CHAPTER 5

VALUE DISTRIBUTION OF DIFFERENTIAL MONOMIALS OF MEROMORPHIC FUNCTIONS SATISFYING GENERALIZED DIFFERENTIAL EQUATIONS
CHAPTER 5

VALUE DISTRIBUTION OF DIFFERENTIAL MONOMIALS OF MEROMORPHIC FUNCTIONS SATISFYING GENERALIZED, FIRST, SECOND AND FOURTH PAINLEVE DIFFERENTIAL EQUATIONS

In this Chapter, applying Nevanlinna theory we study the value distribution properties of differential monomials of transcendental meromorphic solutions of generalized first, second and the fourth Painleve differential equations.

5.1 INTRODUCTION

In this paper we consider

(1) Generalized first Painleve differential equation

\[
f^{(m)} = a_1 f^2 + zf, \quad \text{...(5.1.1)}
\]

where \( m \geq 1 \) is an integer and \( a_1 \) is any constant. If \( m = 2 \) and \( a_1 = 6 \), we get first Painleve differential equation.

(2) Generalized second Painleve differential equation

\[
f^{(2m)} = a_2 f^3 + zf + b_2, \quad \text{...(5.1.2)}
\]

where \( m \geq 1 \) is an integer, \( a_2 \) and \( b_2 \) are any constants. If \( m = 1 \) and \( a_2 = 2 \), we get the second Painleve differential equation.

(3) Generalized fourth Painleve differential equation

\[
f f^{(2m)} = \sum_{i=1}^{m} (-1)^{i+1} f^{(2m-i)} f^{(i)} + \frac{(-1)^m \left( f^{(m)} \right)^2}{2} + \sum_{i=1}^{2m-1} k_i f^{2m+3-i} + k_{2m} zf^3 + k_{2m+1}(z^2 - \alpha) f^2 + B, \quad \text{...(5.1.3)}
\]
where \( m \geq 1 \) is an integer and \( k_1, k_2 \ldots k_{2m+1} \) are constants. If \( m = 1 \), \( k_1 = 3/2, k_2 = 4, k_3 = 2 \), we get fourth Painleve differential equation.

We now introduce **differential monomials** and **differential polynomials**

Let \( M(f) = f^{n_0}(f^1)^{n_1} \ldots (f^s)^{n_s} \) be a differential monomial.

**Degree of** \( M(f) = \gamma_M = n_0 + n_1 + \ldots + n_s \).

**Weight of** \( M(f) = \Gamma_M = \sum_{i=0}^{s} (i+1)n_i = n_0 + 2n_1 + 3n_2 + \ldots + (s+1)n_s \). \hspace{1cm} \ldots(5.1.4)\)

If \( M_1, M_2 \ldots M_s \) are differential monomials and if \( a_j \)'s are constants \((a_j \neq 0)\), then \( P(f) = \sum_{j=1}^{s} a_j M_j(f) \) is called a differential polynomial in \( f \).

**Degree of** \( P(f) = \gamma_P = \max_{j} \gamma_{M_j} \).

**Weight of** \( P(f) = \Gamma_P = \max_{j} \Gamma_{M_j} \). \hspace{1cm} \ldots(5.1.5)\)

If \( \gamma_P = \gamma_{M_j} \) and \( \Gamma_P = \Gamma_{M_j} \) for \( (j = 1, 2, 3 \ldots s) \), then \( P(f) \) is called homogeneous differential polynomials in \( f \).

Our main aim here is to obtain some interesting value distribution properties of differential monomials of meromorphic functions satisfying (5.1.1), (5.1.2), and (5.1.3).

Furthermore, in order to obtain counting functions for multiple points, we consider following definitions.

If \( a \in \overline{C} \) is an \( a \)-point of \( f \) of multiplicity \( \geq s \), we set

\[
N_s(r,a,f) = N(r,a,f) - s \overline{N}(r,a,f)
\]
and we define

\[
\theta_s(a,f) = \lim_{r \to \infty} \frac{N_s(r,a,f)}{T(r,f)},
\]

\[
N_1(r, \frac{1}{f-a}) = N\left(r, \frac{1}{f-a}\right) - \overline{N}\left(r, \frac{1}{f-a}\right),
\]

\[
N_1(r,f) = 2N(r,f) - N(r,f'),
\]

\[
\theta_1(g,f) = \lim_{r \to \infty} \frac{N_1\left(r, \frac{1}{f-g}\right)}{T(r,f)},
\]

\[
\Phi_B(f) = \lim_{r \to \infty} \frac{N_B(r,f)}{T(r,f)}
\]

where

\[
N_B(r,f) = N\left(r, \frac{1}{f}\right) + 2N(r,f) - N(r,f'),
\]

\[
\Phi_e(f) = \lim_{r \to \infty} \frac{N\left(r, \frac{1}{f'}\right)}{T(r,f)},
\]

\[
\Phi(M(f))a = \lim_{r \to \infty} \frac{N\left(r, \frac{1}{M(f)-a}\right)}{T(r,f)},
\]

\[
\delta(g,f) = \lim_{r \to \infty} \frac{m\left(r, \frac{1}{f-g}\right)}{T(r,f)},
\]

\[
\Delta(g,f) = \lim_{r \to \infty} \frac{m\left(r, \frac{1}{f-g}\right)}{T(r,f)}.
\]

In 1999, Bhoosnurmath and Chhaya Hombali proved the following two theorems

**Theorem 5.1.A [4]**: All solutions of (5.1.1) are transcendental meromorphic functions having poles of order \(m\). Further, \(\delta(a,f) = 0\) for every \(a \in \overline{C}\). Then
\[(i) \, \theta_l(\infty, f) = \frac{m-1}{m}, \quad (ii) \, \Phi_e(f) = \lim_{r \to \infty} \frac{N(r, f')}{T(r, f)} = \frac{m+1}{m}, \quad \text{and} \]

\[(iii) \, \Phi(f) = \lim_{r \to \infty} \frac{N_l(r)}{T(r, f)} = 2. \]

If \( m = 2n \) where \( n \geq 1 \) then \( \theta_n(a, f) \leq \frac{2n-1}{2(2n-1)} \) where \( a \) is an \( a \)-point of \( f \) of multiplicity \( \geq n \).

