CHAPTER 6
FIXED POINTS AND NORMALITY CRITERIA
In this chapter, we prove some sufficient conditions for a family of meromorphic functions to be normal in a domain.

6.1 INTRODUCTION

Let $C$ be the open complex plane and $D \subset C$ be a domain.

It is an interesting problem to find out criteria for normality of a family of analytic or meromorphic functions. In recent years this problem attracted the attention of a number of researchers worldwide.

In 1969 D. Drasin [10] proved the following normality criterion

**Theorem 6.1.A**: Let $F$ be a family of analytic functions in a domain $D$ and $a(\neq 0), b$ be two finite numbers. If for every $f \in F$, $f' - af^n - b$ has no zero then $F$ is normal, where $n (\geq 3)$ is an integer.

Chen-Fang [8] and Ye [45] independently proved that Theorem 6.1.A also holds for $n = 2$. A number of authors { [6],[27],[28],[29],[33],[50]} extended Theorem 6.1.A to a family of meromorphic functions in a domain. Their results can be combined in the following theorem.

**Theorem 6.1.B**: Let $F$ be a family of meromorphic functions in a domain $D$ and $a(\neq 0), b$ be two finite numbers. If for every $f \in F$, $f' - af^n - b$ has no zero then $F$ is normal, where $n (\geq 3)$ is an integer.

Li [28], Li [29] and Langley [27] proved Theorem 6.1.B for $n \geq 5$, Pang [33] proved for $n = 4$ and Chen-Fang [6], Zalcman [50] proved for $n = 3$. Fang-
Yuan [14] showed that Theorem 6.1.B does not, in general, hold for \( n = 2 \).

For the case \( n = 2 \) they proved the following result.

**Theorem 6.1.C:** Let \( F \) be a family of meromorphic functions in a domain \( D \) and \( a(\neq 0), b \) be two finite numbers. If \( f' - af^2 - b \) has no zero and \( f \) has no simple and double pole for every \( f \in F \), then \( F \) is normal.

Fang-Yuan [14] mentioned the following example from which it appears that the condition for each \( f \in F \) not to have any simple and double pole is necessary for Theorem 6.1.C.

**Example 6.1.1:** Let \( f_n(z) = nz(z^n - 1)^2 \) for \( n = 1, 2, \ldots \) and \( D : |z| < 1 \). Then each \( f_n \) has only a double pole and a simple zero. Also \( f_n' + f_n^2 = n(z^{n-1})^2 \neq 0 \).

Since \( f_n^*(0) = n \to \infty \) as \( n \to \infty \), it follows from Marty's criterion that \( \{f_n\} \) is not normal in \( D \).

Indrajit Lahiri and Shyamali Dewan [25] mentioned the following example from which it suggests that the restriction on the poles of \( f \in F \) may be relaxed at the cost of some restriction imposed on the zeros of \( f \in F \).

**Example 6.1.2:** Let \( f_n(z) = nz^{-2} \) for \( n = 3, 4, \ldots \) and \( D : |z| < 1 \). Then each \( f_n \) has only a double pole and no simple zero. Also we see that \( f_n' + f_n^2 = n(n-2z)z^{-4} \neq 0 \) in \( D \).

Since

\[
    f_n^*(z) = \frac{2n|z|}{|z|^2 + n^2} \leq \frac{2}{n} < 1
\]

in \( D \), it follows from Marty's criterion that the family \( \{f_n\} \) is normal in \( D \).

Indrajit Lahiri and Shyamali Dewan [25] have given some sufficient conditions for a family of meromorphic functions to be normal in a domain by proving following theorem and also have given example to show that given conditions are necessary.
**Theorem 6.1.D**: Let $F$ be a family of meromorphic functions in a domain $D$ such that no $f \in F$ has any simple zero and simple pole. Let

$$E_f = \{z : z \in D \text{ and } f'(z) - af^2(z) = b\},$$

where $a(\neq 0), b$ be two finite numbers.

If there exists a positive number $M$ such that for every $f \in F$, $|f(z)| \leq M$ whenever $z \in E_f$, then $F$ is normal.

Now we extend theorem 6.1.D by replacing constant $a$ by variable $z$ in the following form.

