CHAPTER 3

MINIMAL OPEN SETS AND MINIMAL CONTINUOUS MAPPINGS IN FUZZY TOPOLOGICAL SPACES.

3.1 Introduction.

In the year 1965, L. A. Zadeh [33] introduced the concept of fuzzy subset as a generalization of that of an ordinary subset. The introduction of fuzzy subsets paved the way for rapid research work in many areas of Mathematical Sciences.

In the year 1968, C. L. Chang [4] introduced the concept of fuzzy topological spaces as an application of fuzzy sets to topological spaces. Subsequently, several researchers like J. A. Goguen[11], C. K. Wong[32], R. H. Warren[31], R. Lowen [16], M. Ferraro and D. H. Foster [8], S. R. Malghan and S. S. Benchalli [20] [21], G. Balasubramanian and P. Sundaram [2] and many other authors contributed to the development of the theory and applications of fuzzy topology. The theory of fuzzy topological spaces can be regarded as a generalization theory of topological spaces.

This section is intended to provide a brief introduction to fuzzy subsets and fuzzy topology. The concept of a fuzzy subset, operations on fuzzy subsets, fuzzy subsets induced by mappings and fuzzy topological spaces are discussed in this section, which are subsequently used in this chapter. In section 2, minimal open sets that include maximal open sets are introduced and characterized in fuzzy topological spaces. Section 3 deals with minimal continuous maps that include a class of maximal continuous
maps in fuzzy topological spaces. In section 4, minimal open maps that include a class of maximal open maps are characterized in fuzzy topological spaces. Throughout this chapter (X, T), (Y, S) and (Z, P) denote fuzzy topological spaces on which no separation axioms are assumed unless otherwise explicitly mentioned. For any fuzzy subset A of a fuzzy topological space (X, T), closure of A, interior of A and complement of A is denoted by cl(A), int(A) and A' (or (1-A)) respectively.

An ordinary subset A of a set X can be characterized by a function called characteristic function \( \mu_A: X \to \{0, 1\} \) of A, defined by

\[
\mu_A(x) = 1, \text{ if } x \in A \\
= 0, \text{ if } x \not\in A.
\]

Thus an element \( x \in X \) is in A if \( \mu_A(x) = 1 \) and is not in A if \( \mu_A(x) = 0 \). In general, if X is a set and A is a subset of X then A has the following representation.

\[ A = \{(x, \mu_A(x)) : x \in X \} \]

Here \( \mu_A(x) \) may be regarded as the degree of belongingness of \( x \) to A, which is either 0 or 1. Hence A is the class of objects with degree of belongingness either 0 or 1 only. Prof. L. A. Zadeh [30] introduced class of objects with continuum grades of belongingness ranging between 0 and 1. He called such a class as a fuzzy subset. A fuzzy subset A in X is characterized by a membership function \( \mu_A: X \to [0,1] \) which associates with each point in X, a real number \( \mu_A(x) \) between 0 and 1 which represents the degree or grade of membership or belongingness of \( x \) to A.

**Definition 3.1.1[33]:** A fuzzy subset A in a set X is a function \( A: X \to [0,1] \).
A fuzzy subset in $X$ is empty iff its membership function is identically zero on $X$ and is denoted by $0$ or $\mu_\emptyset$. The set $X$ can be considered as a fuzzy subset of $X$ whose membership function is identically $1$ on $X$ and is denoted by $\mu_X$ or $1_X$. In fact every subset of $X$ is a fuzzy subset of $X$ but not conversely. Hence the concept of a fuzzy subset is a generalization of the concept of a subset.

**Definition 3.1.2[33]:** If $A$ and $B$ are any two fuzzy subsets of a set $X$, then $A$ is said to be included in $B$ or $A$ is contained in $B$ iff $A(x) \leq B(x)$ for all $x$ in $X$. Equivalently, $A \leq B$ iff $\mu_A(x) \leq \mu_B(x)$ for all $x$ in $X$.

Every fuzzy subset is included in itself and empty fuzzy subset is included in every fuzzy subset.

**Definition 3.1.3[33]:** Two fuzzy subsets $A$ and $B$ are said to be equal if $A(x) = B(x)$ for every $x$ in $X$. Equivalently $A = B$ if $\mu_A(x) = \mu_B(x)$ for every $x$ in $X$.

**Definition 3.1.4[33]:** The complement of a fuzzy subset $A$ in a set $X$, denoted by $A^c$ or $\sim A$, is the fuzzy subset of $X$ defined by $A^c(x) = 1 - A(x)$ for all $x$ in $X$. Note that $(A^c)^c = A$.

**Definition 3.1.5[33]:** The union of two fuzzy subsets $A$ and $B$ in $X$, denoted by $A \vee B$, is a fuzzy subset in $X$ defined by

$$(A \vee B)(x) = \max \{\mu_A(x), \mu_B(x)\} \text{ for all } x \text{ in } X.$$ 

Equivalently, $\mu_{A \vee B}(x) = \max \{\mu_A(x), \mu_B(x)\}$ for all $x$ in $X$.

In general, the union of a family of fuzzy subsets $\{A_\lambda: \lambda \in \Lambda\}$ is a fuzzy subset denoted by $(\bigvee_{\lambda \in \Lambda} A_\lambda)(x) = \sup_{\lambda \in \Lambda} \{A_\lambda(x)\}$ for all $x$ in $X$.
Definition 3.1.6[33]: The intersection of two fuzzy subsets $A$ and $B$ in $X$, denoted by $A \wedge B$, is a fuzzy subset in $X$ defined by

$$(A \wedge B)(x) = \min \{\mu_A(x), \mu_B(x)\} \text{ for all } x \text{ in } X.$$ 

Equivalently, $\mu_{A \wedge B}(x) = \max \{\mu_A(x), \mu_B(x)\} \text{ for all } x \text{ in } X.$

In general, the intersection of a family of fuzzy subsets $\{A_\lambda : \lambda \in \Lambda\}$ is a fuzzy subset denoted by $$(\bigwedge_{\lambda \in \Lambda} A_\lambda)(x) = \inf_{\lambda \in \Lambda} \{A_\lambda(x)\} \text{ for all } x \text{ in } X.$$ 

Remark 3.1.7[33]: Following are some of the basic properties of union and intersection of fuzzy subsets of $X$.

Let $X$ be any set and $A$, $B$, $C$ be fuzzy subsets of $X$, then

1) $A \cup (B \cup C) = (A \cup B) \cup (A \cup C)$
2) $A \cap (B \cap C) = (A \cap B) \cap (A \cap C)$
3) $A \cup 0 = A$, where $0$ is the empty fuzzy subset.
4) $A \cap 0 = 0$, where $0$ is the empty fuzzy subset.
5) $A \cup X = X$
6) $A \cap X = A$
7) $1 - (A \cup B) = (1 - A) \cap (1 - B)$
8) $1 - (A \cap B) = (1 - A) \cup (1 - B)$
9) $A - B = A \cap (1 - B)$

Thus the above properties are clear extensions of the basic set theoretic properties to fuzzy subsets.

Note that the following properties that are true in the case of set theory are no longer true for fuzzy subsets in general.
1) \( A \land (1 - A) = 0 \). This is true for \( A = 0 \) or \( A = X \).

2) \( A \lor (1 - A) = X \). This is true for \( A = 0 \) or \( A = X \).

**Definition 3.1.8[33]**: Let \( f: X \to Y \) be a mapping from a set \( X \) into a set \( Y \). Let \( A \) be a fuzzy set in \( X \) and \( B \) be a fuzzy set in \( Y \).

1) The inverse image of \( B \) under \( f \), written as \( f^{-1}(B) \) is a fuzzy set in \( X \), defined by \[ (f^{-1}(B))(x) = B(f(x)) = (B \circ f)(x) \] for each \( x \in X \).

2) The image of \( A \) under \( f \), written as \( f(A) \) is a fuzzy set in \( Y \), defined by \[ (f(A))(y) = \sup \{ A(z) : z \in f^{-1}(y) \} \] for each \( y \in Y \), where \( f^{-1}(y) = \{ x \in X : f(x) = y \} \).

**Theorem 3.1.9[4]**: Let \( f \) be a mapping from a set \( X \) into a set \( Y \). The following are true.

1) \( f^{-1}(1 - B) = 1 - f^{-1}(B) \) for any fuzzy set \( B \) in \( Y \).

2) \( f(1 - A) \geq 1 - f(A) \) for any fuzzy set \( A \) in \( X \).

3) \( A \leq B \) implies \( f(A) \leq f(B) \) for any two fuzzy sets \( A, B \) in \( X \).

4) \( C \leq D \) implies \( f^{-1}(C) \leq f^{-1}(D) \) for any two fuzzy sets \( C, D \) in \( Y \).

5) \( A \leq f^{-1}[f(A)] \) for any fuzzy set \( A \) in \( X \).

6) \( B \geq f[f^{-1}(B)] \) for any fuzzy set \( B \) in \( Y \).

7) Let \( g \) be a function from \( Y \) to \( Z \). Then \( (g \circ f)^{-1}(C) = f^{-1}[g^{-1}(C)] \) for any fuzzy set \( C \) in \( Z \).

**Theorem 3.1.10[31]**: Let \( f \) be a function from a set \( X \) into a set \( Y \). If \( A, A_i, i \in I \) are fuzzy sets in \( X \), and \( B, B_j, j \in J \) are fuzzy sets in \( Y \), then the following results are true.

1) \( f[ f^{-1}(B) ] = B \), when \( f \) is onto.
2) \[ f(\bigwedge_{i \in I} A_i) \leq \bigwedge_{i \in I} f(A_i). \]

3) \[ f^{-1}(\bigwedge_{j \in J} B_j) = \bigwedge_{j \in J} f^{-1}(B_j). \]

4) \[ f(\bigvee_{i \in I} A_i) = \bigvee_{i \in I} f(A_i). \]

5) \[ f^{-1}(\bigvee_{j \in J} B_j) = \bigvee_{j \in J} f^{-1}(B_j). \]

6) \[ f[ f^{-1}(B) \wedge A] = B \wedge f(A). \]

**Definition 3.1.11[11]**: A fuzzy set on \( X \) is 'Crisp' if it takes only the values 0 and 1 on \( X \).

**Definition 3.1.12[4]**: Let \( X \) be a set and \( T \) be a family of fuzzy subsets of \( X \). \( T \) is called a fuzzy topology on \( X \) iff \( T \) satisfies the following conditions.

1) \( \mu_{\emptyset}, \mu_X \in T \). That is 0, 1 \( \in T \).

2) If \( G_i \in T \) for \( i \in I \) then \( \bigvee_{i \in I} G_i \in T \).

3) If \( G, H \in T \) then \( G \wedge H \in T \).

The pair \((X, T)\) is called a fuzzy topological space (abbreviated as \( fts \)). The members of \( T \) are called fuzzy open sets and a fuzzy set \( A \) in \( X \) is said to be closed iff \( 1 - A \) is an fuzzy open set in \( X \).

**Remark 3.1.13[4]**: Every topological space is a fuzzy topological space but not conversely.

**Definition 3.1.14[31]**: Let \( X \) be a \( fts \) and \( A \) be a fuzzy subset in \( X \). Then \( \bigwedge \{ B \colon B \text{ is a closed fuzzy set in } X \text{ and } B \geq A \} \) is called the closure of \( A \) and is denoted by \( \overline{A} \) or \( cl(A) \).

**Theorem 3.1.15[31]**: Let \( A \) and \( B \) be two fuzzy sets in a \( fts \) \((X, T)\). Then the following results are true.
1) \( \overline{A} \) is a closed fuzzy set in X.
2) \( \overline{A} \) is the least closed fuzzy set in X, which is greater than or equal to A.
3) A is closed iff \( A = \overline{A} \).
4) \( 0 = 0 \), where 0 is the empty fuzzy set.
5) \( A = A \).
6) \( A \lor B = \overline{A \land B} \).
7) \( A \land B \geq \overline{A \lor B} \).
8) \( A \leq B \) implies \( \overline{A} \leq \overline{B} \).

**Definition 3.1.16[4]**: Let A and B be two fuzzy sets in a fuzzy topological space \((X, T)\) and let \( A \geq B \). Then B is called an interior fuzzy set of A if there exists \( G \in T \) such that \( A \geq G \geq B \). The least upper bound of all interior fuzzy sets of A is called the interior of A and is denoted by \( A^0 \).