**Theorem 5.1.B [5]:** Let \( f(z) \) be a solution of (5.1.2). Then \( f \) is meromorphic and all poles of \( f(z) \) are of order \( m \) and \( \delta(a, f) = 0 \) for every \( a \in \mathbb{C} \),

\[(i) \, \theta_l(\infty, f) = \frac{m-1}{m}, \quad (ii) \, \Phi_e(f) = \lim_{r \to \infty} \frac{N(r, f')}{T(r, f)} = \frac{m+1}{m}, \]

\[(iii) \, \Phi(f) = \lim_{r \to \infty} \frac{N_l(r)}{T(r, f)} = 2 \quad \text{and} \quad (iv) \, \theta_n(a, f) \leq \frac{m}{2m+1} \]

where \( a \neq 0, \infty \) is an \( a \)-point of \( f \) of multiplicity \( \geq m \).

The main results of chapter 5 are as follows

**Theorem 5.1.1:** Let \( \psi_j \) be a transcendental meromorphic solution of generalized first, second and fourth Painleve differential equations, for \( j = 1, 2, 4 \) and let \( n \in \mathbb{N} \cup \{0\} \). Then

\[(i) \, \Delta[p(z), \psi_j^{(n)}] = \lim_{r \to \infty} \frac{m}{T(r, \psi_j^{(n)})} = 0, \quad b_2, B \neq 0 \quad \text{for} \quad j = 2, 4 \]

and for all polynomials \( p(z) \),

\[(ii) \, \Delta[0, M(\psi_j)] = \lim_{r \to \infty} \frac{m}{r, M(\psi_j)} = 0 \quad \text{for} \quad j = 1, 2, 4 \]

\[(iii) \, \Delta(\infty, P(\psi_j)) = 0 \quad \text{for} \quad j = 1, 2, 4 \]
for all differential monomial $M(f)$ and homogeneous differential polynomial $P(f)$ defined as above.

In order to further investigate the value distribution of $\psi_j, j = 1, 2, 4$, we need some more definition.

**Definition:** Let $f$ be a meromorphic function, let $\alpha \in \mathcal{L}_f$ and let $n \in \mathbb{N} \cup \{0\}$, then

$$\Phi(n)(f, \alpha) = \lim_{r \to \infty} \frac{N\left(\frac{1}{r}, f(n) - \alpha\right)}{T(r, f)},$$

where $\mathcal{L}_f = \{h \text{ meromorphic} : T(r, h) = S(r, f)\}$ and

$$\Phi(M(f), \alpha) = \lim_{r \to \infty} \frac{N\left(\frac{1}{r}, \frac{M(f) - \alpha}{r}\right)}{T(r, f)}$$

**Theorem 5.1.2:** Let the assumption of Theorem 5.1.1 hold. Then

(i) $\Phi(n)(\psi_j, p(z)) = \frac{m+n}{m}$, for $j = 1, 2$ and $b_2 \neq 0$ for $j = 2$

(ii) If $B \neq 0$, then $\Phi(n)(\psi_4, p(z)) = n + 1$

(iii) $\Phi(M(\psi_j), p(z)) \leq \frac{(m-1)\gamma_M + \Gamma_M}{m}$, for $j = 1, 2$ and $b_2 \neq 0$ for $j = 2$

(iv) $\Phi(M(\psi_4), p(z)) \leq \Gamma_M$ if $B \neq 0$

**Theorem 5.1.3:** Let the assumption of Theorem 5.1.1 hold. Then

(i) $T\left(r, P(\psi_j)\right) = \frac{(m-1)\gamma_p + \Gamma_p}{m} T(r, \psi_j) + O(\log r)$ for $j = 1, 2$

(ii) $T\left(r, P(\psi_4)\right) = \Gamma_p \cdot T(r, \psi_4) + O(\log r)$

In particular $T\left(r, \psi_j^{(n)}\right) = \frac{m+n}{m} T(r, \psi_j) + O(\log r)$ for $j = 1, 2$
As we can see from Theorem 5.1.2 and Theorem 5.1.3, the value distribution of the \( n^{th} \) derivative of the first, second and fourth generalized Painlevé differential equations is extremely regular. The following theorem demonstrates this regularity even further.

**Theorem 5.1.4:** Let the assumption of Theorem 5.1.1 hold and let \( b_2, B \neq 0 \).

Then

\[
(\text{i}) \; \Phi_{e}(\psi_j^{(n)}) = \frac{m+n+1}{m+n} \text{ for } j = 1, 2
\]
\[
(\text{ii}) \; \Phi_{e}(\psi_4^{(n)}) = \frac{n-2}{n+1}
\]
\[
(\text{iii}) \; \Phi_{e}(p(\psi_j)) \leq \frac{(m-1)\gamma_p + \Gamma_p + 1}{(m-1)\gamma_p + \Gamma_p} \text{ for } j = 1, 2 \quad \text{and} \quad (\text{iv}) \; \Phi_{e}(p(\psi_4)) \leq \frac{\Gamma_p + 1}{\Gamma_p}
\]

We also have the following version of the Second Main Theorem for the solutions of equation (5.1.1), (5.1.2) and (5.1.3).