**Theorem 6.1.1**: Let $F$ be a family of transcendental (rational) meromorphic functions in a domain $D$ such that if all zeros and poles of $f \in F$ are multiple, except possibly finitely many (if $f \in F$ have no simple zeros and no simple pole). Let $E_f = \{z : z \in D \text{ and } f' + zf^2 = a\}$ where $a$ is a finite number. If there exists a positive number $M$ such that for every $f \in F$, $|f(z)| \leq M$ whenever $z \in E_f$, then $F$ is normal.

The following examples show that the condition of Theorem 6.1.1 on the zeros and poles are necessary.

**Example 6.1.3**: Let $f_n(z) = n \tan n z^2$ for $n = 1, 2, \ldots$ and $D: |z| < \pi$. Then $f_n$ has only simple zeros and simple poles. Here $E_f = \{0\}$ and $|f_n(0)| = 0$. Since $f_n(0) = 0$ and for $z \neq 0$ $f_n(z) \to \infty$ as $n \to \infty$, it follows that the family $\{f_n\}$ is not normal.

**Example 6.1.4**: $f_n(z) = \frac{1}{nz}$ for $n = 2, 3, \ldots$ and $D: |z| < 1$. Then $f_n$ has simple pole and no simple zeros. Here $f_n' + zf_n^2 \neq 0$ and since $f_n^4(0) = n \to \infty$ as $n \to \infty$, by Marty criterion the family $\{f_n\}$ is not normal.
Example 6.1.5: \( f_n(z) = -nz \) for \( n = 2, 3, \ldots \) and \( D: |z| < 1 \). Then \( f_n \) has only simple zero and no simple poles. Here \( E_f = \left\{ \frac{1}{n^{1/2}} \right\} \) for \( n = 2, 3, \ldots \) and \( D: |z| < 1 \).

Also we see that for \( z \in E_f \), \( \left| f_n'(z) \right| = \frac{1}{n^{3/2}} \to 0 \) as \( n \to \infty \) and \( f_n''(0) = n \to \infty \) as \( n \to \infty \), by Marty criterion the family \( \{f_n\} \) is not normal.

Drasin also proved the following normality criterion which involves differential polynomials.

**Theorem 6.1.E:** Let \( F \) be a family of analytic functions in a domain \( D \) and \( a_0, a_1, \ldots, a_{k-1} \) be finite constants, where \( k \) is a positive integer. Let

\[
H(f) = f^{(k)} + a_{k-1}f^{(k-1)} + \ldots + a_1f + a_0f
\]

If for every \( f \in F \)

(i) \( f \) has no zero

(ii) \( H(f) - 1 \) has no zero of multiplicity less than \( k + 2 \),

then \( F \) is normal.

Recently Fang-Yuan [14] proved that Theorem 6.1.E remains valid even if \( H(f) - 1 \) has only multiple zeros for every \( f \in F \). In the next theorem Indrajit Lahiri and Shyamali Dewan [25] extended Theorem 6.1.E to a family of meromorphic functions which also improves a result of Fang-Yuan [14] and also we give some examples to show that given conditions are necessary.

**Theorem 6.1.F:** Let \( F \) be a family of meromorphic functions in a domain \( D \) and

\[
H(f) = f^{(k)} + a_{k-1}f^{(k-1)} + \ldots + a_1f + a_0f
\]

where \( a_0, a_1, \ldots, a_{k-1} \) are finite constants and \( k \) is a positive integer. Let

\[ E_f = \{ z: z \in D \text{ and } z \text{ is a simple zero of } H(f) - 1 \} \, . \]
If for every $f \in F$

(i) $f$ has no zero

(ii) $f$ has no pole of multiplicity less than $k + 3$

(iii) there exists a positive constant $M$ such that $|f(z)| \geq M$ whenever $z \in E_f$, then $F$ is normal.

In the second theorem of this chapter, we extend Theorem 6.1.F for fixed points.

**Theorem 6.1.2:** Let $F$ be a family of meromorphic functions in a domain $D$ and

$$H(f) = f^{(k)} + a_{k-1}f^{(k-1)} + \ldots + a_1f^{(1)} + a_0f$$

where $a_0, a_1, \ldots, a_{k-1}$ are finite constants and $k$ is a positive integer.