**Theorem 3.1.17[4]. [31]**: Let X be an fts and A and B be two fuzzy sets in X. The following results hold good.
1) \( A^0 \) is an open fuzzy set in X.
2) \( A^0 \) is the largest open fuzzy set in X which is less than or equal to A.
3) A is open iff \( A = A^0 \).
4) \( A \leq B \) iff \( A^0 \leq B^0 \).
5) \( (A^0)^0 = A^0 \).
6) \( A^0 \land B^0 = (A \land B)^0 \).
7) \( A^0 \lor B^0 \leq (A \lor B)^0 \).
8) \( (1 - A)^0 = 1 - \overline{A} \).
9) \( \overline{1 - A} = 1 - A^0 \).
Definition 3.1.18[31]: Let \((X, T)\) be an fts and let \(A\) be a crisp subset of \(X\). Then the family \(T_A = \{G/A : G \in T\}\) is a fuzzy topology on \(A\), where \(G/A\) is the restriction of \(G\) to \(A\). The fuzzy topology \(T_A\) is called the relative fuzzy topology on \(A\) or the fuzzy topology on \(A\) induced by the fuzzy topology \(T\) on \(X\). Also \((A, T_A)\) is called the fuzzy subspace of \((X, T)\).

Definition 3.1.19[4]: Let \(f: X \to Y\) be a function from a fts \((X, T)\) to a fts \((Y, S)\). Then \(f\) is said to be fuzzy continuous (briefly \(f\) - continuous) iff for each \(B \in S\), \(f^{-1}(B) \in T\).

Definition 3.1.20[32]: A function \(f : X \to Y\) from a fts \(X\) into a fts \(Y\) is said to be \(f\) -open (resp. \(f\) -closed) iff for each fuzzy open (resp. fuzzy closed) set \(A\) in \(X\), \(f(A)\) is a fuzzy open (resp. fuzzy closed) set in \(Y\).

3.2 Fuzzy Minimal Open Sets in Fuzzy Topological Spaces.

In this section, fuzzy minimal open sets that include a class of fuzzy maximal open sets are defined and characterized in fuzzy topological spaces. Some of the basic properties of such classes of fuzzy sets have been studied.

Definition 3.2.1: A nonempty fuzzy open set \(A\) of a fuzzy topological space \((X, T)\) is said to be a fuzzy minimal open (briefly \(f\)-m. open) set if any fuzzy open set which is contained in \(A\) is either 0 or \(A\).

Lemma 3.2.2: (i) If \(A\) is any fuzzy minimal open set and \(\alpha\) is a fuzzy open set, then \(A \land \alpha = 0\) or \(A \leq \alpha\).

(ii) If \(A\) and \(B\) are \(f\)- minimal open sets then \(A \land B = 0\) or \(A = B\).
Proof: (i) Let $A$ be any fuzzy minimal open set and $\alpha$ be a fuzzy open set. If $A \land \alpha = 0$ then there is nothing to prove. But if $A \land \alpha \neq 0$, then we have to prove that $A \leq \alpha$. Now $A \land \alpha \neq 0$ implies $A \land \alpha \leq A$ and $A \land \alpha \leq \alpha$. Therefore $A \land \alpha \leq A$ and $A$ is f-minimal open set implies that $A \land \alpha = 0$ or $A \land \alpha = A$. But $A \land \alpha \neq 0$ then $A \land \alpha = A$ implies $A \leq \alpha$.

(ii) Since every f- minimal open set is fuzzy open set it follows from (i) that $A \leq B$ and $B \leq A$. Therefore $A = B$.

Theorem 3.2.3: Let $A$ and $A_i$ be f- minimal open sets for any $i \in \Lambda$. If $A \leq \bigvee_{i \in \Lambda} A_i$ then there exists an element $j$ of $\Lambda$ such that $A = A_j$.

Proof: Let $A \leq \bigvee_{i \in \Lambda} A_i$. Then $A = A \land \bigvee_{i \in \Lambda} A_i$ which implies $A = \bigvee_{i \in \Lambda} [A \land A_i]$. Since $A$ and $A_i$ are f-minimal open sets, by the Lemma 3.2.2 either $A \land A_i = 0$ or $A = A_i$. Now if $A \land A_i = 0$ then $A = 0$, which contradicts the fact that $A$ is f-minimal open set. Therefore if $A \land A_i \neq 0$. Then there exists an element $j$ of $\Lambda$ such that $A = A_j$.

Theorem 3.2.4: Let $A$ and $A_i$ be f- minimal open sets for any $i \in \Lambda$. If $A \neq A_i$ for any element $i \in \Lambda$ then $\bigvee_{i \in \Lambda} A_i \land A = 0$.

Proof: Suppose $\bigvee_{i \in \Lambda} A_i \land A \neq 0$. Then there exists an element $i \in \Lambda$ such that $A_i \land A \neq 0$. By the Lemma 3.2.2 $A_i = A$, which contradicts the fact that $A \neq A_i$. Therefore $\bigvee_{i \in \Lambda} A_i \land A = 0$.

Theorem 3.2.5: If $A_i$ is a fuzzy minimal open set for any element $i$ of $\Lambda$ and $A_i \neq A_j$ for any elements $i$ and $j$ of $\Lambda$ with $i \neq j$, then for any element $j$ of $\Lambda$, $\bigvee_{i \in \Lambda \setminus \{j\}} A_i \land A_j = 0$, where $|\Lambda| \geq 2$. 

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Proof: Suppose that \[ \bigvee_{i \in J} A_i \cap A_j \neq 0. \] Then \[ \bigvee_{i \in J} [A_i \cap A_j] \neq 0. \] Therefore \[ A_i \cap A_j \neq 0. \] By the Lemma 3.2.2 \( A_i = A_j \). This is a contradiction to the hypothesis. Therefore for any element \( j \) of \( \Lambda \), \[ \bigvee_{i \in J} A_i \cap A_j = 0. \]

Theorem 3.2.6: If \( A_i \) is a fuzzy minimal open set for any element \( i \) of \( \Lambda \) and \( A_i \neq A_j \) for any elements \( i \) and \( j \) of \( \Lambda \) with \( i \neq j \) and if \( \Gamma \) is a proper nonempty subset of \( \Lambda \), then \[ \bigvee_{i \in \Lambda \backslash \Gamma} A_i \cap [\bigvee_{m \in \Gamma} A_m] = 0. \]

Proof: Suppose \[ \bigvee_{i \in \Lambda \backslash \Gamma} A_i \cap [\bigvee_{m \in \Gamma} A_m] \neq 0. \] Then \([A_i \cap A_m] \neq 0 \) for \( i \in \Lambda \backslash \Gamma \) and \( m \in \Gamma \) \( \Rightarrow A_i \cap A_m \neq 0 \) for some \( i \in \Lambda \) and \( m \in \Gamma \).

\( \Rightarrow A_i = A_m \) by the Lemma 3.2.2. Hence we have a contradiction to the fact that \( A_i \neq A_m \). Therefore \[ \bigvee_{i \in \Lambda \backslash \Gamma} A_i \cap [\bigvee_{m \in \Gamma} A_m] = 0. \]

Theorem 3.2.7: If \( A_i \) and \( A_m \) are fuzzy minimal open sets for any elements \( i \in \Lambda \) and \( m \in \Gamma \) and if there exists an element \( n \in \Gamma \) such that \( A_i \neq A_n \) for any element \( i \in \Lambda \), then \[ \bigvee_{m \in \Gamma} A_m \neq [\bigvee_{i \in \Lambda} A_i]. \]

Proof: Suppose that there exists an element \( n \in \Gamma \) satisfying the condition \( A_i \neq A_n \) for any element \( i \in \Lambda \) such that \[ [\bigvee_{m \in \Gamma} A_m] < [\bigvee_{i \in \Lambda} A_i]. \]

\( \Rightarrow A_n < [\bigvee_{m \in \Gamma} A_i] \) for some \( n \in \Gamma \).

\( \Rightarrow A_n = A_j \) for some \( j \in \Lambda \), by the Theorem 3.2.3. This is a contradiction to the fact that \( A_n \neq A_j \) for any \( j \in \Lambda \). Therefore \[ [\bigvee_{m \in \Gamma} A_m] \neq [\bigvee_{i \in \Lambda} A_i]. \]
Theorem 3.2.8: If $A_i$ is a fuzzy minimal open set for any element $i$ of $\Lambda$ and $A_i \neq A_j$ for any elements $i$ and $j$ of $\Lambda$ with $i \neq j$ and if $\Gamma$ is a proper nonempty subset of $\Lambda$, then $\bigvee_{m \in \Gamma} A_m \not\leq \bigvee_{i \in \Lambda} A_i$.

Proof: Let $k$ be any element of $\Lambda \setminus \Gamma$. Then $A_k$ is a fuzzy minimal open set of the family $\{A_k: k \in \Lambda \setminus \Gamma\}$ of fuzzy minimal open sets. Then,

$$A_i \land \bigvee_{m \in \Gamma} A_m = \bigvee_{m \in \Gamma} [A_i \land A_m] = 0 \text{ and}$$

$$A_i \land \bigvee_{i \in \Lambda} A_i = \bigvee_{i \in \Lambda} [A_i \land A_i] = A_i \downarrow$$

If $\bigvee_{i \in \Lambda} A_i = \bigvee_{i \in \Lambda} A_i$, then $0 = A_k$. This is a contradiction to the fact that $A_i$ is a fuzzy minimal open set. Therefore $\bigvee_{m \in \Gamma} A_m \neq \bigvee_{i \in \Lambda} A_i$.

Hence $\bigvee_{m \in \Gamma} A_m \not\leq \bigvee_{i \in \Lambda} A_i$.

Theorem 3.2.9: Assume that $|\Lambda| \geq 2$. If $A_i$ is a fuzzy minimal open set for any element $i$ of $\Lambda$ and $A_i \neq A_j$ for any elements $i$ and $j$ of $\Lambda$ with $i \neq j$, then

i) $A_j \not\leq \bigvee_{i \in \Lambda \setminus \{j\}} A_i$, for some element $j$ of $\Lambda$.

ii) $\bigvee_{i \in \Lambda \setminus \{j\}} A_i \neq 1$, for any element $j$ of $\Lambda$.

Proof: i) Let $j$ be any element of $\Lambda$. By hypothesis $A_i \neq A_j$ for any elements $i$ and $j$ of $\Lambda$ with $i \neq j$. Then by the Theorem 3.2.4,

$$[\bigvee_{i \in \Lambda} A_i] \land A_j = 0,$$

which is true for any $j \in \Lambda$.

$$\Rightarrow \bigvee_{i \in \Lambda} (A_i \land A_j) = 0,$$

for some elements $i$ and $j$ of $\Lambda$.

$$\Rightarrow A_i \land A_j = 0,$$

by the Lemma 3.2.2.

$$\Rightarrow A_i \not\leq 1 - A_j$$

$$\Rightarrow \bigvee_{i \in \Lambda \setminus \{j\}} A_i \not\leq 1 - A_j$$

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Therefore $A_j < 1 - \bigvee_{i \in A(j)} A_i$ for some element $j$ of $\Lambda$.

ii) Let $j$ be any element of $\Lambda$ such that $\bigvee_{i \in A(j)} A_i = 1$.

$\Rightarrow A_j = 0$. This is a contradiction to the fact that $A_i$ is fuzzy minimal open set.

Therefore $\bigvee_{i \in A(j)} A_i \neq 1$, for any element $j$ of $\Lambda$.

**Corollary 3.2.10:** If $A_i$ is a fuzzy minimal open set for any element $i$ of $\Lambda$ and $A_i \neq A_j$ for any elements $i$ and $j$ of $\Lambda$ with $i \neq j$. If $|\Lambda| \geq 3$, then $A_i \vee A_j \neq 1$ for any elements $i$ and $j$ of $\Lambda$ with $i \neq j$.

**Proof:** Follows from the Theorem 3.2.9(ii).

**Theorem 3.2.11:** (Recognition Principle for fuzzy minimal open sets).
Assume that $|\Lambda| \geq 2$. If $A_i$ is a fuzzy minimal open set for any element $i$ of $\Lambda$ and $A_i \neq A_j$ for any elements $i$ and $j$ of $\Lambda$ with $i \neq j$. Then, $A_j = (\bigvee_{i \in \Lambda} A_i) \wedge (1 - \bigvee_{i \in A(j)} A_i)$ for any element $j$ of $\Lambda$.