**Theorem 5.1.5:** Let \( p_1(z), p_2(z), p_3(z), \ldots, p_n(z) \) be distinct complex polynomials and \( b_2, B \neq 0 \). Then for all \( n \in \mathbb{N} \cup \{0\} \), we have

\[
\sum_{k=1}^{g} m \left( r, \frac{1}{\psi_j^{(n)} - p_k(z)} \right) + N_B \left( r, \psi_j^{(n)} \right) = \frac{2(m+n)}{m} T(r, \psi_j) + O(\log r) \quad \text{for } j = 1, 2
\]

and

\[
\sum_{k=1}^{g} m \left( r, \frac{1}{\psi_4^{(n)} - p_k(z)} \right) + N_B \left( r, \psi_4^{(n)} \right) = (2n+2) T(r, \psi_4) + O(\log r)
\]

Thus, Nevanlinna's second fundamental theorem goes into asymptotic equality.
Theorem 5.1.6: Let \( \psi_j, j=1,2,4 \) be a transcendental meromorphic solutions of generalized first, second and fourth Painleve differential equation, respectively. Then

\[
(I) \quad \Phi_B[P(\psi_j)] = \lim_{r \to \infty} \frac{N_B[r,P(\psi_j)]}{T[r,P(\psi_j)]} \leq 2, \quad \forall \ j=1,2,4
\]

(ii) \( \nu(\infty, P(\psi_j)] = \lim_{r \to \infty} \frac{N_j[r,P(\psi_j)]}{T[r,P(\psi_j)]} = \frac{(m-1)\gamma_p + \Gamma_p - 1}{(m-1)\gamma_p + \Gamma_p} \) for \( j=1,2 \)

and (iii) \( \nu(\infty, P(\psi_4)] = \lim_{r \to \infty} \frac{N_4[r,P(\psi_4)]}{T[r,P(\psi_4)]} = \frac{\Gamma_p - 1}{\Gamma_p} \)

5.2 LEMMAS

We require following lemmas for the proof of our main theorem.

Lemma 5.2.1 [9, Clunie]: Let \( f \) be a transcendental meromorphic solution of finite order of the differential equation \( f^n P(z,f) = Q(z,f) \), where \( P(z,f) \) and \( Q(z,f) \) are polynomials in \( f \) and its derivatives with rational coefficients. If the total degree of \( Q(z,f) \) is \( \leq n \), then

\[
m(r,P(z,f)) = O(\log r).
\]

Lemma 5.2.2 [32, Mohon'ko]: Let \( P(z,u) \) be a polynomial in \( u \) and its derivatives with rational coefficients and assume that \( u = f \) is a transcendental meromorphic solution of finite order of the differential equation \( P(z,u) = 0 \).

If \( P(z,0) \neq 0 \), then

\[
m\left( r, \frac{1}{f} \right) = O(\log r).
\]
The following lemma is fundamental for our considerations of the value distribution of differential polynomials of the solution of (5.1.1), (5.1.2), and (5.1.3).

**Lemma 5.2.3**: Let $j=1,2,4$, let $\psi_j$ denote arbitrary transcendental meromorphic solutions of (5.1.1), (5.1.2) and (5.1.3). Let $p(z)$ be a polynomial and let $n \in \mathbb{N} \cup \{0\}$. Then

$$m(r, \psi_j^{(n)}) = O(\log r) \quad \text{...(5.2.1)}$$

$$m\left(r, \frac{1}{\psi_j^{(n)} - p(z)} \right) = O(\log r) \quad \text{...(5.2.2)}$$

$$m\left(r, P(\psi_j) \right) = O(\log r) \quad \text{...(5.2.3)}$$

and

$$m\left(r, \frac{1}{M(\psi_j)} \right) = O(\log r), \quad \text{...(5.2.4)}$$

where $M(f), P(f)$ are differential monomials and polynomials in $f$.

**Proof**: We consider the following cases for (5.2.1)

**Case 1**: $n = 0$ and $j = 1$

From equation (5.1.1), we get

$$\psi_j(a_1 \psi_1) = \psi_1^{(m)} - z$$

By Lemma 5.2.1, we get

$$m(r, \psi_1) = O(\log r)$$

Hence (5.2.1) holds for $n = 0$ and $j = 1$.

**Case 2**: $n = 0$ and $j = 2$

By (5.1.2), we get

$$\psi_2^{(2m)} = a_2 \psi_2^3 + z \psi_2 + b_2$$

$$\psi_2(a_2 \psi_2^2 + z) = \psi_2^{(2m)} - b_2$$

By lemma 5.2.1, we get

$$m(r, \psi_2) = O(\log r)$$
Hence (5.2.1) holds for \( n = 0 \) and \( j = 2 \).

**Case 3:** \( n = 0 \) and \( j = 4 \)

By (5.1.3), we can write

\[
k_1 \psi_4^{2m+2} = P(z, \psi_4, \psi_4', \psi_4'', \ldots, \psi_4^{(2m)}),
\]

where \( P \) is a differential polynomial in \( \psi_4 \) of degree \( 2m + 1 \)

\[
\psi_4^{2m+1}(k_1 \psi_4) = P(z, \psi_4, \psi_4', \psi_4'', \ldots, \psi_4^{(2m)}).
\]

Then by Lemma 5.2.1, we get

\[
m(r, \psi_4) = O(\log r).
\]

Therefore, we observe that the case \( n = 0 \) follows by Lemma 5.2.1 for (5.2.1).