Let $E_f = \{ z : z \in D \text{ and } z \text{ is a simple zero of } H(f) - z \}$

If for every $f \in F$

(i) $f$ has no pole of multiplicity less than $k + 2$

(ii) $f$ has no zero

(iii) there exists a positive constant $M$ such that $|f(z)| \geq M$ whenever $z \in E_f$,

then $F$ is normal.

The following examples show that conditions of Theorem 6.1.2 are necessary.

**Example 6.1.6:** $f_n(z) = nz^2$ for $n = 1, 2, \ldots$ in $D : |z| < 1$. Here $H(f) = f' - f'$ and $M = 1/3$. Then $f_n$ has a zero at $z = 0$ and $E_{f_n} = \left\{ \frac{2n}{2n+1} \right\}$ for $n = 1, 2, \ldots$. 
so
\[ \left| f_n \left( \frac{2n}{2n+1} \right) \right| \geq M \text{ for } n = 1, 2, \ldots \]

Since \( f_n(0) = 0 \) and for \( z \neq 0 \) \( f_n(z) \to \infty \) as \( n \to \infty \), it follows that the family \( \{f_n\} \) is not normal.

**Example 6.1.7**: \( f_n(z) = \frac{1}{nz} \) for \( n = 1, 2, \ldots \) in \( D : |z| < 1 \). \( H(f) = f^* \), then
\[ E_n = \{2/n\}^4 \text{ for } n = 3, 4, \ldots \]

For \( z \in E_n \), \( f_n(z) \to 0 \) as \( n \to \infty \) and since \( f_n^*(0) = n \to \infty \) as \( n \to \infty \), by Marty criterion the family \( \{f_n\} \) is not normal.

In connection with Theorem 6.1.A Chen-Fang [6] proposed the following conjecture

**Conjecture 6.1.1**: Let \( F \) be a family of meromorphic functions in a domain \( D \). If for every function \( f \in F \), \( f^{(k)} - af - b \) has no zero in \( D \) then \( F \) is normal, where \( a(\neq 0), b \) are two finite numbers and \( k, n \geq (k + 2) \) are positive integers.

In response to this conjecture Xu [41] proved the following result.

**Theorem 6.1.G**: Let \( F \) be a family of meromorphic functions in a domain \( D \) and \( a(\neq 0), b \) be two finite constants. If \( k \) and \( n \) are positive integers such that \( n \geq k + 2 \) and for every \( f \in F \)

(i) \( f^{(k)} - af - b \) has no zero

(ii) \( f \) has no simple zero,

then \( F \) is normal.

The condition (ii) of Theorem 6.1.G can be dropped if we choose \( n \geq k + 4 \) ([33],[38]). Also some improvement of Theorem 6.1.G can be found in [46].
In the next theorem Indrajit Lahiri and Shyamali Dewan [25] investigated the situation when the power of $f$ is negative in condition (i) of Theorem 6.1.F and given some examples to show that given conditions are necessary.

**Theorem 6.1.H**: Let $F$ be a family of meromorphic functions in a domain $D$ and $a(\neq 0), b$ be two finite numbers. Suppose that

$$E_f = \{z : z \in D \text{ and } f^{(k)}(z) + af^{-n}(z) = b\},$$

where $k, n(>k)$ are positive integers. If for every $f \in F$

(i) $f$ has no zero of multiplicity less than $k$

(ii) there exists a positive number $M$ such that for every $f \in F$, $|f(z)| \geq M$ whenever $z \in E_f$, then $F$ is normal.

In the next theorem of this chapter, we replace constant 'a' by $z$ in the Theorem 6.1.H and obtain following theorem

**Theorem 6.1.3**: Let $F$ be a family of meromorphic functions in a domain $D$ and $a$ be finite number. Suppose that

$$E_f = \{z : z \in D \text{ and } f^{(k)}(z) - zf^{-n}(z) = a\}$$

where $k, n(\geq k)$ are positive integers.

If for every $f \in F$

(i) $f$ has no zero of multiplicity less than $k$

(ii) there exists a positive number $M$ such that for every $f \in F$, $|f(z)| \geq M$ whenever $z \in E_f$, then $F$ is normal.