**Proof:** Let $j$ be any element of $\Lambda$, then

$$(\bigvee_{i \in \Lambda} A_i) \wedge (1 - \bigvee_{i \in A(j)} A_i) = [(\bigvee_{i \in A(j)} A_i) \vee A_j] \wedge [1 - (\bigvee_{i \in \Lambda} A_i)]$$

$$= [(\bigvee_{i \in A(j)} A_i) \wedge [1 - \bigvee_{i \in \Lambda} A_i]] \vee [A_j \wedge (1 - \bigvee_{i \in \Lambda} A_i)]$$

$$= 0 \vee A_j$$

$$= A_j$$ for any element $j$ of $\Lambda$.

**Theorem 3.2.12:** If $A_i$ is a fuzzy minimal open set for any element $i$ of $\Lambda$ and $A_i \neq A_j$ for any elements $i$ and $j$ of $\Lambda$ with $i \neq j$ and if $\bigvee_{i \in \Lambda} A_i$ is a fuzzy closed set then $A_i$ is a fuzzy closed set for any element $i$ of $\Lambda$. 

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Proof: Let \( j \) be any element of \( A \). Then by the Theorem 3.2.11

\[ A_j = (\bigvee_{i \in A} A_i) \land (1 - \bigvee_{i \in A \setminus j} A_i) \]

\[ = (\bigvee_{i \in A} A_i) \land \left( \bigwedge_{i \in A \setminus j} (1 - A_i) \right) \]

\[ = \text{Fuzzy closed} \land \text{fuzzy closed} = \text{fuzzy closed set}. \]

Theorem 3.2.13: Assume that \( |\Lambda| \geq 2 \). If \( A_i \) is a fuzzy minimal open set for any element \( i \) of \( \Lambda \) and \( A_i \neq A_j \) for any elements \( i \) and \( j \) of \( \Lambda \) with \( i \neq j \) and if \( \bigvee_{i \in \Lambda} A_i = 1 \), then \( \{ A_i / i \in \Lambda \} \) is the set of all fuzzy minimal open sets of a fuzzy topology \((X, T)\).

Proof: Suppose that there exists another fuzzy minimal open set \( A_m \) of a fuzzy topology \((X, T)\) which is not equal to \( A_i \) for any element \( i \) of \( \Lambda \). Then,

\[ 1 = \bigvee_{i \in \Lambda} A_i = \bigvee_{i \in \Lambda \setminus \{m\}} A_i \neq 1, \text{ by the Theorem 3.2.9(ii)}. \]

This contradicts our assumption. Therefore \( \{ A_i / i \in \Lambda \} \) is the set of all fuzzy minimal open sets of fuzzy topology \((X, T)\).

Definition 3.2.14: A nonempty fuzzy open set \( A \) of a fuzzy topological space \((X, T)\) is said to be fuzzy maximal open (briefly f- m_o open) set if any fuzzy open set which contains \( A \) is either 1 or \( A \).

Lemma 3.2.15: (i) If \( A \) is any fuzzy maximal open set and \( \alpha \) is fuzzy open set, then \( A \lor \alpha = 1 \) or \( \alpha \leq A \).

(ii) If \( A \) and \( B \) are f- maximal open sets then \( A \lor B = 1 \) or \( A = B \).

Proof: (i) Let \( A \) be any fuzzy maximal open set and \( \alpha \) be a fuzzy open set. If \( A \lor \alpha = 1 \) then there is nothing to prove. But if \( A \lor \alpha \neq 1 \), then we have to prove that \( \alpha \leq A \). Now \( A \lor \alpha \neq 1 \) implies \( A \leq A \lor \alpha \) and \( \alpha \leq A \lor \alpha \).
Therefore $A \leq A \vee \alpha$ and $A$ is $f$-maximal open set implies that $A \vee \alpha = 1$ or $A \vee \alpha = A$. But $A \vee \alpha \neq 1$ then $A \vee \alpha = A$ which implies $\alpha \leq A$.

(ii) Since every $f$-maximal open set is a fuzzy open set, it follows from (i) that $A \leq B$ and $B \leq A$. Therefore $A = B$.

**Theorem 3.2.16:** If $A$, $B$ and $C$ are fuzzy maximal open sets such that $A \neq B$ and if $A \wedge B < C$, then $A = C$ or $B = C$.

**Proof:** Let $A$, $B$ and $C$ be fuzzy maximal open sets such that $A \neq B$ and $A \wedge B < C$. If $A = C$ then there is nothing to prove. But if $A \neq C$, then we have to prove that $B = C$.

Now $B \wedge C = B \wedge (C \wedge 1)$

$$= B \wedge [C \wedge (A \vee B)]$$

by the Lemma 3.2.15 $A \vee B = 1$.

$$= B \wedge [(C \wedge A) \vee (C \wedge B)]$$

$$= (B \wedge C \wedge A) \vee (B \wedge C \wedge B)$$

$$= (B \wedge A) \vee (B \wedge C), \text{ by hypothesis.}$$

$$= B \wedge (A \vee C) = B \wedge 1 = B.$$ 

$\Rightarrow B \wedge C = B$ implies $B \leq C$. From the Definition of fuzzy maximal open sets, it follows that $B = C$.

**Theorem 3.2.17:** If $A$, $B$ and $C$ are fuzzy maximal open sets which are different from each other, then $A \wedge B \neq A \wedge C$.

**Proof:** Let $A$, $B$ and $C$ be any fuzzy maximal open sets which are different from each other such that $A \wedge B < A \wedge C$, then we see that 

$$(A \wedge B) \vee (B \wedge C) < (A \wedge C) \vee (B \wedge C)$$

$$= (A \vee C) \wedge B < (A \vee B) \wedge C$$

$$= 1 \wedge B < 1 \wedge C$$

by the Lemma 3.2.15.
It follows that $B = C$ from the Definition of fuzzy maximal open sets. This contradicts the fact $A \neq B \neq C$.

Therefore $A \land B \not< A \land C$.

**Theorem 3.2.18:** Assume that $|\Lambda| \geq 2$. If $A_i$ is a fuzzy maximal open set for any element $i$ of $\Lambda$ and $A_i \neq A_j$ for any elements $i$ and $j$ of $\Lambda$ with $i \neq j$. Then,

(i) $1- \bigwedge_{i \in \Lambda \setminus \{j\}} A_i < A_j$, for any element $j$ of $\Lambda$.

(ii) $\bigwedge_{i \in \Lambda \setminus \{j\}} A_i \neq 0$

**Proof:** (i) Let $j$ be any element of $\Lambda$. Since $A_i$ and $A_j$ are fuzzy maximal open sets with $i \neq j$, by the Lemma 3.2.15 we have,

$$A_i \lor A_j = 1 \Rightarrow 1- A_i < A_j = 1- A_j < \bigwedge_{i \in \Lambda \setminus \{j\}} A_i.$$ 

Therefore $1- \bigwedge_{i \in \Lambda \setminus \{j\}} A_i < A_j$, for any element $j$ of $\Lambda$.

(ii) Let $j$ be any element of $\Lambda$ such that $\bigwedge_{i \in \Lambda \setminus \{j\}} A_i = 0$ then from (i) $A_j = 1$, this contradicts the fact that $A_j$ is fuzzy maximal open set. Therefore $\bigwedge_{i \in \Lambda \setminus \{j\}} A_i \neq 0$.

**Corollary 3.2.19:** If $A_i$ is a fuzzy maximal open set for any element $i$ of $\Lambda$ and $A_i \neq A_j$ for any elements $i, j$ of $\Lambda$ with $i \neq j$. If $|\Lambda| \geq 3$ then $A_i \lor A_j \neq 0$, for any element $i$ and $j$ of $\Lambda$ with $i \neq j$.

**Proof:** Follows from the Theorem 3.2.18 (ii).

**Theorem 3.2.20:** If $A_i$ is a fuzzy maximal open set for any element $i$ of $\Lambda$ and $A_i \neq A_j$ for any elements $i, j$ of $\Lambda$ with $i \neq j$. Assume that $|\Lambda| \geq 2$, then $\bigwedge_{i \in \Lambda \setminus \{j\}} A_i < A_j < \bigwedge_{i \in \Lambda \setminus \{j\}} A_j$ for any element $j$ of $\Lambda$. 

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Proof: Let \( j \) be any element of \( \Lambda \) such that \( \bigwedge_{i \in \Lambda \setminus \{j\}} A_i < A_j \).

Then \( 1 = (1 - \bigwedge_{i \in \Lambda \setminus \{j\}} A_i) \lor (\bigwedge_{i \in \Lambda \setminus \{j\}} A_i) < A_j \) by the Theorem 3.2.18(i), which implies that \( 1 < A_j \), which is a contradiction to our assumption.

Therefore \( \bigwedge_{i \in \Lambda \setminus \{j\}} A_i \not< A_j \) \hspace{1cm} (i)

Again let \( A_j < \bigwedge_{i \in \Lambda \setminus \{j\}} A_i \), then \( A_j < A_i \) for some element \( i \) of \( \Lambda \), which implies \( A_j = A_i \) by the definition of fuzzy maximal open set. This contradicts our assumption. Therefore \( A_j \not< \bigwedge_{i \in \Lambda \setminus \{j\}} A_i \) \hspace{1cm} (ii).

From (i) and (ii) \( \bigwedge_{i \in \Lambda \setminus \{j\}} A_i \not< A_j \not< \bigwedge_{i \in \Lambda \setminus \{j\}} A_i \).

Corollary 3.2.21: If \( A_i \) is a fuzzy maximal open set for any element \( i \) of \( \Lambda \) and \( A_i \not= A_j \) for any elements \( i \) and \( j \) of \( \Lambda \) with \( i \not= j \) and if \( \Gamma \) is a proper nonempty subset of \( \Lambda \), then \( \bigwedge_{i \in \Lambda \setminus \Gamma} A_i \not< \bigwedge_{i \in \Lambda \setminus \Gamma} A_i \) \hspace{1cm} (i).

Proof: Let \( k \) be any element of \( \Gamma \), then by the Theorem 3.2.20

\( \bigwedge_{i \in \Lambda \setminus \Gamma} A_i \not< A_i \Rightarrow \bigwedge_{i \in \Lambda \setminus \Gamma} A_i \not< \bigwedge_{k \in \Gamma} A_i \) \hspace{1cm} (i).

Similarly \( \bigwedge_{k \in \Gamma} A_i \not< \bigwedge_{i \in \Lambda \setminus \Gamma} A_i \) \hspace{1cm} (ii).

From (i) and (ii) we have \( \bigwedge_{i \in \Lambda \setminus \Gamma} A_i \not< \bigwedge_{k \in \Gamma} A_i \not< \bigwedge_{i \in \Lambda \setminus \Gamma} A_i \).

Theorem 3.2.22 (Decomposition theorem):

Assume that \( |\Lambda| \geq 2 \). If \( A_i \) is a fuzzy maximal open set for any element \( i \) of \( \Lambda \) and \( A_i \not= A_j \) for any elements \( i \) and \( j \) of \( \Lambda \) with \( i \not= j \), then for any element \( j \) of \( \Lambda \), \( A_j = (\bigwedge_{i \in \Lambda} A_i) \lor (1 - \bigwedge_{i \in \Lambda \setminus \{j\}} A_i) \).
Proof: Let $j$ be any element of $A$, then by the Theorem 3.2.18 (i)
\[
(\bigwedge_{i \in A} A_i) \lor (1- \bigwedge_{i \in A \setminus \{j\}} A_i) = ((\bigwedge_{i \in A \setminus \{j\}} A_i)^\lor A_j) \lor (1- \bigwedge_{i \in A \setminus \{j\}} A_i)
\]
\[
= [(\bigwedge_{i \in A \setminus \{j\}} A_i) \lor (1- \bigwedge_{i \in A \setminus \{j\}} A_i)] \land [A_j \lor (1- \bigwedge_{i \in A \setminus \{j\}} A_i)]
\]
\[
= 1 \land A_j \text{ by the Theorem 3.2.18.}
\]

Theorem 3.2.23: If $A_i$ is a fuzzy maximal open set for any element $i$ of $A$
and $A_i \neq A_j$ for any elements $i$ and $j$ of $A$ with $i \neq j$. If $\Gamma$ is a proper
nonempty subset of $A$, then $\bigwedge_{i \in A} A_i \leq \bigwedge_{k \in \Gamma} A_k$.