Hence, by the lemma of the logarithmic derivative, we obtain

\[
m\left( r, \frac{\psi_j^{(n)}}{\psi_j} \right) \leq m\left( r, \frac{\psi_j^{(n)}}{\psi_j} \right) + m(r, \psi_j) = O(\log r)
\]

for all \( j = 1, 2 \) and \( 4 \) and for all \( n \in \mathbb{N} \cup \{0\} \).

**Next we consider the assertion (5.2.2).**

**Case 4:** \( p(z) = 0, \ n = 0 \) and \( j = 1 \)

By Lemma 5.2.2, we have

\[
m\left( r, \frac{1}{\psi_1} \right) = O(\log r)
\]

**Case 5:** \( p(z) = 0, \ n = 1 \) and \( j = 1 \)

Differentiating (5.1.1) and dividing by \( \psi_1' \), we have

\[
\frac{\psi_1^{(m+1)}}{\psi_1} - 2k \psi_1 = \frac{1}{\psi_1}.
\]

Hence, by lemma of the logarithmic derivative and by (5.2.1) and (5.2.5), we get
Case 6: \( p(z) = 0 \), \( n = 1,2,\ldots,k \) and \( j = 1 \)

Assume now that we have proved

\[ m\left( r, \frac{1}{\psi_1^{(n)}} \right) = O(\log r) \] \hspace{1cm} ...(5.2.6)

for all \( n = 1,2,\ldots,k \).

Now we consider the case \( p(z) = 0 \), \( j = 1 \) \( n = k + 1 \).

By differentiating equation (5.1.1), \( 2k \) times, we obtain

\[ \psi_1^{(2k+m)} = k \sum_{j=0}^{2k} \binom{2k}{j} \psi_1^{(j)} \psi_1^{(2k-j)}, \]

so that,

\[ \frac{\psi_1^{(k)}}{\psi_1^{(k+1)}} = \frac{1}{2k} \left( \frac{\psi_1^{(2k+m)}}{\psi_1^{(k+1)}} \right) - \sum_{j=0}^{2k} \frac{\binom{2k}{j} \psi_1^{(j)} \psi_1^{(2k-j)}}{\psi_1^{(k+1)}}, \] \hspace{1cm} ...(5.2.7)

where \( Q(\psi_1) \) is a differential polynomial in \( \psi_1 \) and its derivatives with constant coefficients. Moreover, every term of \( Q \) has some \( \psi_1^{(l)} \) as a factor, where \( l \geq k + 1 \).

Hence by (5.2.1), (5.2.6), (5.2.7) and by lemma of the logarithmic derivative, we have

\[ m\left( r, \frac{1}{\psi_1^{(k+1)}} \right) \leq m\left( r, \frac{1}{\psi_1^{(k)}} \right) + m\left( r, \frac{\psi_1^{(k)}}{\psi_1^{(k+1)}} \right) \]

\[ = m\left( r, \frac{\psi_1^{(k)}}{\psi_1^{(k+1)}} \right) + O(\log r) = O(\log r) \]
Thus, \[ m \left( r, \frac{1}{\psi_1(n)} \right) = O(\log r) \] 
...(5.2.8)
for all \( n \in \mathbb{N} \cup \{0\} \) by the induction principle.

**Case 7:** \( p(z) = 0, \ j = 2 \) and \( n = 0 \)

By Lemma 5.2.2, we have \[ m \left( r, \frac{1}{\psi_2} \right) = O(\log r) \] 
...(5.2.9)
since we assumed \( b_2 \neq 0 \).

**Case 8:** \( p(z) = 0, \ j = 2 \) and \( n = 1 \)

Differentiating (5.1.2) and dividing by \( \psi_2' \), we get \[ \frac{\psi_2}{\psi_2'} = \frac{\psi_2(2m+1)}{\psi_2'} - 3a_2\psi_2^2 - z \]  
...(5.2.10)
we can write \[ m \left( r, \frac{1}{\psi_1} \right) \leq m \left( r, \frac{\psi_2}{\psi_2'} \right) + m \left( r, \frac{1}{\psi_2} \right) \]

Hence by (5.2.1), (5.2.9), (5.2.10) and by the lemma of the logarithmic derivative, we have \[ m \left( r, \frac{1}{\psi_2} \right) \leq O(\log r) \]  

**Case 9:** \( p(z) = 0, \ n = 1,2,\ldots,k \) and \( j = 2 \)

Assume now that we have proved 
\[ m \left( r, \frac{1}{\psi_2(n)} \right) = O(\log r) \]  
for all \( n = 1,2,\ldots,k \).  
...(5.2.11)
We consider next case \( p(z) = 0, \ j = 2 \) and \( n = k + 1 \).