Following examples show that the conditions of theorem 3 are necessary
Example 6.1.8 : Let $f_p(z) = \frac{p}{z}$ for $p = 1, 2, 3, \ldots$ and $D : |z| < 1$, $n = 2$, $k = 1$, $a = 0$. Here $f_p(z)$ has only simple zero at the origin and $E_{f_p} = \{1\}$ so that $|f_p(z)| \to \infty$ as $p \to \infty$ whenever $z \in E_f$. Since $f_p^*(0) = p^2 \to \infty$ as $p \to \infty$, by Marty criterion the family $\{f_p\}$ is not normal.

Example 6.1.9 : Let $f_p(z) = p z^2$ for $p = 1, 2, 3, \ldots$ and $D : |z| < 1$, $n = 1$, $k = 1$, $a = 0$. Then $f_p(z)$ has only double zero at the origin and $E_{f_p} = \left\{ \pm \frac{1}{\sqrt{2}} p \right\}$ so that $|f_p(z)| \to 0$ as $p \to \infty$ whenever $z \in E_f$. Since $f_p(0) = 0$ and $z \neq 0$, $f_p(z) \to \infty$ as $p \to \infty$ it follows that the family $\{f_p\}$ is not normal.

6.2 LEMMAS

For the proof of the above theorems, we require the following lemmas:

Lemma 6.2.1 [7],[40] : Let $F$ be a family of meromorphic functions in a domain $D$ and let the zeros of $f$ be of multiplicity not less than $k$ (a positive integer) for each $f \in F$. If $F$ is not normal at $z_0 \in D$ then for $0 \leq \alpha < k$ there exist a sequence of complex numbers $z_j \to z_0$, a sequence of functions $f_j \in F$ and a sequence of positive numbers $\rho_j \to 0$ such that

$$g_j(\xi) = \rho_j^{-\alpha} f_j(z_j + \rho_j \xi)$$

converges spherically and locally uniformly to a nonconstant meromorphic function $g(\xi)$ in $C$. Moreover the order of $g$ is not greater than two and the zeros of $g$ are of multiplicity not less than $k$.

Note 1: If each $f \in F$ has no zero then $g$ also has no zero and in this case we can choose $\alpha$ to be any finite real numbers.
W. Bergweiler and X. Pang [2] proved the following theorem:

**Lemma 6.2.2**: Let $f$ be a transcendental meromorphic function of finite order and let $P$ be a polynomial, $P \not= 0$. Suppose that all zeros of $f$ are multiple, except possibly finitely many. Then $f' - P$ has infinitely many zeros.

**Lemma 6.2.3**: Let $f$ be a non-constant rational function in $C$. Then $f' - z$ must have some zeros.

**Proof**: Let $f = p/q$, where $p, q$ are polynomials of degree $m, n$ respectively and $p, q$ have no common factor.

We now consider the following cases.

Let $\psi = f' - z = \frac{p'q - pq'}{q^2} - z = \frac{p'q - pq' - zq^2}{q^2} = \frac{p_1}{q_1}$, say,

where $p_i$ and $q_i$ are polynomials of degree $m_i$ and $n_i$.

**Case 1**: Let $m < n$

Clearly $m_i > n_i$. Therefore $\psi$ is non-constant. If $\psi$ has no zero then $p_i$ and $q_i$ share 0 CM (counting multiplicities) and so $p_i = A q_i$, where $A$ is a constant. Therefore $\psi = A$, which is impossible. So $\psi = f' - z$ must have some zeros.

**Case 2**: Let $m = n$

Clearly $m_i > n_i$ and so $\psi$ is non-constant. Therefore as Case 1, $\psi = f' - z$ must have some zeros.

**Case 3**: Let $m > n$

Subcase (i) Let $m = n + 1$
Clearly $m > n$, and so $\psi$ is non-constant. Therefore as Case 1, $\psi = f' - z$ must have some zeros.

Subcase (ii) Let $m \geq n + 2$

Clearly $m > n$, and so $\psi$ is non-constant. Therefore as Case 1, $\psi = f' - z$ must have some zeros. This proves the lemma 6.2.3

W. Bergweiler [3] proved the following theorem

**Lemma 6.2.4:** Let $f$ be a transcendental meromorphic function of finite order and $R$ is a polynomial which does not vanish identically, then $f^2 f' - R$ has infinitely many zeros for every $n \in N$.