Proof: Since $\Gamma \neq \emptyset$, there exists an element $s \notin \Gamma$ and an element $j \in \Gamma$.
If $\Gamma$ contains only one element, then we have $\bigwedge_{i \in A} A_i < A_j$.
If $\bigwedge_{i \in A} A_i = A_j$ then $A_j < A_i$ for any element $i$ of $\Lambda$. Since $A_i$ is a fuzzy
maximal open set for any element $i$ of $\Lambda$, we have $A_j = A_i$ which
contradicts our assumption. Therefore $\bigwedge_{i \in A} A_i \leq A_j$.

If $|\Gamma| > 2$ then by the Theorem 3.2.22,
\[
A_i = (\bigwedge_{i \in A} A_i) \lor (1- \bigwedge_{i \in A \setminus \{j\}} A_i)
\]
\[
A_j = (\bigwedge_{k \in \Gamma} A_k) \lor (1- \bigwedge_{k \in \Gamma \setminus \{j\}} A_k).
\]
If $\bigwedge_{i \in A} A_i = \bigwedge_{k \in \Gamma} A_k$, then
\[
\bigwedge_{i \in A} A_i = \bigwedge_{i \in A} A_i < \bigwedge_{i \in A \setminus \{j\}} A_i < \bigwedge_{k \in \Gamma} A_k.
\]
Therefore we have $\bigwedge_{i \in A} A_i = \bigwedge_{k \in \Gamma} A_k < \bigwedge_{i \in A \setminus \{j\}} A_i$. 

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Thus we see that $A_i = A_j$. It follows that $A_i = A_j$ with $s \neq j$. This contradicts our assumption. Therefore $\bigwedge_{i \in \Lambda} A_i \neq \bigwedge_{i \in \Lambda} A_i$.

**Theorem 3.2.24:** If $A_i$ is a fuzzy maximal open set for any element $i$ of $\Lambda$ and $A_i \neq A_j$ for any elements $i$ and $j$ of $\Lambda$ with $i \neq j$ and if $\bigwedge_{i \in \Lambda} A_i$ is a fuzzy closed set, then $A_j$ is a fuzzy closed set for any element $i$ of $\Lambda$.

**Proof:** By theorem 3.2.22 we have,

$A_j = (\bigwedge_{i \in \Lambda} A_i) \lor (1 - \bigwedge_{i \notin \{j\}} A_i)$, for any element $j$ of $\Lambda$.

$= (\bigwedge_{i \in \Lambda} A_i) \lor (\bigvee_{i \in \Lambda \setminus \{j\}} (1 - A_i))$. Since $\Lambda$ is a finite set, we see that $(\bigvee_{i \in \Lambda \setminus \{j\}} (1 - A_i))$ is a fuzzy closed set. Hence $A_j$ is fuzzy closed set for any element $j$ of $\Lambda$.

**Theorem 3.2.25:** Assume that $|\Lambda| \geq 2$. If $A_i$ is a fuzzy maximal open set for any element $i$ of $\Lambda$ and $A_i \neq A_j$ for any elements $i$ and $j$ of $\Lambda$ with $i \neq j$. If $\bigwedge_{i \in \Lambda} A_i = 1$, then $\{A_i / i \in \Lambda\}$ is the set of all fuzzy maximal open sets of a fuzzy topological space $(X, T)$.

**Proof:** Suppose that there exists another fuzzy maximal open set $A_m$ of a fuzzy topological space $(X, T)$, which is not equal to $A_i$ for any element $i$ of $\Lambda$. Then, $0 = \bigwedge_{i \in \Lambda} A_i = \bigwedge_{i \in (\Lambda \setminus \{m\})} A_i \neq 0$, by the Theorem 3.2.18(ii).

This contradicts our assumption. Therefore $\{A_i / i \in \Lambda\}$ is the set of all fuzzy maximal open sets of fuzzy topological space $(X, T)$.

**Proposition 3.2.26:** Let $A$ and $B$ be any fuzzy subsets of $X$. If $A \lor B = 1$, $A \land B$ is a fuzzy closed set and $A$ is fuzzy open then $B$ is fuzzy closed set.
Proof: If $A \lor B = 1 \Rightarrow 1 - A < B$, then

$$(A \land B) \lor (1 - A) = [A \lor (1 - A)] \land [B \lor (1 - A)]$$

$$= 1 \land [B \lor (1 - A)] = [B \lor (1 - A)] = B = \text{fuzzy closed set.}$$

Since $A$ is fuzzy open set, $1 - A$ is fuzzy closed set, $(A \land B) \lor (1 - A)$ is fuzzy closed set. Therefore $B$ is fuzzy closed set.

**Proposition 3.2.27:** If $A_i$ is a fuzzy open set for any element $i$ of $\Lambda$ and $A_i \lor A_j = 1$ for any elements $i$ and $j$ of $\Lambda$ with $i \neq j$ and if $\bigwedge_{\kappa \in \Lambda(i)} A_i$ is a fuzzy closed set then $\bigwedge_{\kappa \in \Lambda(i)} A_i$ is a fuzzy closed set for any element $i$ of $\Lambda$.

**Proof:** Let $j$ be any element of $\Lambda$. Since $A_i \lor A_j = 1$ for any elements $i$ and $j$ of $\Lambda$ with $i \neq j$ we have $A_j \lor (\bigwedge_{\kappa \in \Lambda(j)} A_i) = \bigwedge_{\kappa \in \Lambda(j)} (A_j \lor A_i) = 1$.

Since $A_j \lor (\bigwedge_{\kappa \in \Lambda(j)} A_i) = \bigwedge_{\kappa \in \Lambda} A_i$ is a fuzzy closed set by our assumption.

Therefore $\bigwedge_{\kappa \in \Lambda(i)} A_i$ is a fuzzy closed set for any element $i$ of $\Lambda$.

**Theorem 3.2.28:** If $A_i$ is a fuzzy maximal open set for any element $i$ of $\Lambda$ and $A_i \neq A_j$ for any elements $i$ and $j$ of $\Lambda$ with $i \neq j$ and if $\bigwedge_{\kappa \in \Lambda(i)} A_i$ is a fuzzy closed set then $\bigwedge_{\kappa \in \Lambda(i)} A_i$ is a fuzzy closed set for any element $i$ of $\Lambda$.

**Proof:** By hypothesis $A_i \neq A_j$, then by the Lemma 3.2.15(ii) $A_i \lor A_j = 1$ for any elements $i$ and $j$ of $\Lambda$ with $i \neq j$. By Proposition 3.2.27 it follows that $\bigwedge_{\kappa \in \Lambda(j)} A_i$ is a fuzzy closed set for any element $i$ of $\Lambda$.

**Definition 3.2.29:** A nonempty fuzzy closed set $B$ of a fts $(X, T)$ is said to be fuzzy minimal closed set if any fuzzy closed set which is contained in $B$ is either $0$ or $B$. 83
Definition 3.2.30: A nonempty fuzzy closed set B of a fts (X, T) is said to be fuzzy maximal closed set if any fuzzy closed set which contains B is either 1 or B.

Remark 3.2.31: Fuzzy minimal open (resp. fuzzy minimal closed) sets and fuzzy maximal closed (resp. fuzzy maximal open) sets are complements of each other.

3.3 Fuzzy Minimal Continuous Maps in Fuzzy Topological Spaces.

In this section, fuzzy minimal continuous map that includes the class of fuzzy maximal continuous maps, fuzzy minimal irresolute maps and fuzzy maximal irresolute maps are introduced in fuzzy topological spaces. Some of their basic properties have been obtained in this section.

Definition 3.3.1: A mapping \( f: X \rightarrow Y \), from a fts X into a fts Y is said to be fuzzy minimal continuous (briefly f-m continuous) map if the inverse image of every fuzzy minimal open set in Y is a fuzzy open set in X.

Theorem 3.3.2: Every fuzzy continuous map is fuzzy minimal continuous.

Proof: Let \( f: X \rightarrow Y \), from a fts X into a fts Y be fuzzy continuous map and let \( \alpha \) be any fuzzy minimal open set in Y. As every fuzzy minimal open set is fuzzy open, \( \alpha \) is fuzzy open set. Since \( f \) is fuzzy continuous map, \( f^{-1}(\alpha) \) is fuzzy open set in X. Therefore \( f \) is fuzzy minimal continuous map.

Remark 3.3.3: Converse of the Theorem 3.3.2 need not be true.
Example 3.3.4: Let $X = Y \{a, b, c\}$; $I = [0, 1]$ and the functions $\alpha, \beta, \gamma, \delta : X \to I$ be defined as,

$$
\alpha(x) = \begin{cases} 1, & \text{if } x = a \\ 0, & \text{otherwise} \end{cases}, \quad \beta(x) = \begin{cases} 1, & \text{if } x = b \\ 0, & \text{otherwise} \end{cases},
$$

$$
\gamma(x) = \begin{cases} 1, & \text{if } x = a, b \\ 0, & \text{otherwise} \end{cases}, \quad \delta(x) = \begin{cases} 1, & \text{if } x = a, c \\ 0, & \text{otherwise} \end{cases}.
$$

Consider $T = \{0, 1, \alpha, \beta, \gamma\}$ and $S = \{0, 1, \alpha, \gamma, \delta\}$. Here $(X, T)$ and $(Y, S)$ are fuzzy topological spaces. Let the mapping $f : X \to Y$, from a fts $X$ into a fts $Y$ be defined by $f(a) = a$, $f(b) = b$, $f(c) = c$. Then $f$ is fuzzy minimal continuous map but not fuzzy continuous map, as the inverse image of the fuzzy open set $\delta$ in $(Y, S)$ is $\lambda : X \to I$ defined by,

$$
\lambda(x) = \begin{cases} 1, & \text{if } x = a, c \\ 0, & \text{otherwise} \end{cases}
$$

is not a fuzzy open set in $(X, T)$.

Theorem 3.3.5: If $f : X \to Y$, from a fts $X$ into a fts $Y$ is any mapping, then the following are equivalent.

(i) $f$ is fuzzy minimal continuous map.

(ii) The inverse image of every fuzzy maximal closed set in $Y$ is a fuzzy closed set in $X$.

Proof: (i) $\Rightarrow$ (ii). Let $\alpha$ be any fuzzy maximal closed set in $Y$. Then $(1 - \alpha)$ is a fuzzy minimal open set in $Y$. Since $f$ is fuzzy minimal continuous map, $f^{-1}(1 - \alpha) = [1 - f^{-1}(\alpha)]$ is a fuzzy open set in $X$. Therefore $f^{-1}(\alpha)$ is a fuzzy closed set in $X$. Thus the inverse image of every fuzzy maximal closed set in $Y$ is a fuzzy closed set in $X$.

(ii) $\Rightarrow$ (i). Let $\beta$ be any fuzzy minimal open set in $Y$. Then $\beta^c$ is a fuzzy maximal closed set in $Y$. From (ii) $f^{-1}(1 - \beta) = [1 - f^{-1}(\beta)]$ is a fuzzy closed
set in X. Therefore \( f^{-1}(\beta) \) is a fuzzy open set in X. Hence \( f \) is fuzzy minimal continuous map.

**Theorem 3.3.6:** If \( f: X \to Y \), from a fts X into a fts Y is fuzzy minimal continuous mapping, then \( f[\text{cl} (\alpha)] \leq \text{cl} [f (\alpha)] \), for every fuzzy closed set \( \alpha \) in X.

**Proof:** Let \( f: X \to Y \), from a fts X into a fts Y be fuzzy minimal continuous mapping and \( \alpha \) be any fuzzy closed set in X, then \( \text{cl} (\alpha) = \alpha \). Therefore \( f[\text{cl} (\alpha)] = f(\alpha) \leq \text{cl} [f(\alpha)] \).

**Remark 3.3.7:** Converse of the Theorem 3.3.6 need not be true.

**Example 3.3.8:** Let \( X = Y = \{a, b, c\}; I = [0, 1] \) and the functions \( \alpha, \beta, \gamma, \delta: X \to I \) be defined as,

\[
\alpha(x) = \begin{cases} 
1, & \text{if } x = a \\
0, & \text{otherwise}
\end{cases} \quad \beta(x) = \begin{cases} 
1, & \text{if } x = b \\
0, & \text{otherwise}
\end{cases} \\
\gamma(x) = \begin{cases} 
1, & \text{if } x = a, b \\
0, & \text{otherwise}
\end{cases} \quad \delta(x) = \begin{cases} 
1, & \text{if } x = a, c \\
0, & \text{otherwise}
\end{cases}
\]

1- \( \alpha = \{(a, 0), (b, 1), (c, 1)\}; 1- \beta = \{(a, 1), (b, 0), (c, 1)\}; \\
1- \gamma = \{(a,0),(b,0),(c,1)\}; 1- \delta = \{(a,0), (b,1),(c, 0)\}.