By differentiating equation (5.1.2), \( 3k \) times, we obtain by applying Leibniz theorem

\[
\psi_2^{(2m+3k)} = k \sum_{j=0}^{3k} \binom{3k}{j} \left( \sum_{m=0}^{j} \frac{\psi_2^{(3k-j)} \psi_2^{(m)} \psi_2^{(j-m)} \psi_2^{(3k-j)} + z \psi_2^{(3k)} + 3k \psi_2^{(3k-1)}}{\psi_2^{(k+1)}} \right)
\]

\[
= k \sum_{j=0}^{3k} \binom{3k}{j} \left( \sum_{m=0}^{j} \frac{\psi_2^{(m)} \psi_2^{(j-m)} \psi_2^{(3k-j)} + z \psi_2^{(3k)} + 3k \psi_2^{(3k-1)}}{\psi_2^{(k+1)}} \right) + \left( \binom{3k}{2k} \frac{\psi_2^{(3k-2k)} \psi_2^{(k)} \psi_2^{(2k-2k)}}{\psi_2^{(k+1)}} + \frac{z \psi_2^{(3k)} + 3k \psi_2^{(3k-1)}}{\psi_2^{(k+1)}} \right)
\]

Dividing throughout by \( \psi_2^{(k+1)} \), we get

\[
\frac{\psi_2^{(2m+3k)}}{\psi_2^{(k+1)}} = k \sum_{j=0}^{3k} \binom{3k}{j} \left( \sum_{m=0}^{j} \frac{\psi_2^{(3k-j)} \psi_2^{(m)} \psi_2^{(j-m)} \psi_2^{(3k-j)} + z \psi_2^{(3k)} + 3k \psi_2^{(3k-1)}}{\psi_2^{(k+1)}} \right)
\]

\[
= k \sum_{j=0}^{3k} \binom{3k}{j} \left( \sum_{m=0}^{j} \frac{\psi_2^{(m)} \psi_2^{(j-m)} \psi_2^{(3k-j)} + z \psi_2^{(3k)} + 3k \psi_2^{(3k-1)}}{\psi_2^{(k+1)}} \right) + \frac{\psi_2^{(3k-2k)} \psi_2^{(k)} \psi_2^{(2k-2k)}}{\psi_2^{(k+1)}} + \frac{z \psi_2^{(3k)} + 3k \psi_2^{(3k-1)}}{\psi_2^{(k+1)}}
\]

\[
\left( \frac{\psi_2^{(k)}}{\psi_2^{(k+1)}} \right)^3 = \frac{\psi_2^{(2m+3k)}}{\psi_2^{(k+1)}} - k \sum_{j=0}^{3k} \binom{3k}{j} \left( \sum_{m=0}^{j} \frac{\psi_2^{(3k-j)} \psi_2^{(m)} \psi_2^{(j-m)} \psi_2^{(3k-j)} + z \psi_2^{(3k)} + 3k \psi_2^{(3k-1)}}{\psi_2^{(k+1)}} \right)
\]

\[
\frac{\psi_2^{(3k)} \psi_2^{(2k-2k)} \psi_2^{(k)}}{\psi_2^{(k+1)}} + \frac{z \psi_2^{(3k)} + 3k \psi_2^{(3k-1)}}{\psi_2^{(k+1)}}
\]

...(5.2.12)

Every term in the numerator of the second term on the right hand side of (5.2.12) has at least one function \( \psi_2^{(l)} \) as a factor such that \( l \geq k + 1 \).

Hence by (5.2.1), (5.2.9) and by the lemma of the logarithmic derivative, we get
Thus
\[
m \left( r, \frac{1}{\psi_2(k+1)} \right) \leq m \left( r, \frac{\left( \psi_2(k) \right)^3}{\psi_2(k+1)} \right) + m \left( r, \frac{1}{\psi_2(k+1)} \right)
\]
\[= m \left( r, \frac{\left( \psi_2(k) \right)^3}{\psi_2(k+1)} \right) + O(\log r) = O(\log r)
\]

Thus
\[
m \left( r, \frac{1}{\psi_2(n)} \right) = O(\log r)
\]...

for all \( n \in \mathbb{N} \cup \{0\} \) by the induction principle.

**Case 10:** \( p(z) = 0, \ j = 4 \) and \( n = 0 \)

By Lemma 5.2.2, we have
\[
m \left( r, \frac{1}{\psi_4} \right) = O(\log r), \quad \text{since } B \neq 0
\]

**Case 11:** \( p(z) = 0, \ j = 4 \) and \( n = 1 \)

Differentiating (5.1.3), we get
\[
\psi_4 \psi_4^{(2m+1)} = \sum_{i=1}^{2m-1} k_i (2m+3-i) \psi_4^{(2m+2-i)} \psi_4 + 3k_2m z \psi_4 \psi_4' + k_2m \psi_4^3
\]
\[+ 2k_2m+1 \psi_4 \psi_4' (z^2 - \alpha) + 2k_2m+1 z \psi_4^2
\]

Dividing throughout by \( \psi_4 \), we get
\[
\psi_4^{(2m+1)} = \sum_{i=1}^{2m-1} k_i (2m+3-i) \psi_4^{(2m+1-i)} \psi_4 + 3k_2m z \psi_4 \psi_4' + k_2m \psi_4^2
\]
\[+ 2k_2m+1 \psi_4' (z^2 - \alpha) + 2k_2m+1 z \psi_4
\]...

