text{...(1.2.6)}\]

Adding (1.2.5) and (1.2.6), we get
\[
T(r, f) + T(r, g) \leq \bar{N}(r, f) + \bar{N}(r, g) + (k + 1)\left[ \bar{N}\left( r, -\frac{1}{f} \right) + \bar{N}\left( r, -\frac{1}{g} \right) \right] \\
+ 2\bar{N}\left( r, -\frac{1}{f(k) - 1} \right) - N_0\left( r, -\frac{1}{f(k+1)} \right) \\
- N_0\left( r, -\frac{1}{g(k+1)} \right) + S(r, f) + S(r, g)
\]
\[\text{...(1.2.7)}\]

From (1.2.3), (1.2.4), (1.2.7) and since
\[
N\left( r, -\frac{1}{f(k) - 1} \right) \leq T(r, f) + k\bar{N}(r, f) + S(r, f),
\]
we get
\[
T(r, f) + T(r, g) \leq \bar{N}(r, f) + \bar{N}(r, g) + (k + 1)\left[ \bar{N}\left( r, -\frac{1}{f} \right) + \bar{N}\left( r, -\frac{1}{g} \right) \right] + \bar{N}(r, f) + \bar{N}(r, g) \\
+ \bar{N}\left( r, -\frac{1}{f} \right) + \bar{N}\left( r, -\frac{1}{g} \right) + T(r, f) + k\bar{N}(r, f) + S(r, f) + S(r, g).
\]
Hence, \( T(r,g) \leq (k + 2) \left[ N(r,f) + N(r,g) + \frac{1}{f} + \frac{1}{g} \right] + S(r,f) + S(r,g). \)

Without loss of generality, we suppose that there exists a set \( I \) with infinite measure such that \( T(r,f) \leq T(r,g) \) for \( r \in I \).

Hence, we obtain that

\[
T(r,g) \leq (k + 2) \left[ 4 - \left( \Theta(0,f) + \Theta(0,g) + \Theta(\infty,f) + \Theta(\infty,g) \right) + \varepsilon \right] + S(r,g)
\]

for \( r \in I \) and \( r \) sufficiently large. Therefore

\[
T(r,g) \leq (k + 2) \left[ 4 - \Delta + \varepsilon \right] + S(r,g)
\]

for \( r \in I \) and \( r \) sufficiently large.

\[
\Rightarrow 1 \leq (k + 2)(4 - \Delta)
\]

\[
\Rightarrow 1 \leq 4k + 8 - \Delta(k + 2)
\]

\[
\Rightarrow \Delta(k + 2) \leq 4k + 7
\]

\[
\Rightarrow \Delta \leq \frac{4k + 7}{k + 2} \quad \text{for } r \in I \text{ and } r \text{ sufficiently large,}
\]

which is a contradiction to our hypothesis \( \Delta > \frac{4k + 7}{k + 2} \).

Thus, \( \Phi(z) = 0 \).

Therefore by (1.2.1), we have

\[
\frac{f(k+2)}{f(k+1)} - 2 \frac{f(k+1)}{f(k) - 1} \equiv \frac{g(k+2)}{g(k+1)} - 2 \frac{g(k+1)}{g(k) - 1}.
\]

By solving this, we obtain

\[
\frac{1}{f(k) - 1} = \frac{b g(k) + a - b}{g(k) - 1},
\]

where \( a \) and \( b \) are two constants and \( a \neq 0 \).
Next we consider three cases

**Case 1:** \(a = b\).

If \(b = -1\), then from (1.2.8), we obtain that
\[ g^{(k)} f^{(k)} = 1. \]

If \(b \neq -1\), then from (1.2.8), we obtain that
\[ \frac{1}{f^{(k)} - 1} = \frac{bg^{(k)}}{g^{(k)} - 1} \quad \Rightarrow \quad \frac{1}{f^{(k)}} = \frac{bg^{(k)}}{(1+b)g^{(k)} - 1} \]

By Lemma of logarithmic derivative and First fundamental theorem, we obtain the following inequality
\[
\begin{aligned}
N \left( r, \frac{1}{f^{(k)}} \right) &\leq N \left( r, \frac{f}{f^{(k)}} \right) + N \left( r, \frac{1}{f} \right) \\
&\leq N \left( r, \frac{f^{(k)}}{f} \right) + N \left( r, \frac{1}{f} \right) + S(r, f)
\end{aligned}
\]