Consider \( T = \{0, 1, \alpha, \beta, \gamma\} \) and \( S = \{0, 1, \alpha, \gamma, \delta\} \). Here \((X, T)\) and \((Y, S)\) are fuzzy topological spaces. Let the mapping \( f: X \to Y \), from a fts X into a fts Y be defined by \( f(a) = c, f(b) = b, f(c) = a \). Here for every fuzzy closed set \( \lambda \) in X, \( f[\text{cl}(\lambda)] = f(\lambda) \leq \text{cl} [f(\lambda)] \). But \( f: X \to Y \) is not fuzzy minimal continuous map.
Theorem 3.3.9: If $f: X \to Y$, from a fts X into a fts Y is fuzzy minimal continuous mapping, then $\text{cl} [f^{-1}(\beta)] \leq f^{-1} [\text{cl} (\beta)]$, for every fuzzy maximal closed set $\beta$ in Y.

Proof: Let $f: X \to Y$, from a fts X into a fts Y be fuzzy minimal continuous mapping and $\beta$ be any fuzzy maximal closed set in Y. Then by the Theorem 3.3.5 $f^{-1}(\beta)$ is a closed set in X. By the Theorem 3.3.6, $f[\text{cl} (f^{-1}(\beta))] \subseteq \text{cl} [f(f^{-1}(\beta))] \Rightarrow f[\text{cl} (f^{-1}(\beta))] \subseteq \text{cl}(\beta)$. Hence $\text{cl} [f^{-1}(\beta)] \leq f^{-1} [\text{cl} (\beta)]$.

Remark 3.3.10: Converse of the Theorem 3.3.9 need not be true.

Example 3.3.11: In Example 3.3.8, for every fuzzy maximal closed set $\beta$ in Y, $\text{cl} [f^{-1} (\beta)] \leq f^{-1} [\text{cl} (\beta)]$ but $f: X \to Y$ is not fuzzy minimal continuous map.

Remark 3.3.12: Composition of fuzzy minimal continuous maps need not be fuzzy minimal continuous.

Example 3.3.13: Let $X = Y = Z = \{a, b, c\}$; $I = [0, 1]$ and the functions $\alpha, \beta, \gamma, \delta: X \to I$ be defined as,

$\alpha(x) = \begin{cases} 1, & \text{if } x = a \\ 0, & \text{otherwise} \end{cases}$

$\beta(x) = \begin{cases} 1, & \text{if } x = b \\ 0, & \text{otherwise} \end{cases}$

$\gamma(x) = \begin{cases} 1, & \text{if } x = a, b \\ 0, & \text{otherwise} \end{cases}$

$\delta(x) = \begin{cases} 1, & \text{if } x = a, c \\ 0, & \text{otherwise} \end{cases}$

Consider $T = \{0, 1, \alpha, \beta, \gamma\}$, $S = \{0, 1, \alpha, \beta, \delta\}$ and $P = \{0, 1, \alpha, \delta\}$. Here $(X, T)$, $(Y, S)$ and $(Z, P)$ are fuzzy topological spaces. Let the mapping $f: X \to Y$, from a fts X into a fts Y be defined by $f(a) = a$, $f(b) = b$, $f(c) = c$ and $h: Y \to Z$ from a fts Y into a fts Z be defined by $h(a) = h(c) = a$ and $h(b) = b$. Clearly $f$ and $h$ are fuzzy minimal continuous maps but $hof: X \to Z$
is not a fuzzy minimal continuous map. Since the inverse image of fuzzy minimal open set \( \alpha \) in \((Z, P)\) is \( X: Y \rightarrow I\), defined by

\[
\lambda(x) = \begin{cases} 
1, & \text{if } x = a, c \\
0, & \text{otherwise}
\end{cases}
\]

Again the inverse image of fuzzy set \( \lambda(x) \) in \((Y, S)\) is \( \mu: X \rightarrow I\), defined by

\[
\mu(x) = \begin{cases} 
1, & \text{if } x = a, c \\
0, & \text{otherwise}
\end{cases}
\]

is not a fuzzy open set in \((X, T)\).

Therefore \( \text{h} \circ \text{f}: X \rightarrow Z \) is not fuzzy minimal continuous map.

**Theorem 3.3.14:** If \( f: X \rightarrow Y \), from a fts X into a fts Y is fuzzy continuous map and \( h: Y \rightarrow Z \), from a fts Y into a fts Z is fuzzy minimal continuous map, then \( \text{h} \circ \text{f}: X \rightarrow Z \) is fuzzy minimal continuous map.

**Proof:** Let \( h: Y \rightarrow Z \), from a fts Y into a fts Z be fuzzy minimal continuous map and \( \beta \) be any fuzzy minimal open set in Z. Then \( h^{-1}(\beta) \) is a fuzzy open set in Y. But \( f: X \rightarrow Y \), from a fts X into a fts Y is fuzzy continuous map. Therefore \( f^{-1}[h^{-1}(\beta)] \) is a fuzzy open set in X. That is \( [h \circ f]^{-1}(\beta) \) is a fuzzy open set in X. Hence \( \text{h} \circ \text{f}: X \rightarrow Z \) is fuzzy minimal continuous map.

**Definition 3.3.15:** A mapping \( f: X \rightarrow Y \), from fts X into fts Y is said to be fuzzy maximal continuous (briefly f-m continuous) map if the inverse image of every fuzzy maximal open set in Y is a fuzzy open set in X.

**Theorem 3.3.16:** Every fuzzy continuous map is fuzzy maximal continuous.

**Proof:** Let \( f: X \rightarrow Y \), from a fts X into a fts Y be fuzzy continuous map and \( \alpha \) be any fuzzy maximal open set in Y. As every fuzzy maximal open set is fuzzy open set, \( \alpha \) is a fuzzy open set. Since \( f \) is fuzzy continuous map, \( f^{-1}(\alpha) \) is a fuzzy open set in X. Thus \( f \) is fuzzy maximal continuous map.
Remark 3.3.17: Converse of the Theorem 3.3.16 need not be true.

Example 3.3.18: Let $X = Y \{a, b, c\}; I = [0, 1]$ and the functions $\alpha, \beta, \gamma: X \to I$ be defined as,

\[
\begin{align*}
\alpha(x) &= \begin{cases} 
1, &\text{if } x = a \\
0, &\text{otherwise}
\end{cases} \\
\beta(x) &= \begin{cases} 
1, &\text{if } x = b \\
0, &\text{otherwise}
\end{cases} \\
\gamma(x) &= \begin{cases} 
1, &\text{if } x = a, b \\
0, &\text{otherwise}
\end{cases}
\end{align*}
\]

Consider $T = \{0, 1, \alpha, \gamma\}$ and $S = \{0, 1, \alpha, \beta, \gamma\}$. Here $(X, T)$ and $(Y, S)$ are fuzzy topological spaces. Let the mapping $f: X \to Y$, from a fts $X$ into a fts $Y$ be defined by $f(a) = a, f(b) = b, f(c) = c$. Then clearly $f$ is fuzzy maximal continuous map but not fuzzy continuous map, as the inverse image of the fuzzy open set $\beta$ in $(Y, S)$ is $X$:

\[
\begin{align*}
\lambda(x) &= \begin{cases} 
1, &\text{if } x = b \\
0, &\text{otherwise}
\end{cases}
\end{align*}
\]

is not a fuzzy open in $(X, S)$.

Remark 3.3.19: Fuzzy minimal continuous maps and fuzzy maximal continuous maps are independent of each other.

Example 3.3.20: Let $X_1 = X_2 = Y_1 = Y_2 = \{a, b, c\}; I = [0, 1]$ and the functions $\alpha, \beta, \gamma, \delta: X \to I$ be defined as,

\[
\begin{align*}
\alpha(x) &= \begin{cases} 
1, &\text{if } x = a \\
0, &\text{otherwise}
\end{cases} \\
\beta(x) &= \begin{cases} 
1, &\text{if } x = b \\
0, &\text{otherwise}
\end{cases} \\
\gamma(x) &= \begin{cases} 
1, &\text{if } x = a, b \\
0, &\text{otherwise}
\end{cases} \\
\delta(x) &= \begin{cases} 
1, &\text{if } x = a, c \\
0, &\text{otherwise}
\end{cases}
\end{align*}
\]

Consider $T = \{0, 1, \alpha, \gamma\}, S = \{0, 1, \alpha, \beta, \gamma\}, P = \{0, 1, \alpha, \beta, \gamma\}$ and $L = \{0, 1, \alpha, \gamma, \delta\}$. Here $(X_1, T), (Y_1, S), (X_2, P)$ and $(Y_2, L)$ are fuzzy topological spaces. Let the mappings $f: X_1 \to Y_1$, from a fts $X_1$ into a fts $Y_1$.
be defined by \( f(a) = a, f(b) = b, f(c) = c \) and \( h: X_2 \rightarrow Y_2 \), from a fts \( X_2 \) into a fts \( Y_2 \) be defined by \( h(a) = a, h(b) = b \) and \( h(c) = c \).

Clearly \( f \) is fuzzy maximal continuous map but not fuzzy minimal continuous map and \( h \) is fuzzy minimal continuous map but not fuzzy maximal continuous map.

**Theorem 3.3.21:** If \( f: X \rightarrow Y \), from a fts \( X \) into a fts \( Y \) is any mapping, then the following are equivalent.

(i) \( f \) is fuzzy maximal continuous map.

(ii) The inverse image of every fuzzy minimal closed set in \( Y \) is a fuzzy closed set in \( X \).

Proof: (i) \( \Rightarrow \) (ii). Let \( \alpha \) be any fuzzy minimal closed set in \( Y \). Then \( \alpha^c \) is a fuzzy maximal open set in \( Y \). Since \( f \) is fuzzy maximal continuous, \( f^{-1}(1- \alpha) = [1- f^{-1}(\alpha)] \) is a fuzzy open set in \( X \). Therefore \( f^{-1}(\alpha) \) is a fuzzy closed set in \( X \). Hence the inverse image of every fuzzy minimal closed set in \( Y \) is a fuzzy closed set in \( X \).

(ii) \( \Rightarrow \) (i). Let \( \beta \) be any fuzzy maximal open set in \( Y \). Then \( \beta^c \) is a fuzzy minimal closed set in \( Y \). From (ii) \( f^{-1}(1- \beta) = [1- f^{-1}(\beta)] \) fuzzy closed set in \( X \). Therefore \( f^{-1}(\beta) \) is a fuzzy open set in \( X \). Hence \( f \) is fuzzy maximal continuous map.

**Theorem 3.3.22:** If \( f: X \rightarrow Y \), from a fts \( X \) into a fts \( Y \) is fuzzy maximal continuous mapping, then we have the following.

(i) \( f[\text{cl}(\alpha)] \leq \text{cl}[f(\alpha)] \), for every fuzzy closed set \( \alpha \) in \( X \).

(ii) \( \text{cl}[f^{-1}(\beta)] \leq f^{-1}[\text{cl}(\beta)] \), for every fuzzy minimal closed set \( \beta \) in \( Y \).
**Proof:** (i) Let \( f: X \rightarrow Y \), from a fts \( X \) into a fts \( Y \) be any fuzzy maximal continuous mapping and \( \alpha \) be any fuzzy closed set in \( X \), then \( \text{cl}(\alpha) = \alpha \). Therefore \( f[\text{cl}(\alpha)] = f(\alpha) \leq \text{cl}[f(\alpha)] \).

(ii) Let \( f: X \rightarrow Y \), from a fts \( X \) into a fts \( Y \) be any fuzzy maximal continuous mapping and \( \beta \) be any fuzzy minimal closed set in \( Y \), then by the Theorem 3.3.21 \( f^{-1}(\beta) \) is a fuzzy closed set in \( X \). Therefore from (i), \( f[\text{cl}(f^{-1}(\beta))] \leq \text{cl}[f(f^{-1}(\beta))] \) implies \( f[\text{cl}(f^{-1}(\beta))] \leq \text{cl}(\beta) \). Hence \( \text{cl}[f^{-1}(\beta)] \leq f^{-1}[\text{cl}(\beta)] \).