Differentiating equation (5.2.14), we get
\[
\psi_4^{(2m+2)} = \sum_{i=1}^{2m-1} k_i (2m+3-i) \left[ \psi_4^{(2m+1-i)} \psi_4' + (2m+1-i) \psi_4^{2m-i} \omega^2 \right]
\]
\[+ 3k_2m \left[ z \psi_4 \psi_4' + \psi_4 \psi_4' \psi_4 + \psi_4 \psi_4' \right] + 2k_2m \psi_4 \psi_4'
\]
\[+ 2k_2m+1 \left[ z \psi_4' + (z^2 - \alpha) \psi_4' \right] + 2k_2m+1 z \psi_4'
\]
Dividing above equation by $2k_{2m+1}\psi_4'$

$$\frac{\psi_4^{(2m+2)}}{2k_{2m+1}\psi_4'} = \frac{\sum_{i=1}^{2m-1} k_i(2m+3-i)\psi_4^{2m+1-i}\psi_4' + \sum_{i=1}^{2m-1} k_i(2m+3-i)(2m+1-i)\psi_4^{2m-i}(\psi')^2}{2k_{2m+1}\psi_4'} + \frac{3k_{2m}z\psi_4\psi_4'}{2k_{2m+1}\psi_4'} + \frac{3k_{2m}z(\psi')^2}{2k_{2m+1}\psi_4'}$$

$$+ \frac{3k_{2m}z\psi_4'}{2k_{2m+1}\psi_4'} + \frac{3k_{2m}z\psi_4'}{2k_{2m+1}\psi_4'} + \frac{4k_{2m+1}z\psi_4'}{2k_{2m+1}\psi_4'}$$

$$+ \frac{2k_{2m+1}(z^2-\alpha)\psi_4'}{2k_{2m+1}\psi_4'} + \frac{2k_{2m+1}z\psi_4'}{2k_{2m+1}\psi_4'}$$

We can write

$$m \left( r, \frac{1}{\psi_4'} \right) = m \left( r, \frac{\psi_4}{\psi_4'} \right) + m \left( r, \frac{1}{\psi_4} \right).$$

Hence by (5.2.1) and (5.2.15), we get

$$m \left( r, \frac{1}{\psi_4} \right) \leq O(\log r).$$

Case 12: $\rho(z) \equiv 0$, $n = 1,2,...,k$ and $j = 4$

We assume that

$$m \left( r, \frac{1}{\psi_4(n)} \right) = O(\log r) \quad \ldots (5.2.16)$$

has been proved for all $n = 1,2,...,k$, we consider next case

$p(z) \equiv 0$, $j = 4$ and $n = k + 1.$
Differenting equation (5.2.14), $3k - 1$ times, we have

$$
\psi_4^{(2m+3k)} = 4k^{2m-1} \sum_{j=1}^{3k-1} \left( 3k - 1 \right)_{j} \left( \psi_4^{(3k-j-1)} \left( \psi_4^{(2m+1)} \right)^j \right)
$$

$$
+ 2m-2 \sum_{i=1}^{3k-1} \left( 3k - 1 \right)_{j} \left( \psi_4^{(3k-j-1)} \left( \psi_4^{(2m+1)} \right)^j \right)
$$

$$
+ 3k^{2m} \sum_{j=0}^{3k-1} \left( 3k - 1 \right)_{j} \left( \psi_4^{(3k-j-1)} \psi_4 \right)
$$

$$
+ k^{2m} \sum_{j=0}^{3k-1} \left( 3k - 1 \right)_{j} \left( \psi_4^{(3k-j-1)} \psi_4 \right)
$$

and so

$$
\frac{4k^{2m-1} \left( \psi_4 \right)^3}{\psi_4^{(k+1)}} = \psi_4^{(2m+3k)} \frac{\psi_4^{(2m+3k)}}{\psi_4^{(k+1)}} - \frac{Q(z, \psi_4)}{\psi_4^{(k+1)}} \quad \text{(5.2.17)}
$$

where $Q(z, f)$ is a differential polynomial in $f$ and its derivatives with polynomial coefficients. Moreover, every term of $Q$ has some $\psi_4^{(l)}$ as a factor, where $l \geq k + 1$.

Hence by (5.2.1), (5.2.15) and (5.2.17) and the lemma of logarithmic derivative, we get
Therefore 

\[ m \left( r, \frac{1}{\psi_4^{(k+1)}} \right) \leq m \left( r, \frac{\psi_4^{(k)}}{\psi_4^{(k+1)}} \right) + m \left( r, \frac{1}{\psi_4^{(k)}} \right) = m \left( r, \frac{\psi_4^{(k)}}{\psi_4^{(k+1)}} \right) + O(\log r) = O(\log r) \]

Therefore

\[ m \left( r, \frac{1}{\psi_4^{(n)}} \right) = O(\log r), \quad \cdots \text{(5.2.18)} \]

for all \( n \in N \cup \{0\} \) by the induction principle.

**Case 13:** Suppose now that \( p(z) \neq 0 \) and that \( \deg p(z) = p \), then we have for all \( n \in N \cup \{0\} \) and for all \( j = 1, 2 \) and 4.

\[ m \left( r, \frac{1}{\psi_j^{(n)}} \right) \leq m \left( r, \frac{\psi_j^{(n)}}{\psi_j^{(n-1)}} \frac{p+1}{p} \right) + m \left( r, \frac{1}{\psi_j^{(n-1)}} \right) = O(\log r) \]

by (5.2.8), (5.2.13) and (5.2.18) and by the lemma of logarithmic derivative.