**Lemma 6.2.5:** Let $f$ be a non-constant rational function in $C$. Then $f^2 f' - z$ must have some zeros.

Proof: Let $g = f^{n+1}/(n+1)$. Then $g$ is a non-constant rational function in $C$. So by Lemma 6.2.3, $g' - z = f^2 f' - z$ must have some zeros. This proves the lemma 6.2.5.

**Lemma 6.2.6:** Let $f$ be a non-constant meromorphic function in $C$ such that $f$ has no zero of multiplicity $< k$. Suppose that $\Psi = \frac{f^n f^{(k)}}{z}$, where $k$, $n$ are positive integers. If $n \geq k \geq 2$, then $f^* f^{(k)}$ has infinitely many fix points.

Proof: Since $f$ has no zero of multiplicity $< k$, we get

$$\Psi = \frac{f^n f^{(k)}}{z} \neq \text{constant.}$$

By second fundamental theorem, we have
\[
T(r, \Psi) \leq N(r, \Psi) + N\left(r, \frac{1}{\Psi}\right) + N\left(r, \frac{1}{\Psi^2}\right) + S(r, \Psi)
\]
\[
= N\left(r, \frac{f^n f^{(k)}}{z}\right) + N\left(r, \frac{1}{f^n f^{(k)} / z}\right) + N\left(r, \frac{1}{\Psi^2}\right) + S(r, \Psi)
\]
\[
\leq \frac{1}{n + k + 1} N\left(r, \frac{f^n f^{(k)}}{z}\right) + \frac{1}{nk} N\left(r, \frac{1}{f^n f^{(k)} / z}\right) + \frac{1}{\Psi^2} + S(r, \Psi)
\]
\[
\leq \frac{nk + n + k + 1}{nk (n + k + 1)} T(r, \Psi) + N\left(r, \frac{1}{\Psi^2}\right) + S(r, \Psi)
\]
\[
\left(1 - \frac{nk + n + k + 1}{nk (n + k + 1)}\right) T(r, \Psi) \leq N\left(r, \frac{1}{\Psi^2}\right) + S(r, \Psi)
\]

Since \( n \geq k \geq 2 \), we get \( \Psi^{-1} \) has infinitely many zeros i.e \( f^n f^{(k)} \) has infinitely many fix points.

**Lemma 6.2.7**: Let \( f \) be a non constant meromorphic function in \( C \) such that \( f^i \neq 0 \). Then \( f^{(k)} - z f^n \) must have some zeros, where \( k \) and \( n (\geq k) \) are positive integers.

Proof: First we assume that \( k = 1 \). Then by Lemmas 6.2.4 and 6.2.5, we see that \( f^n f' - z \) must have some zeros. Since a zero of \( f^n f' - z \) is not a pole or a zero of \( f \), it follows that a zero of \( f^n f' - z \) is a zero of \( f' - z f^n \).

Now we assume that \( k \geq 2 \). Then by Lemma 6.2.6, \( f^n f^{(k)} - z \) must have some zeros. Since a zero of \( f^n f^{(k)} - z \) is a zero of \( f^{(k)} - z f^n \), The lemma 6.2.7 is proved.

**Lemma 6.2.8**: Let \( f \) be a non constant meromorphic function in \( C \) such that \( f \) has no zeros and also \( f \) has no pole of multiplicity less than \( k + 2 \). Then \( f^{(k)} - z \) must have some simple zero, where \( k \) is a positive integer.

Proof: Since \( f \) has no pole of multiplicity less than \( k + 2 \), we get
\[ \Psi = \frac{f^{(k)}}{z} \neq \text{constant.} \]

Then by second fundamental theorem

\[
T(r,\Psi) \leq N(r,\Psi) + N\left(r, \frac{1}{\Psi}\right) + N\left(r, \frac{1}{\Psi - 1}\right) + S(r,\Psi)
\]

\[
= N\left(r, \frac{f^{(k)}}{z}\right) + N\left(r, \frac{1}{f^{(k)} / z}\right) + N\left(r, \frac{1}{\Psi - 1}\right) + S(r,\Psi)
\]

\[
\leq \frac{1}{k + 3 + k} N\left(r, \frac{f^{(k)}}{z}\right) + \left[kN(r, f) + (k + 1)N\left(r, \frac{1}{f}\right)\right] + N\left(r, \frac{1}{\Psi - 1}\right) + S(r,\Psi)
\]

\[
\leq \frac{1}{2k + 3} N\left(r, \frac{f^{(k)}}{z}\right) + \frac{k}{2k + 3} N\left(r, \frac{f^{(k)}}{z}\right) + N\left(r, \frac{1}{\Psi - 1}\right) + S(r,\Psi)
\]