**Remark 3.3.23:** Converse of the Theorem 3.3.22 need not be true.

**Example 3.3.24:** Let \( X = Y \{a, b, c\}; I = [0, 1] \) and the functions \( \alpha, \beta, \gamma, \delta: X \rightarrow I \) be defined as,

\[
\alpha(x) = \begin{cases} 
1, & \text{if } x = a \\
0, & \text{otherwise}
\end{cases} \quad \beta(x) = \begin{cases} 
1, & \text{if } x = b \\
0, & \text{otherwise}
\end{cases} \\
\gamma(x) = \begin{cases} 
1, & \text{if } x = a, b \\
0, & \text{otherwise}
\end{cases} \quad \delta(x) = \begin{cases} 
1, & \text{if } x = a, c \\
0, & \text{otherwise}
\end{cases}
\]

\( 1^- \alpha = \{(a, 0), (b, 1), (c, 1)\}; 1^- \beta = \{(a, 1), (b, 0), (c, 1)\}; \\
1^- \gamma = \{(a, 0), (b, 0), (c, 1)\}; 1^- \delta = \{(a, 0), (b, 1), (c, 0)\}. \)

Consider \( T = \{0, 1, \alpha, \beta, \gamma\} \) and \( S = \{0, 1, \alpha, \gamma, \delta\} \). Here \((X, T)\) and \((Y, S)\) are fuzzy topological spaces. Let the mapping \( f: X \rightarrow Y \), from a fts \( X \) into a fts \( Y \) be defined by \( f(a) = c, f(b) = b, f(c) = a \).

Here for every fuzzy closed set \( \lambda \) in \( X \), \( f[\text{cl}(\lambda)] \leq \text{cl}[f(\lambda)] \) and for every fuzzy minimal closed set \( \mu \) in \( Y \), \( \text{cl}[f^{-1}(\mu)] \leq f^{-1}[\text{cl}(\mu)] \). But \( f: X \rightarrow Y \) is not fuzzy maximal continuous map.

**Remark 3.3.25:** Composition of fuzzy maximal continuous map need not be fuzzy maximal continuous.
Example 3.3.26: Let \( X = Y = Z = \{a, b, c\}; I = [0, 1] \) and the functions \( \alpha, \beta, \gamma, \delta: X \to I \) be defined as,

\[
\alpha(x) = \begin{cases} 
1, & \text{if } x = a \\
0, & \text{otherwise}
\end{cases}
\]

\[
\beta(x) = \begin{cases} 
1, & \text{if } x = b \\
0, & \text{otherwise}
\end{cases}
\]

\[
\gamma(x) = \begin{cases} 
1, & \text{if } x = a, b \\
0, & \text{otherwise}
\end{cases}
\]

\[
\delta(x) = \begin{cases} 
1, & \text{if } x = a, c \\
0, & \text{otherwise}
\end{cases}
\]

Consider \( T = \{0, 1, \beta, \delta\}, S = \{0, 1, \alpha, \gamma\} \) and \( P = \{0, 1, \alpha, \delta\} \). Here \((X, T), (Y, S)\) and \((Z, P)\) are fuzzy topological spaces. Let the mappings \( f: X \to Y \), from a fts \( X \) into a fts \( Y \) be defined by \( f(a) = a, f(b) = f(c) = b \) and \( h: Y \to Z \) from a fts \( Y \) into a fts \( Z \) be defined by \( h(a) = a \) and \( h(c) = h(b) = b \).

Clearly \( f \) and \( h \) are fuzzy maximal continuous maps but \( h \circ f: X \to Z \) is not fuzzy maximal continuous map. Since the inverse image of fuzzy maximal open set \( \delta \) in \((Z, P)\) is

\[
X: Y \to I, \text{ defined by } X(x) = \begin{cases} 
-1, & \text{if } x = a \\
0, & \text{otherwise}
\end{cases}
\]

Again the inverse image of fuzzy set \( \lambda \) (x) in \((Y, S)\) is \( \mu: X \to I \), defined by

\[
\mu(x) = \begin{cases} 
1, & \text{if } x = a \\
0, & \text{otherwise}
\end{cases}
\]

is not a fuzzy open set in \((X, T)\).

Therefore \( h \circ f: X \to Z \) is not fuzzy maximal continuous map.

Theorem 3.3.27: If \( f: X \to Y \), from a fts \( X \) into a fts \( Y \) is fuzzy continuous map and \( h: Y \to Z \), from a fts \( Y \) into a fts \( Y \) is fuzzy maximal continuous map, then \( h \circ f: X \to Z \) is fuzzy maximal continuous map.

Proof: Let \( h: Y \to Z \), from a fts \( Y \) into a fts \( Z \) be fuzzy maximal continuous map and \( \beta \) be any fuzzy maximal open set in \( Z \). Then \( h^{-1}(\beta) \) is a fuzzy open set in \( Y \). But \( f: X \to Y \), from a fts \( X \) into a fts \( Y \) is fuzzy continuous map. 
Therefore \( f^{-1}[h^{-1}(\beta)] \) is a fuzzy open set in \( X \). That is \([h \circ f]^{-1}(\beta)\) is a fuzzy open set in \( X \). Hence \( h \circ f: X \rightarrow Z \) is fuzzy maximal continuous map.

**Definition 3.3.28:** A mapping \( f: X \rightarrow Y \), from a fts \( X \) into a fts \( Y \) is said to be fuzzy minimal irresolute (briefly f-mi irresolute) map if the inverse image of every fuzzy minimal open set in \( Y \) is a fuzzy minimal open set in \( X \).

**Theorem 3.3.29:** Every fuzzy minimal irresolute map is fuzzy minimal continuous map.

**Proof:** Let \( f: X \rightarrow Y \), from a fts \( X \) into a fts \( Y \) be fuzzy minimal irresolute map and let \( \alpha \) be any fuzzy minimal open set in \( Y \). Then \( f^{-1}(\alpha) \) is a fuzzy minimal open set in \( X \). As every fuzzy minimal open set is a fuzzy open set, \( f^{-1}(\alpha) \) is a fuzzy open set in \( X \). Therefore \( f \) is fuzzy minimal continuous map.

**Remark 3.3.30:** Converse of the Theorem 3.3.29 need not be true.

**Example 3.3.31:** Let \( X = Y = \{a, b, c, d\} \); \( I = [0, 1] \) and the functions \( \alpha, \beta, \gamma, \delta: X \rightarrow I \) be defined as,

\[
\alpha(x) = \begin{cases} 
1, & \text{if } x = a \\
0, & \text{otherwise}
\end{cases} \quad \beta(x) = \begin{cases} 
1, & \text{if } x = b \\
0, & \text{otherwise}
\end{cases} \\
\gamma(x) = \begin{cases} 
1, & \text{if } x = a, b \\
0, & \text{otherwise}
\end{cases} \quad \delta(x) = \begin{cases} 
1, & \text{if } x = a, b, c \\
0, & \text{otherwise}
\end{cases}
\]

Consider \( T = \{0, 1, \alpha, \beta, \gamma\} \) and \( S = \{0, 1, \gamma, \delta\} \). Here \((X, T)\) and \((Y, S)\) are fuzzy topological spaces. Let the mapping \( f: X \rightarrow Y \), from a fts \( X \) into a fts \( Y \) be defined by \( f(a) = a, f(b) = b, f(c) = c \) and \( f(d) = d \). Then \( f \) is fuzzy minimal continuous map but not fuzzy minimal irresolute map, as the inverse image of a fuzzy minimal open set \( \gamma \) in \((Y, S)\) is \( \lambda: X \rightarrow I \) defined by,
\( \lambda(x) = \begin{cases} 1, & \text{if } x = a, b \\ 0, & \text{otherwise} \end{cases} \) is not a fuzzy minimal open set in \((X, T)\).

**Theorem 3.3.32:** If \( f: X \to Y \), from a fts \( X \) into a fts \( Y \) is any mapping, then the following are equivalent.

(i) \( f \) is fuzzy minimal irresolute map.

(ii) The inverse image of every fuzzy maximal closed set in \( Y \) is a fuzzy maximal closed set in \( X \).

**Proof:** (i) \( \Rightarrow \) (ii). Let \( \alpha \) be any fuzzy maximal closed set in \( Y \). Then \( \alpha^c \) is a fuzzy minimal open set in \( Y \). Since \( f \) is fuzzy minimal irresolute map, 
\[ f^{-1}(1- \alpha) = [1 - f^{-1}(\alpha)] \] 
is a fuzzy minimal open set in \( X \). Therefore \( f^{-1}(\alpha) \) is a fuzzy maximal closed set in \( X \). Hence the inverse image of every fuzzy maximal closed set in \( Y \) is fuzzy maximal closed set in \( X \).

(ii) \( \Rightarrow \) (i). Let \( \beta \) be any fuzzy minimal open set in \( Y \), then \( \beta^c \) is a fuzzy maximal closed set in \( Y \). From (ii) 
\[ f^{-1}(1- \beta) = [1 - f^{-1}(\beta)] \] 
is a fuzzy maximal closed set in \( X \). Therefore \( f^{-1}(\beta) \) is a fuzzy minimal open set in \( X \).

Hence \( f \) is fuzzy minimal irresolute map.

**Theorem 3.3.33:** If \( f: X \to Y \), from a fts \( X \) into a fts \( Y \), is fuzzy minimal irresolute mapping, then the following are true.

(i) \( f[\text{cl}(\alpha)] \leq \text{cl}[f(\alpha)] \), for every fuzzy maximal closed set \( \alpha \) in \( X \).

(ii) \( \text{cl}[f^{-1}(\beta)] \leq f^{-1}[\text{cl}(\beta)] \), for every fuzzy maximal closed set \( \beta \) in \( Y \).

**Proof:** (i) Let \( f: X \to Y \), from a fts \( X \) into a fts \( Y \) be fuzzy minimal irresolute map and \( \alpha \) be any fuzzy maximal closed set in \( X \). Since every fuzzy maximal closed set is a fuzzy closed set, \( \alpha \) is fuzzy closed set in \( X \). Then \( \text{cl}(\alpha) = \alpha \). Therefore 
\[ f[\text{cl}(\alpha)] = f(\alpha) \leq \text{cl}[f(\alpha)] \].
(ii) Let \( f: X \to Y \), from a fts \( X \) into a fts \( Y \) be fuzzy minimal irresolute map and \( \beta \) be any fuzzy maximal closed set in \( Y \), then by the Theorem 3.3.32, \( f^{-1}(\beta) \) is fuzzy maximal closed set in \( X \). Therefore from (i), \( f[\text{cl} (f^{-1}(\beta))] \leq \text{cl} [f(f^{-1}(\beta))] \Rightarrow f[\text{cl} (f^{-1}(\beta))] \leq \text{cl}(\beta) \). Hence \( \text{cl} [f^{-1}(\beta)] \leq f^{-1}[\text{cl} (\beta)] \).

**Remark 3.3.34:** Converse of the Theorem 3.3.33 need not be true.

**Example 3.3.35:** Let \( X = Y = \{a, b, c\} \); \( I = [0, 1] \) and the functions \( \alpha, \beta, \gamma, \delta: X \to I \) be defined as,

\[
\alpha(x) = \begin{cases} 1, & \text{if } x = a \\ 0, & \text{otherwise} \end{cases}, \\
\beta(x) = \begin{cases} 1, & \text{if } x = b \\ 0, & \text{otherwise} \end{cases}, \\
\gamma(x) = \begin{cases} 1, & \text{if } x = a, b \\ 0, & \text{otherwise} \end{cases}, \\
\delta(x) = \begin{cases} 1, & \text{if } x = a, c \\ 0, & \text{otherwise} \end{cases}
\]

\(1- \alpha = \{(a, 0), (b, 1), (c, 1)\}; 1- \beta = \{(a, 1), (b, 0), (c, 1)\}; 1- \gamma = \{(a,0),(b,0),(c,1)\}; 1- \delta = \{(a,0), (b,1),(c, 0)\}\).

Consider \( T = \{0, 1, \alpha, \beta, \gamma\} \) and \( S = \{0, 1, \alpha, \gamma, \delta\} \). Then \((X, T)\) and \((Y, S)\) are fuzzy topological spaces. Let the mapping \( f: X \to Y \), from a fts \( X \) into a fts \( Y \) by \( f(a) = c, f(b) = b, f(c) = a \).