Therefore
\[
\begin{aligned}
N \left( r, \frac{1}{g^{(k)} - (1/(1+b))} \right) &\leq k \left( N(r, f) + (k+1) N \left( r, \frac{1}{f} \right) + S(r, f) \right) \\
&\leq k N(r, f) + (k+1) N \left( r, \frac{1}{f} \right) + S(r, f).
\end{aligned}
\]

From (1.2.9) and by of Lemma 1.2.1., we have
\[ T(r, g) \leq \overline{N}(r, g) + (k + 1)\overline{N} \left( r, \frac{1}{g} \right) + \overline{N} \left( r, \frac{1}{g(k) - (1/(1+b))} \right) - N_0 \left( r, \frac{1}{g(k+1)} \right) + S(r, g), \]

\[ \leq \overline{N}(r, g) + (k + 1)\overline{N} \left( r, \frac{1}{g} \right) + k \overline{N}(r, f) + (k + 1)\overline{N} \left( r, \frac{1}{f} \right) + S(r, f) + S(r, g), \]

\[ \leq (k + 1) \left[ \overline{N}(r, f) + \overline{N} \left( r, \frac{1}{f} \right) + \overline{N}(r, g) + \overline{N} \left( r, \frac{1}{g} \right) \right] + S(r, g) + S(r, f), \]

\[ \leq (k + 1)[4 - \Theta(0, f) + \Theta(0, g) + \Theta(\infty, f) + \Theta(\infty, g)] + \varepsilon]T(r, g) + S(r, g) \]

for \( r \in \mathcal{I} \) and \( r \) sufficiently large.

Therefore

\[ T(r, g) \leq (k + 1)[4 - \Delta + \varepsilon]T(r, g) + S(r, g) + S(r, f) \]

for \( r \in \mathcal{I} \) and \( r \) sufficiently large.

That is

\[ 1 \leq (k + 1)[4 - \Delta] \leq (k + 2)[4 - \Delta] \quad \Rightarrow \quad \Delta \leq \frac{4k + 7}{k + 2}, \]

which is contradiction to our hypothesis \( \Delta > \frac{4k + 7}{k + 2} \).

**Case 2:** \( b \neq 0 \) and \( a \neq b \)

(i) If \( b \neq -1 \), then from (1.2.8), we obtain that

\[ f^{(k)} - (1 + 1/b) = \frac{-a}{b^2 \left[ g^{(k)} + \frac{a-b}{b} \right]} \]

Therefore

\[ \overline{N} \left[ r, \left( \frac{1}{g^{(k)} + \frac{a-b}{b}} \right) \right] = \overline{N} \left( r, f^{(k)} - (1 + 1/b) \right) = \overline{N}(r, f^{(k)}) = \overline{N}(r, f). \]

By of Lemma 1.2.1, we have
\[ T(r, g) \leq \overline{N}(r, g) + (k + 1)\overline{N}\left(r, \frac{1}{g(k)}\right) + \overline{N}\left(r, \frac{1}{g(k) + a - b}\right) - N_0\left(r, \frac{1}{g(k+1)}\right) + S(r, g), \]

\[ \leq \overline{N}(r, g) + (k + 1)\overline{N}\left(r, \frac{1}{g}\right) + \overline{N}(r, f) + S(r, g), \]

\[ \leq (k + 1)\left[ \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r, g) + \overline{N}\left(r, \frac{1}{g}\right)\right] + S(r, g), \]

By using the argument as in case 1, we get a contradiction.

(ii) If \( b = -1 \), then from (1.2.8), we obtain

\[ f^{(k)}(k) = \frac{a}{-g^{(k)} + a + 1}. \]

\[ \overline{N}\left[r, \left(\frac{1}{g^{(k)} - (1 + a)}\right)\right] = \overline{N}(r, f^{(k)}) = \overline{N}(r, f). \]

From Lemma 1.2.1, we have

\[ T(r, g) \leq \overline{N}(r, g) + (k + 1)\overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g(k) - (1 + a)}\right) - N_0\left(r, \frac{1}{g(k+1)}\right) + S(r, g), \]

\[ \leq \overline{N}(r, g) + (k + 1)\overline{N}\left(r, \frac{1}{g}\right) + \overline{N}(r, f) - N_0\left(r, \frac{1}{g(k+1)}\right) + S(r, g), \]

\[ \leq (k + 1)\left[ \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r, g) + \overline{N}\left(r, \frac{1}{g}\right)\right] + S(r, g), \]

That is

\[ T(r, g) \leq (k + 1)[4 - \Delta + \varepsilon T(r, g)] + S(r, g) \]

for \( r \in I \) and \( r \) sufficiently large.