If possible, suppose that \( f^{(k)} - z \) has no simple zero, then

\[
T(r,\Psi) \leq \frac{1}{2k + 3} N\left(r, \frac{f^{(k)}}{z}\right) + \frac{k}{2k + 3} N\left(r, \frac{f^{(k)}}{z}\right) + \frac{1}{2} N\left(r, \frac{1}{\Psi - 1}\right) + S(r,\Psi)
\]

\[
\leq \frac{1}{2k + 3} T(r,\Psi) + \frac{k}{2k + 3} T(r,\Psi) + \frac{1}{2} T(r,\Psi) + S(r,\Psi)
\]

\[
\leq \left[\frac{4k + 5}{4k + 6}\right] T(r,\Psi) + S(r,\Psi)
\]

That is \[ \left[\frac{1}{4k + 6}\right] T(r,\Psi) \leq S(r,\Psi) \]

Since \( k \geq 1 \), we get a contradiction. Therefore \( f^{(k)} - z \) must have some simple zeros.

### 6.3 PROOFS OF THE THEOREMS

**PROOF OF THEOREM 6.1.1**

If possible suppose that \( F \) is not normal at \( z_0 \in D \). Then \( F = \left\{ \frac{1}{f} : f \in F \right\} \) is not normal at \( z_0 \in D \). Let \( \alpha = 1 \), then by Lemma 6.2.1, there exist a sequence of
functions $f_j \in F$, a sequence of complex numbers $z_j \to z_0$ and a sequence of positive numbers $\rho_j \to 0$ such that $g_j(\zeta) = \rho_j^{-1} f^{-1}(z_j + \rho_j \zeta)$ converges spherically and locally uniformly to a non constant meromorphic function $g(\zeta)$ in $C$. Also, the order of $g$ does not exceed two and $g$ has multiple zeros.

Again by Hurwitz's Theorem $g$ has multiple poles.

By Lemmas 6.2.3 and 6.2.4, we see that there exists $\zeta_0 \in C$ such that

$$g'(\zeta_0) - \zeta_0 = 0 \quad \text{...(6.3.1)}$$

Since $\zeta_0$ is not a pole of $g$, it follows that $g_j(\zeta)$ converges uniformly to $g(\zeta)$ in some neighborhood of $\zeta_0$.

We also see that $-\frac{1}{g'(\zeta)} \{g'(\zeta) - \zeta\}$ is the uniform limit of $\rho_j^{-1} \left\{f_j'(\zeta) + \zeta f_j^2 - a\right\}$ in some neighborhood of $\zeta_0$. In view of (6.3.1) and Hurwitz's Theorem there exists a sequence $\zeta_j \to \zeta_0$ such that $f_j'(\zeta_j) + \zeta_j f_j^2 - a = 0$.

So by the given condition, we have

$$|g_j(\zeta_j)| = \frac{1}{\rho_j f_j(z_j + \rho_j \zeta_j)} \leq \frac{1}{\rho_j M}.$$ 

Since $\zeta_0$ is not a pole of $g$, there exists a positive number $K$ such that in some neighborhood of $\zeta_0$, we get $|g(\zeta)| \leq K$.

Since $g_j(\zeta)$ converges uniformly to $g(\zeta)$ in some neighborhood of $\zeta_0$. We get for all large values of $j$ and for all $\zeta$ in that neighborhood of $\zeta_0$

$$|g_j(\zeta) - g(\zeta)| < 1$$

Since $\zeta_j \to \zeta$, we get for all large values of $j$

$$K \geq |g(\zeta_j)| \geq |g_j(\zeta_j)| - |g(\zeta_j) - g_j(\zeta_j)| \geq \frac{1}{\rho_j M} - 1,$$

which is contradiction. This proves the theorem.
PROOF OF THEOREM 6.1.2

Let \( a = k \). If possible suppose that \( F \) is not normal at \( z_0 \in D \). Then by Lemma 6.2.1 and Note 6.2.1 there exists a sequence of functions \( f_j \in F \), a sequence of complex numbers \( z_j \to z_0 \), and a sequence of positive numbers \( \rho_j \to 0 \) such that

\[
g_j(\zeta) = \rho_j^{-k} f_j(z_j + \rho_j \zeta)
\]

converges spherically and locally uniformly to a non-constant meromorphic function \( g(\zeta) \) in \( C \). Now by conditions (i) and (ii) and Hurwitz’s theorem we see that \( g(\zeta) \) has no zero and has no pole of multiplicity less than \( k + 2 \).