Here for every fuzzy maximal closed set \( v \) in \( X \), \( f[\text{cl} (v)] = f(v) \leq \text{cl} [f(v)] \) and for every fuzzy maximal closed set \( \mu \) in \( Y \), \( \text{cl} [f^{-1}(\mu)] \leq f^{-1}[\text{cl} (\mu)] \). But \( f: X \to Y \) is not fuzzy minimal irresolute map since the inverse image of fuzzy minimal open set \( \alpha \) in \((Y, S)\) is \( \lambda: X \to I \) defined by,

\[
\lambda(x) = \begin{cases} 1, & \text{if } x = c \\ 0, & \text{otherwise} \end{cases}
\]

is not fuzzy minimal open set in \((X, T)\).

**Theorem 3.3.36:** If \( f: X \to Y \), from a fts \( X \) into a fts \( Y \) is fuzzy minimal irresolute map and \( h: Y \to Z \), from a fts \( Y \) into a fts \( Z \) is fuzzy minimal irresolute map, then \( hof: X \to Z \) is a fuzzy minimal irresolute map.

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Proof: Let \( h: Y \rightarrow Z \), from a fts \( Y \) into a fts \( Z \) be fuzzy minimal irresolute map and \( \beta \) be fuzzy minimal open set in \( Z \). Then \( h^{-1}(\beta) \) is fuzzy minimal open set in \( Y \). But \( f: X \rightarrow Y \), from a fts \( X \) into a fts \( Y \) is fuzzy minimal irresolute map. Therefore \( f^{-1}[h^{-1}(\beta)] \) is fuzzy minimal open set in \( X \). That is \( (h \circ f)^{-1}(\beta) \) is a fuzzy minimal open set in \( X \). Hence \( h \circ f: X \rightarrow Z \) is fuzzy minimal irresolute map.

Definition 3.3.37: A mapping \( f: X \rightarrow Y \), from fts \( X \) into fts \( Y \) is said to be fuzzy maximal irresolute (briefly f- ma irresolute) map if the inverse image of every fuzzy maximal open set in \( Y \) is a fuzzy maximal open set in \( X \).

Theorem 3.3.38: Every fuzzy maximal irresolute map is fuzzy maximal continuous map.

Proof: Let \( f: X \rightarrow Y \), from a fts \( X \) into a fts \( Y \) be fuzzy maximal irresolute map and let \( \alpha \) be any fuzzy maximal open set in \( Y \). Then \( f^{-1}(\alpha) \) is a fuzzy maximal open set in \( X \). As every fuzzy maximal open set is fuzzy open set, \( f^{-1}(\alpha) \) is a fuzzy open set in \( X \). Therefore \( f \) is fuzzy maximal continuous map.

Remark 3.3.39: Converse of the Theorem 3.3.38 need not be true.

Example 3.3.40: Let \( X = Y \{a, b, c, d\}; I = [0, 1] \) and the functions \( \alpha, \beta, \gamma, \delta: X \rightarrow I \) be defined as,

\[
\alpha(x) = \begin{cases} 1, & \text{if } x = a \\ 0, & \text{otherwise} \end{cases}, \quad \beta(x) = \begin{cases} 1, & \text{if } x = b \\ 0, & \text{otherwise} \end{cases}
\]

\[
\gamma(x) = \begin{cases} 1, & \text{if } x = a, b \\ 0, & \text{otherwise} \end{cases}, \quad \delta(x) = \begin{cases} 1, & \text{if } x = a, b, c \\ 0, & \text{otherwise} \end{cases}
\]
Consider $T = \{0, 1, \alpha, \beta, \gamma, \delta\}$ and $S = \{0, 1, \alpha, \beta, \gamma\}$. Here $(X, T)$ and $(Y, S)$ are fuzzy topological spaces. Let the mapping $f: X \rightarrow Y$, from a fts $X$ into a fts $Y$ be defined by $f(a) = a$, $f(b) = b$, $f(c) = c$ and $f(d) = d$. Then $f$ is fuzzy maximal continuous map but not fuzzy maximal irresolute map, as the inverse image of the fuzzy maximal open set $\gamma$ in $(Y, S)$ is $\lambda: X \rightarrow I$ defined by,

$$
\lambda(x) = \begin{cases} 
1, & \text{if } x = a, b \\
0, & \text{otherwise}
\end{cases}
$$

is not a fuzzy maximal open set in $(X, T)$.

**Remark 3.3.41:** Fuzzy maximal irresolute map and fuzzy minimal irresolute map are independent of each other.

**Example 3.3.42:** Let $X = Y = \{a, b, c\}$; $I = [0, 1]$ and the functions $\alpha, \beta, \gamma, \delta: X \rightarrow I$ be defined as,

$$
\alpha(x) = \begin{cases} 
1, & \text{if } x = a \\
0, & \text{otherwise}
\end{cases}, \quad \beta(x) = \begin{cases} 
1, & \text{if } x = b \\
0, & \text{otherwise}
\end{cases},
$$

$$
\gamma(x) = \begin{cases} 
1, & \text{if } x = a, b \\
0, & \text{otherwise}
\end{cases}, \quad \delta(x) = \begin{cases} 
1, & \text{if } x = a, c \\
0, & \text{otherwise}
\end{cases}
$$

Consider $T = \{0, 1, \alpha, \gamma\}$, $S = \{0, 1, \alpha, \beta, \gamma\}$, $P = \{0, 1, \alpha, \beta, \gamma\}$ and $L = \{0, 1, \alpha, \gamma, \delta\}$. Here $(X_1, T)$, $(Y_1, S)$, $(X_2, P)$ and $(Y_2, L)$ are fuzzy topological spaces. Let the mappings $f: X_1 \rightarrow Y_1$, from a fts $X_1$ into a fts $Y_1$ be defined by $f(a) = a$, $f(b) = b$, $f(c) = c$ and $h: X_2 \rightarrow Y_2$, from a fts $X_2$ into a fts $Y_2$ be defined by $h(a) = a$, $h(b) = b$ and $h(c) = c$.

Clearly $f: X_1 \rightarrow Y_1$ is fuzzy maximal irresolute map but not fuzzy minimal irresolute map and $h: X_2 \rightarrow Y_2$ is fuzzy minimal irresolute map but not fuzzy maximal irresolute map.

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Theorem 3.3.43: If \( f: X \to Y \), from a fts \( X \) into a fts \( Y \) is any mapping, then the following are equivalent.

(i) \( f \) is fuzzy maximal irresolute map.

(ii) The inverse image of every fuzzy minimal closed set in \( Y \) is a fuzzy minimal closed set in \( X \).

Proof: (i) \( \Rightarrow \) (ii). Let \( f: X \to Y \), from a fts \( X \) into a fts \( Y \) be fuzzy maximal irresolute map and \( \alpha \) be any fuzzy minimal closed set in \( Y \). Then \( 1-\alpha \) is a fuzzy maximal open set in \( Y \). Since \( f \) is fuzzy maximal irresolute map, \( f^{-1}(1-\alpha) = [1-f^{-1}(\alpha)] \) is a fuzzy maximal open set in \( X \). Therefore \( f^{-1}(\alpha) \) is a fuzzy minimal closed in \( X \). Hence the inverse image of every fuzzy minimal closed set in \( Y \) is a fuzzy minimal closed set in \( X \).

(ii) \( \Rightarrow \) (i). Let \( \beta \) be any fuzzy maximal open set in \( Y \). Then \( \beta^c \) is a fuzzy minimal closed set in \( Y \). From (ii) \( f^{-1}(1-\beta) = [1-f^{-1}(\beta)] \) is a fuzzy minimal closed set in \( X \). Therefore \( f^{-1}(\beta) \) is a fuzzy maximal open set in \( X \). Hence \( f \) is fuzzy maximal irresolute map.

Theorem 3.3.44: If \( f: X \to Y \), from a fts \( X \) into a fts \( Y \) is fuzzy maximal irresolute mapping, then the following are true.

(i) \( f[\text{cl} (\alpha)] \leq \text{cl} [f (\alpha)] \), for every fuzzy minimal closed set \( \alpha \) in \( X \).

(ii) \( \text{cl} [f^{-1}(\beta)] \leq f^{-1}[\text{cl}(\beta)] \), for every fuzzy minimal closed set \( \beta \) in \( Y \).

Proof: (i) Let \( f: X \to Y \), from a fts \( X \) into a fts \( Y \) be fuzzy maximal irresolute mapping and \( \alpha \) be any fuzzy minimal closed set in \( X \). Since every fuzzy minimal closed set is a fuzzy closed set, \( \alpha \) is fuzzy closed set in \( X \). Then \( \text{cl} (\alpha) = \alpha \). Therefore \( f[\text{cl} (\alpha)] = f(\alpha) \leq \text{cl} [f(\alpha)] \).

(ii) Let \( f: X \to Y \), from a fts \( X \) into a fts \( Y \) be fuzzy maximal irresolute mapping and \( \beta \) be any fuzzy minimal closed set in \( Y \). Then from the
Theorem 3.3.43 \( f^{-1}(\beta) \) is a fuzzy minimal closed set in \( X \). Therefore from (i),
\[
f[\operatorname{cl}(f^{-1}(\beta))] \leq \operatorname{cl}[f(f^{-1}(\beta))] = f[\operatorname{cl}(f^{-1}(\beta))] \leq \operatorname{cl}(\beta).
\]
Hence \( \operatorname{cl}[f^{-1}(\beta)] \leq f^{-1}[\operatorname{cl}(\beta)] \).

Remark 3.3.45: Converse of the Theorem 3.3.44 need not be true.

Example 3.3.46: Let \( X = Y \{a, b, c\}; I = [0, 1] \) and the functions \( \alpha, \beta, \gamma, \delta : X \rightarrow I \) be defined as,
\[
\alpha(x) = \begin{cases} 
1, & \text{if } x = a \\
0, & \text{otherwise}
\end{cases},
\]
\[
\beta(x) = \begin{cases} 
1, & \text{if } x = b \\
0, & \text{otherwise}
\end{cases},
\]
\[
\gamma(x) = \begin{cases} 
1, & \text{if } x = a, b \\
0, & \text{otherwise}
\end{cases},
\]
\[
\delta(x) = \begin{cases} 
1, & \text{if } x = a, c \\
0, & \text{otherwise}
\end{cases}.
\]

Consider \( T = \{0, 1, \alpha, \beta, \gamma\} \) and \( S = \{0, 1, \alpha, \gamma, \delta\} \). Then \( (X, T) \) and \( (Y, S) \) are fuzzy topological spaces. Let the mapping \( f : X \rightarrow Y \), from a fts \( X \) into a fts \( Y \) be defined by \( f(a) = c, f(b) = b, f(c) = a \).

Here for every fuzzy minimal closed set \( v \) in \( X \), \( f[\operatorname{cl}(v)] = f(v) \leq \operatorname{cl}[f(v)] \) and for every fuzzy minimal closed set \( \mu \) in \( Y \), \( \operatorname{cl}[f^{-1}(\mu)] \leq f^{-1}[\operatorname{cl}(\mu)] \). But \( f : X \rightarrow Y \) is not fuzzy maximal irresolute map, since the inverse image of fuzzy maximal open set \( \delta \) in \( (Y, S) \) is \( \lambda : X \rightarrow I \) defined by,
\[
\lambda(x) = \begin{cases} 
1, & \text{if } x = a, c \\
0, & \text{otherwise}
\end{cases},
\]
is not a fuzzy maximal open set in \( (X, T) \).

Theorem 3.3.47: If \( f : X \rightarrow Y \), from a fts \( X \) into a fts \( Y \) is fuzzy maximal irresolute map and \( h : Y \rightarrow Z \), from a fts \( Y \) into a fts \( Z \) is fuzzy maximal irresolute map, then \( hof : X \rightarrow Z \) is fuzzy maximal irresolute map.
Proof: Let $h: Y \to Z$, from a fts $Y$ into a fts $Z$ be fuzzy maximal irresolute map and $\beta$ be any fuzzy maximal open set in $Z$. Then $h^{-1}(\beta)$ is a fuzzy maximal open set in $Y$. But $f: X \to Y$ is fuzzy maximal irresolute map. Therefore $f^{-1}[h^{-1}(\beta)]$ is fuzzy maximal open set in $X$. That is $[h \circ f]^{-1}(\beta)$ is a fuzzy maximal open set in $X$. Hence $hof: X \to Z$ is fuzzy maximal irresolute map.