**Case 14:** Consider the assertion (5.2.3)

From theorem 0.2.3, we have

\[ m \left( r, \prod_{j=1}^{n} f_j \right) \leq \sum_{j=1}^{n} m(r, f_j) \]

\[ m \left( r, \prod_{j=1}^{n} f_j \right) \leq \sum_{j=1}^{n} m(r, f_j) + \log P. \]

By above inequalities and assertion (5.2.1), we get

\[ m(r, \prod_{j=1}^{n} f_j) \leq O(\log r). \]

**Case 15:** Consider the assertion (5.2.4). By the Lemma of addition and multiplication for \( m(r, f) \) and assertion (5.2.2) for \( p(z) = 0 \), we get
Lemma 5.2.4: Let \( \psi_j, j=1,2,4 \) be a transcendental meromorphic solution of (5.1.1), (5.1.2) and (5.1.3) respectively. Then

(i) \( N(r,M(\psi_j)) = \frac{(m-1)\gamma + \Gamma}{m} T(r,\psi_j) + O(\log r) \) for \( j=1,2 \)

(ii) \( N(r,M(\psi_4)) = \Gamma T(r,\psi_4) + O(\log r) \)

Proof: By (5.1.4), we have

Let \( M(f) = f^{n_0}(f')^{n_1}(f'')^{n_2}...(f^{(s)})^{n_s} \) be a differential monomial.

Degree of \( M(f) = \gamma_M = n_0 + n_1 + ... + n_s \).

Weight of \( M(f) = \Gamma_M = \sum_{i=0}^{s} (i+1)n_i = n_0 + 2n_1 + 3n_2 + ... + (s+1)n_s \).

(i) Since all poles of \( \psi_j \) for \( j=1,2 \), are of order \( m \), we get \( M(\psi_j) \) has a pole of order

\[
= mn_0 + (m+1)n_1 + (m+2)n_2 + ... + (m+s)n_s
\]

\[
= (m-1 +1)n_0 + (m-1 + 2)n_1 + ... + (m-1 + s +1)n_s
\]

\[
= (m-1)\gamma + \Gamma
\]

Therefore,

\[
N(r,M(\psi_j)) = \frac{(m-1)\gamma + \Gamma}{m} N(r,\psi_j) = \frac{(m-1)\gamma + \Gamma}{m} T(r,\psi_j) + O(\log r) \quad \text{for} \quad j=1,2
\]

(ii) Since \( \psi_4 \) has poles of order 1, then \( M(\psi_4) \) has a pole of order

\[
= n_0 + 2n_1 + 3n_2 + ... + (s+1)n_s = \Gamma
\]

Therefore

\[
N(r,M(\psi_4)) = \Gamma N(r,\psi_4) = \Gamma T(r,\psi_4) + O(\log r)
\]

5.3 PROOFS OF THEOREMS

PROOF OF THEOREM 5.1.1
The assertion follows immediately by lemma 5.2.3.

**PROOF OF THEOREM 5.1.2**

(i) \[ N \left( r, \frac{1}{\psi_j^{(n)} - p(z)} \right) = T \left( r, \psi_j^{(n)} - p(z) \right) - m \left( r, \frac{1}{\psi_j^{(n)} - p(z)} \right) \]
\[ = T \left( r, \psi_j^{(n)} \right) + O(\log r) \]
\[ = N \left( r, \psi_j^{(n)} \right) + m \left( r, \psi_j^{(n)} \right) + O(\log r) \]
\[ = N \left( r, \psi_j^{(n)} \right) + O(\log r), \text{ by lemma 5.2.3} \]

Since all poles of \( \psi_j \) for \( j = 1,2 \) are of order \( m \), we get
\[ N \left( r, \psi_j^{(n)} \right) = \frac{m+n}{m} N \left( r, \psi_j \right) = \frac{m+n}{m} T \left( r, \psi_j \right) + O(\log r) \]

Therefore
\[ N \left( r, \frac{1}{\psi_j^{(n)} - p(z)} \right) = \frac{m+n}{m} T \left( r, \psi_j \right) + O(\log r) \]

And so
\[ \Phi_{(n)}(\psi_j, p(z)) = \frac{m+n}{m} \text{ for } j = 1, 2 \]

(ii) Since all poles of \( \psi_4 \) are simple, we get
\[ N \left( r, \psi_4^{(n)} \right) = (n+1) N \left( r, \psi_4 \right) = (n+1) T \left( r, \psi_4 \right) + O(\log r) \]
\[ N \left( r, \frac{1}{\psi_4^{(n)} - p(z)} \right) = (n+1) T \left( r, \psi_4 \right) + O(\log r) \]

and so
\[ \Phi_{(n)}(\psi_4, p(z)) = n+1 \]

(iii) \[ N \left( r, \frac{1}{M(\psi_j) - p(z)} \right) = T \left( r, M(\psi_j) - p(z) \right) - m \left( r, \frac{1}{M(\psi_j) - p(z)} \right) \]
\[ \leq T \left( r, M(\psi_j) \right) + O(\log r) \]
\[ = N \left( r, M(\psi_j) \right) + m \left( r, M(\psi_j) \right) + O(\log r) \]
\[ = N \left( r, M(\psi_j) \right) + O(\log r), \text{ by lemma 3} \]
Next, $\psi_j$ for $j = 1, 2$ and by Lemma 5.2.4 (i), we get

$$N\left(r, M(\psi_j)\right) = \frac{(m-1)\gamma_M + \Gamma_M}{m} T(r, \psi_j) + O(\log r)$$

and so

$$\Phi(M(\psi_j), p(z)) \leq \frac{(m-1)\gamma_M + \Gamma_M}{m}, \text{ for } j = 1, 2$$

(iv) For $\psi_4$, by Lemma 5.2.4 (ii), we get

$$N\left(r, \Gamma_M T(r, \psi_4) + O(\log r) \right)$$

and so

$$\Phi(M(\psi_4), p(z)) \leq \Gamma_M$$

**PROOF OF THEOREM 5.1.3**

From Theorem 5.1.2, easily we can prove assertion of theorem 5.1.3.