By using the argument as in case 1, we get a contradiction.

Case 3 : \( b = 0 \). From (1.2.8), on integration gives
\[ f = \frac{1}{a} g + p(z), \]

...(1.2.10)

where \( p(z) \) is a polynomial.

If \( p(z) \neq 0 \), then by Second Fundamental Theorem for three small functions

\[
T(r, f) \leq \overline{N} \left( r, \frac{1}{f} \right) + \overline{N} \left( r, \frac{1}{f - p(z)} \right) + \overline{N}(r, f) + S(r, f)
\]

we have

\[
\leq \overline{N} \left( r, \frac{1}{f} \right) + \overline{N} \left( r, \frac{1}{g} \right) + \overline{N}(r, f) + \overline{N}(r, g) + S(r, f).
\]

From (1.2.10), we obtain

\[
T(r, f) = T(r, g) + S(r, f).
\]

Therefore

\[
T(r, f) \leq [4 - \Delta + \varepsilon] T(r, f) + S(r, f)
\]

\[
\leq (k + 2)[4 - \Delta + \varepsilon] T(r, f) + S(r, f).
\]

Next by using the argument as in case 1, we get a contradiction.

Therefore

\[
p(z) \equiv 0.
\]

From (1.2.10), we get

\[
f = \frac{1}{a} g.
\]

If \( a \neq 1 \), then by \( f^{(k)} \) and \( g^{(k)} \) sharing the value 1 CM, we deduce from above equation that \( g^{(k)} \neq 1 \), then by of Lemma 1.2.1, we get

\[
T(r, g) \leq \overline{N}(r, g) + (k + 1) \overline{N} \left( r, \frac{1}{g} \right) - \overline{N} \left( r, \frac{1}{g(k+1)} \right) + S(r, g),
\]

\[
\leq (k + 1) \left[ \overline{N}(r, g) + \overline{N} \left( r, \frac{1}{g} \right) + \overline{N}(r, f) + \overline{N} \left( r, \frac{1}{f} \right) \right] + S(r, g).
\]

Next, we can obtain a contradiction as in case 2.
Thus, we get that $a = 1$, that is, $f \equiv g$. The proof of the Lemma 1.2.2 is completed.

**Lemma 1.2.3:** (Theorem 3 of [43] ) Let $f$ and $g$ be two non-constant entire functions, $n \geq 1$. If $f^n f' g^n g' = 1$, then $f(z) = c_1 e^{cz}$ and $g(z) = c_2 e^{-cz}$, where $c, c_1, c_2$ are constants and $(c_1 c_2)^{n+1} e^2 = -1$.

1.3 PROOF OF THEOREM 1.1.1

Consider $F = \frac{f^{n+1}}{n+1}$ and $G = \frac{g^{n+1}}{n+1}$, then $F' = f^n f'$ and $G' = g^n g'$.

We have

$$N \left( r, \frac{1}{F} \right) = N \left( r, \frac{1}{f^{n+1}} \right) \leq \frac{1}{s(n+1)} N \left( r, \frac{1}{F} \right) \leq \frac{1}{s(n+1)} [T(r,F) + O(1)]$$

Therefore

$$N \left( r, \frac{1}{F} \right) \leq \frac{1}{s(n+1)} [T(r,F) + O(1)].$$

Therefore

$$\Theta(0,F) = 1 - \lim_{r \to \infty} \frac{N \left( r, \frac{1}{F} \right)}{T(r,F)} \geq 1 - \frac{1}{s(n+1)} = \frac{sn+s-1}{s(n+1)}.$$  

Similarly

$$\Theta(0,G) \geq \frac{sn+s-1}{s(n+1)}, \quad \Theta(\infty,F) \geq \frac{sn+s-1}{s(n+1)}, \quad \Theta(\infty,G) \geq \frac{sn+s-1}{s(n+1)}.$$  

Therefore

$$\Delta = \Theta(0,F) + \Theta(0,G) + \Theta(\infty,F) + \Theta(\infty,G) \geq 4 \frac{(sn+s-1)}{s(n+1)}.$$  

Since $(n+1)s > 12$, we get $\Delta > \frac{11}{3}$. Therefore by Lemma 1.2.2 (when $k = 1$),
we get either \( F'G' = 1 \) or \( F = G \).