Now by Lemma 6.2.8, \( g^{(i)}(\zeta) - z \) has a simple zero at a point \( \zeta_0 \in C \). Since \( \zeta_0 \) is not a pole of \( g(\zeta) \), in some neighborhood of \( \zeta_0 \), \( g_j(\zeta) \) converges uniformly to \( g(\zeta) \).

Since

\[
g_j^{(i)}(\zeta) - z + \sum_{i=0}^{k-1} a_i \rho_j^{i+1} g_j^{(i)}(\zeta) = f_j^{(i)}(z_j + \rho_j \zeta) + \sum_{i=0}^{k-1} a_i f_j^{(i)}(z_j + \rho_j \zeta) - z
\]

\[
= H(f_j(z_j + \rho_j \zeta) - z)
\]

and

\[
\sum_{i=0}^{k-1} a_i \rho_j^{i+1} g_j^{(i)}(\zeta)
\]

converges uniformly to zero in some neighborhood of \( \zeta_0 \), it follows that \( g^{(i)}(\zeta) - z \) is the uniform limit of \( H(f_j(z_j + \rho_j \zeta) - z) \).

Since \( \zeta_0 \) is a simple zero of \( g^{(i)}(\zeta) - z \), by Hurwitz’s theorem there exists a sequence \( \zeta_j \to \zeta_0 \) such that \( \zeta_j \) is a simple zero of \( H(f_j(z_j + \rho_j \zeta) - z) \).

So by the given condition \( |f_j(z_j + \rho_j \zeta_j)| \geq M \) for all large values of \( j \).

Hence for all large values of \( j \) we get \( |g_j(\zeta_j)| \geq M/\rho_j^k \) and as the last part of the proof of Theorem 6.1.1 we arrive at a contradiction. This proves the theorem 6.1.2.
PROOF OF THEOREM 6.1.3

Let $\alpha = \frac{k}{1 + n} < 1$. If possible suppose that $F$ is not normal at $z_0 \in D$. Then by Lemma 6.2.1 there exists a sequence of functions $f_j \in F$, a sequence of complex numbers $z_j \to z_0$ and a sequence of positive numbers $\rho_j \to 0$ such that

$$g_j(\xi) = \rho_j^{-a} f_j(z_j + \rho_j \xi)$$

converges spherically and locally uniformly to a non constant meromorphic function $g(\xi)$ in $C$. Also $g$ has no zero of multiplicity less than $k$. So $g^{(k)} \neq 0$ and by Lemma 6.2.7, we get

$$g^{(k)}(\xi_0) - \frac{z}{g'(\xi_0)} = 0$$

for some $\xi_0 \in C$.

Clearly $\xi_0$ is neither a zero nor pole of $g$. So in some neighborhood of $\xi_0$, $g_j(\xi)$ converges uniformly to $g(\xi)$.

Now in some neighborhood of $\xi_0$ we see that $g^{(k)}(\xi) - \xi g^{-a}(\xi)$ is the uniform limit of

$$g_j^{(k)}(\xi) - \xi g_j^{-a}(\xi) - \rho_j^{-a} = \rho_j^{-a} \{f_j^{(k)}(z_j + \rho_j \xi) - (z_j + \rho_j \xi) f_j^{-a}(z_j + \rho_j \xi) - a\}$$

By (6.3.2) and Hurwitz's Theorem there exists a sequence $\xi_j \to \xi_0$ such that for all large values of $j$

$$f_j^{(k)}(z_j + \rho_j \xi_j) - (z_j + \rho_j \xi_j) f_j^{-a}(z_j + \rho_j \xi_j) = a$$

Therefore for all large values of $j$ it follows from the given condition $|g_j(\xi)| \geq M/\rho_j^a$ and as in the last part of the proof of Theorem 6.1.1, we arrive at a contradiction. This proves the theorem 6.1.3.