**PROOF OF THEOREM 5.1.4**

By Lemma 5.2.3 and by the proof of Theorem 5.1.3, we have

(i) \( \Phi_e(\psi_j^{(n)}) = \lim_{r \to \infty} \frac{N \left( r, \frac{1}{\psi_j^{(n)}} \right)}{\Gamma(r, \psi_j^{(n)})} \)

\[= \lim_{r \to \infty} \frac{m + n + 1}{m} \frac{T(r, \psi_j) + O(\log r)}{T(r, \psi_j) + O(\log r)} = \frac{m + n + 1}{m + n} \]

(ii) \( \Phi_e(\psi_4^{(n)}) = \lim_{r \to \infty} \frac{N \left( r, \frac{1}{\psi_4^{(n)}} \right)}{\Gamma(r, \psi_4^{(n)})} \)

\[= \lim_{r \to \infty} \frac{(n + 2)T(r, \psi_4) + O(\log r)}{(n + 1)T(r, \psi_4) + O(\log r)} = \frac{n + 2}{n + 1} \]
\begin{align*}
\Phi_e(P(\psi_j)) &= \lim_{r \to \infty} \frac{N(r, \frac{1}{P(\psi_j)})}{T(r, P(\psi_j))} \quad \text{for } j = 1, 2 \\
&\leq \lim_{r \to \infty} \frac{(m-1)\gamma_p + \Gamma_p + 1}{m} T(r, \psi_j) + O(\log r) \\
&\leq \frac{(m-1)\gamma_p + \Gamma_p + 1}{(m-1)\gamma_p + \Gamma_p}
\end{align*}

Remark 5.3.1; (a) (i) of Theorem 5.1.4 reduces to Theorem 5.1.A in the case $n = 1, \, p(z) = 0$ and $j = 1$

(b) (i) of Theorem 5.1.4 reduces to Theorem 5.1.B in the case $n = 1, \, j = 2$ and $p(z) = 0$

PROOF OF THEOREM 5.1.5

Note that since all poles of $\psi_j, \, j = 1, 2$ are of order $m$, we have

$$N(r, \psi_j^{(n)}) - \overline{N}(r, \psi_j^{(n)}) = \frac{m + n - 1}{m} N(r, \psi_j).$$

Now we have by Lemma 5.2.3 and Theorem 5.1.3 that
Consider the case \( j = 4 \). Note that since all poles of \( \psi_4 \) are simple, we have

\[
N(r, \psi_4^{(n)}) - \overline{N}(r, \psi_4^{(n)}) = n N(r, \psi_4).
\]

Now by Lemma 5.2.3 and Theorem 5.1.3, we have
PROOF OF THEOREM 5.1.6

\[
N_B[r, P(\psi_j)] = N\left( r, \frac{1}{[P(\psi_j)']^T} \right) + 2N[r, P(\psi_j)] - N[r, P(\psi_j)']
\]

\[
= T\left( r, \frac{1}{[P(\psi_j)']^T} \right) - m\left( r, \frac{1}{[P(\psi_j)']^T} \right) + 2T[r, P(\psi_j)]
- 2m[r, P(\psi_j)] - T[r, P(\psi_j)'] + m[r, P(\psi_j)']
\]

\[
\leq 2T(r, P(\psi_j)) + 2T(r, P(\psi_j')) - 2T(r, P(\psi_j)) + O(\log r)
= 2T(r, P(\psi_j)) + O(\log r)
\]

Therefore,

\[
\Phi_B[\psi_j] = \lim_{r \to \infty} \frac{N_B[r, P(\psi_j)]}{T[r, P(\psi_j)]} \leq 2, \quad \forall \, j = 1, 2, 4
\]

(ii) For \( j = 1, 2 \), we have

\[
N_1[r, P(\psi_j)] = 2N[r, P(\psi_j)] - N[r, P(\psi_j)']
\]

\[
= N[r, P(\psi_j)] - \bar{N}[r, P(\psi_j)]
\]

\[
= N[r, P(\psi_j)] - \frac{1}{(m-1)\gamma_p + \Gamma_M}N[r, P(\psi_j)]
\]

\[
= \frac{(m-1)\gamma_p + \Gamma_M - 1}{(m-1)\gamma_p + \Gamma_M}N[r, P(\psi_j)]
\]

\[
= \frac{(m-1)\gamma_p + \Gamma_M - 1}{(m-1)\gamma_p + \Gamma_M}T[r, P(\psi_j)] + O(\log r)
\]

Therefore,

\[
\nu[\infty, P(\psi_j)] = \frac{(m-1)\gamma_p + \Gamma_M - 1}{(m-1)\gamma_p + \Gamma_M} \quad \text{for} \, j = 1, 2
\]
Next for $j = 4$

$$N_i[r, P(\psi_4)] = 2N[r, P(\psi_4)] - N[r, P(\psi_4)^\prime]$$

$$= N[r, P(\psi_4)] - \overline{N}[r, P(\psi_4)]$$

$$= N[r, P(\psi_4)] - \frac{1}{\Gamma} N[r, P(\psi_4)]$$

$$= \frac{\Gamma - 1}{\Gamma} N[r, P(\psi_4)]$$

$$= \frac{\Gamma - 1}{\Gamma} T[r, P(\psi_4)] + O(\log r)$$

Therefore

$$\nu[\infty, P(\psi_4)] = \frac{\Gamma - 1}{\Gamma}$$