Consider the case \( F'G' = 1 \), which gives \( f^n f' g^n g' = 1 \) \( \text{(1.3.1)} \)

Suppose that \( f \) has a pole \( z_o \) (with order \( p \geq s \) say), then \( z_o \) is a zero of \( g \) (with order \( m \geq s \) say).

By (1.3.1), we get \( nm + m - 1 = np + p + 1 \). That is \( (m - p)(n + 1) = 2 \), which is impossible since \( n \geq 2 \) and \( m, p \) are positive integers.

Therefore, we conclude that \( f \) and \( g \) are entire functions. From Lemma 1.2.3, we get \( f(z) = c_1 e^{cz} \) and \( g(z) = c_2 e^{-cz} \), where \( c, c_1, c_2 \) are constants satisfying \( (c_1 c_2)^{n+1} c^2 = -1 \).

Next we consider another case, \( F = G \)

\[
\Rightarrow \quad \frac{f^{n+1}}{n+1} = \frac{g^{n+1}}{n+1} \quad \Rightarrow \quad f^{n+1} = g^{n+1}
\]

\[
\Rightarrow \quad f = dg, \quad \text{for some (n+1)th root of unity } d.
\]

The proof of Theorem 1.1.1 is complete.

**CONSEQUENCES OF THEOREM 1.1.1**: Giving specific values for \( s \), we get the following interesting cases:

(i) If \( s = 1 \) then theorem 1.1.1 reads as follows
Let \( f(z) \) and \( g(z) \) be two non-constant meromorphic functions whose zeros and poles are of multiplicities at least 1. \( n > 1 \) be an integer and if \( f^n f' \) and \( g^n g' \) share the value 1 CM, then either \( f = dg \) for some \((n+1)^{th}\) root of
unity $d$ or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$, where $c, c_1, c_2$ are constants and satisfy $(c_1 c_2)^{n+1} c_2^2 = -1$.

In this case our result is slightly weaker than Chung-Chun Yang and Xinhou Hua result [43].

However, the cases $s \geq 2$ lead to considerable improvement on the value $n$.

(ii) If $s = 2$, then theorem 1.1.1 reads as follows

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions whose zeros and poles are of multiplicities at least 2. $n > 5$ be an integer and if $f^n f'$ and $g^n g'$ share the value 1 CM, then either $f = dg$ for some $(n + 1)^a$ root of unity $d$ or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$, where $c, c_1, c_2$ are constants and satisfy $(c_1 c_2)^{n+1} c_2^2 = -1$.

(iii) If $s = 3$, then theorem 1.1.1 reads as follows

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions whose zeros and poles are of multiplicities at least 3. $n > 3$ be an integer and if $f^n f'$ and $g^n g'$ share the value 1 CM, then either $f = dg$ for some $(n + 1)^a$ root of unity $d$ or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$, where $c, c_1, c_2$ are constants and satisfy $(c_1 c_2)^{n+1} c_2^2 = -1$.

(iv) If $s = 4$ then theorem 1.1.1 reads as follows

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions whose zeros and poles are of multiplicities at least 4. $n > 2$ be an integer and if $f^n f'$ and $g^n g'$ share the value 1 CM, then either $f = dg$ for some $(n + 1)^a$ root of
unity \( d \) or \( g(z) = c_1 e^{cz} \) and \( f(z) = c_2 e^{-cz} \), where \( c, c_1, c_2 \) are constants and satisfy \((c_1 c_2)^{n+1} e^2 = -1\).

(v) If \( s \geq 5 \) then theorem 1.1.1 reads as follows:

Let \( f(z) \) and \( g(z) \) be two non-constant meromorphic functions whose zeros and poles are of multiplicities at least 5. \( n \geq 2 \) be an integer and if \( f^n f' \) and \( g^n g' \) share the value 1 CM, then either \( f = dg \) for some \((n+1)\)th root of unity \( d \) or \( g(z) = c_1 e^{cz} \) and \( f(z) = c_2 e^{-cz} \), where \( c, c_1, c_2 \) are constants and satisfy \((c_1 c_2)^{n+1} e^2 = -1\).