Remark 3.4.48: From the above results we have the following implications.

\begin{center}
\begin{tikzpicture}
\node (m1) at (0,0) {f-\text{m}_i \text{ continuous}};
\node (m2) at (4,0) {f-\text{m}_a \text{ continuous}};
\node (m3) at (0,-4) {f-\text{m}_i \text{ irresolute}};
\node (m4) at (4,-4) {f-\text{m}_a \text{ irresolute}};
\node (m5) at (2,-2) {Fuzzy continuous};
\draw[->] (m1) -- (m2);
\draw[->] (m3) -- (m4);
\draw[->] (m1) -- (m5);
\draw[->] (m5) -- (m2);
\draw[->] (m3) -- (m5);
\draw[->] (m5) -- (m4);
\end{tikzpicture}
\end{center}
3.4 Fuzzy Minimal Open Maps in Fuzzy Topological Spaces.

In this section, fuzzy minimal open maps that includes a class of fuzzy strongly minimal open maps, fuzzy maximal open maps and fuzzy strongly maximal open maps are introduced and studied in fuzzy topological spaces.

**Definition 3.4.1:** A mapping \( f: X \to Y \), from a fts \( X \) into a fts \( Y \) is said to be fuzzy minimal open (briefly f-m* open) map if the image of every fuzzy minimal open set in \( X \) is a fuzzy open set in \( Y \).

**Theorem 3.4.2:** Every fuzzy open map is fuzzy minimal open map.

**Proof:** Let \( f: X \to Y \), from a fts \( X \) into a fts \( Y \) be fuzzy open map and let \( \alpha \) be any fuzzy minimal open set in \( X \). As every fuzzy minimal open set is a fuzzy open set, \( \alpha \) is fuzzy open set. Since \( f \) is fuzzy open map, \( f(\alpha) \) is a fuzzy open set in \( Y \). Therefore \( f \) is fuzzy minimal open map.

**Remark 3.4.3:** Converse of the Theorem 3.4.2 need not be true.

**Example 3.4.4:** Let \( X = Y \{a, b, c\}; I = [0, 1]\) and the functions \( \alpha, \beta, \gamma, \delta: X \to I \) be defined as,

\[
\alpha(x) = \begin{cases} 
1, & \text{if } x = a \\ 
0, & \text{otherwise}
\end{cases} \quad \beta(x) = \begin{cases} 
1, & \text{if } x = b \\ 
0, & \text{otherwise}
\end{cases}
\]

\[
\gamma(x) = \begin{cases} 
1, & \text{if } x = a, b \\ 
0, & \text{otherwise}
\end{cases} \quad \delta(x) = \begin{cases} 
1, & \text{if } x = a, c \\ 
0, & \text{otherwise}
\end{cases}
\]

Consider \( T = \{0, 1, \alpha, \gamma, \delta\} \) and \( S = \{0, 1, \alpha, \beta, \gamma\} \). Here \((X, T)\) and \((Y, S)\) are fuzzy topological spaces. Let the mapping \( f: X \to Y \), from a fts \( X \) into a fts \( Y \) be defined by \( f(a) = a, f(b) = b, f(c) = c \). Then \( f \) is fuzzy minimal open.
map but not fuzzy open map, as the image of a fuzzy open set \( \delta \) in \((X, T)\) is
\[ \lambda: Y \to I \] defined by,
\[ \lambda(x) = \begin{cases} 1, & \text{if } x = a, c \\ 0, & \text{otherwise} \end{cases} \] is not a fuzzy open set in \((Y, S)\).

**Theorem 4.4.5**: If \( f: X \to Y \), from a fts \( X \) into a fts \( Y \) is any mapping, then
the following statements are equivalent.

i) \( f \) is fuzzy minimal open map.

ii) For any fuzzy subset \( \alpha \) of \( Y \) and any fuzzy maximal closed set \( \beta \) in \( X \) containing \( f^{-1}(\alpha) \), there exists a fuzzy closed set \( \delta \) in \( Y \) containing \( \alpha \) such that \( f^{-1}(\delta) < \beta \).

**Proof**: (i) \( \Rightarrow \) (ii). Let \( f: X \to Y \), from a fts \( X \) into a fts \( Y \) be fuzzy minimal open map and let \( \alpha \) be any fuzzy subset of \( Y \) and \( \beta \) be any fuzzy maximal closed set in \( X \) containing \( f^{-1}(\alpha) \). i.e. \( f^{-1}(\alpha) < \beta \). Now \( \beta \) is any fuzzy maximal closed set means \( 1 - \beta \) is a fuzzy minimal open set in \( X \). From (i), \( f(1 - \beta) \) is a fuzzy open set in \( Y \). Therefore \( [1 - f(1 - \beta)] \) is a fuzzy closed set in \( Y \). Let us take \( \delta = [1 - f(1 - \beta)] \).

To prove \( \alpha < \delta \): We have \( f^{-1}(\alpha) < \beta \) which implies \( 1 - \beta < 1 - f^{-1}(\alpha) = f^{-1}(1 - \alpha) \) implies \( f(1 - \beta) < 1 - \alpha \) this implies \( \alpha < 1 - f(1 - \beta) = \delta \). Therefore \( \alpha < \delta \).

To prove \( f^{-1}(\delta) < \beta \): Now \( f^{-1}(\delta) = f^{-1}(1 - f(1 - \beta)) = 1 - f^{-1} [f(1 - \beta)] < \beta \), since \( 1 - \beta < f^{-1} [f(1 - \beta)] \) implies \( 1 - f^{-1} [f(1 - \beta)] < \beta \). Therefore \( f^{-1}(\delta) < \beta \).

(ii) \( \Rightarrow \) (i). Let \( \lambda \) be any fuzzy minimal open set in \( X \), so that \( 1 - \lambda \) is a fuzzy maximal closed set in \( X \). Then \( f^{-1}[1 - f(\lambda)] < 1 - \lambda \). Here \( \alpha = 1 - f(\lambda) \). From (ii) there exists a fuzzy closed set \( \delta \) in \( Y \) containing \([1 - f(\lambda)]\) such that
\( f^{-1}(\delta) < 1 - \lambda = \beta \) implies \( \lambda < 1 - f^{-1}(\delta) \). Now \( [1 - f(\lambda)] < \delta \) implies that \( 1 - \delta < f(\lambda) < f[1 - f^{-1}(\delta)] = 1 - f[f^{-1}(\delta)] < 1 - \delta \).

Therefore \( f(\lambda) = 1 - \delta \), where \( \delta \) is a fuzzy closed set in \( Y \). Then \( 1 - \delta \) is a fuzzy open set in \( Y \). i.e., \( f(\lambda) \) is a fuzzy open set in \( Y \). Therefore \( f: X \to Y \) is fuzzy minimal open map.

**Remark 3.4.6:** Composition of fuzzy minimal open mappings need not be fuzzy minimal open map.

**Example 3.4.7:** Let \( X = Y = Z = \{a, b, c\} \); \( I = [0, 1] \) and the functions \( \alpha, \beta, \gamma, \delta: X \to I \) be defined as,

\[
\alpha(x) = \begin{cases} 1, & \text{if } x = a \\ 0, & \text{otherwise} \end{cases}, \quad \beta(x) = \begin{cases} 1, & \text{if } x = b \\ 0, & \text{otherwise} \end{cases},
\]

\[
\gamma(x) = \begin{cases} 1, & \text{if } x = a, b \\ 0, & \text{otherwise} \end{cases}, \quad \delta(x) = \begin{cases} 1, & \text{if } x = b, c \\ 0, & \text{otherwise} \end{cases}.
\]

Consider \( T = \{0, 1, \alpha, \delta\} \), \( S = \{0, 1, \alpha, \beta, \gamma, \delta\} \) and \( P = \{0, 1, \alpha, \beta, \gamma\} \). Here \( (X, T) \), \( (Y, S) \) and \( (Z, P) \) are fuzzy topological spaces. Let the mappings \( f: X \to Y \), from a fts \( X \) into a fts \( Y \) be defined by \( f(a) = a, f(b) = b, f(c) = c \) and \( h: Y \to Z \), from a fts \( Y \) into a fts \( Z \) be defined by \( h(a) = a, h(b) = b \) and \( h(c) = c \). Clearly \( f \) and \( h \) are fuzzy minimal open maps but \( ho\) is not fuzzy minimal open map. Since the image of a fuzzy minimal open set \( \delta \) in \( (X, T) \) is \( \lambda: Y \to I \), defined by

\[
\lambda(x) = \begin{cases} 1, & \text{if } x = b, c \\ 0, & \text{otherwise} \end{cases}.
\]

Again the image of fuzzy set \( \mu(x) \) in \( (Y, S) \) is \( \mu: X \to I \), defined by

\[
\mu(x) = \begin{cases} 1, & \text{if } x = b, c \\ 0, & \text{otherwise} \end{cases}, \text{ is not a fuzzy open set in } (Z, P).
\]
Therefore \( hof: X \rightarrow Z \) is not fuzzy minimal open map.

**Theorem 3.4.8:** If \( f: X \rightarrow Y \), from a fts X into a fts Y is fuzzy minimal open map and \( h: Y \rightarrow Z \), from a fts Y into a fts Z is fuzzy open map, then \( hof: X \rightarrow Z \) is fuzzy minimal open map.

**Proof:** Let \( f: X \rightarrow Y \), from a fts X into a fts Y be fuzzy minimal open map and \( \beta \) be any fuzzy minimal open set in X. Then \( f (\beta) \) is a fuzzy open set in Y. But \( h: Y \rightarrow Z \) is fuzzy open map. Therefore \( h [f (\beta)] \) is a fuzzy open set in Z. That is \([h \circ f](\beta)\) is a fuzzy open set in Z. Hence \( hof: X \rightarrow Z \) is fuzzy minimal open map.

**Theorem 3.4.9:** If \( f: X \rightarrow Y \), from a fts X into a fts Y and \( h: Y \rightarrow Z \), from a fts Y into a fts Z are any two mappings and \( hof: X \rightarrow Z \), from a fts X into a fts Y is fuzzy minimal open map. Then,

(i) \( h \) is fuzzy minimal open map if \( f \) is fuzzy minimal irresolute map and surjective map.

(ii) \( f \) is fuzzy minimal open map if \( h \) is fuzzy continuous and injective map.

**Proof:**

(i) Let \( \alpha \) be any fuzzy minimal open set in Y. Then, since \( f \) is fuzzy minimal irresolute map, \( f^{-1} (\alpha) \) is a fuzzy minimal open set in X. But \( hof: X \rightarrow Z \) is fuzzy minimal open map. Therefore \( hof[f^{-1} (\alpha)] \) is fuzzy open set in Z. Since \( f \) is surjective map, \( hof[f^{-1} (\alpha)] = h [f(f^{-1} (\alpha))] = h (\alpha) \) is a fuzzy open set in Z. Hence \( h \) is fuzzy minimal open map.

(ii) Let \( \beta \) be any fuzzy minimal open set in X. Since \( hof: X \rightarrow Z \) is fuzzy minimal open map, \( hof(\beta) \) is a fuzzy open set in Z. But \( h \) is fuzzy continuous and injective map. Therefore \( h^{-1} [hof (\beta)] = (h^{-1} h)f (\beta) = f (\beta) \) is a fuzzy open set in Y. Hence \( f \) is fuzzy minimal open map.
Definition 3.4.10: A mapping \( f: X \to Y \), from a fts \( X \) into a fts \( Y \) is said to be a fuzzy strongly minimal open (briefly f-s-m, open) map if the image of every fuzzy minimal open set in \( X \) is a fuzzy minimal open set in \( Y \).

Theorem 3.4.11: Every fuzzy strongly minimal open map is fuzzy minimal open map.

Proof: Let \( f: X \to Y \), from a fts \( X \) into a fts \( Y \) be fuzzy strongly minimal open map and let \( \alpha \) be any fuzzy minimal open set in \( X \). Then \( f(\alpha) \) is a fuzzy minimal open set in \( Y \). As every fuzzy minimal open set is a fuzzy open set, \( f(\alpha) \) is fuzzy open set in \( Y \). Therefore \( f \) is fuzzy minimal open map.

Remark 3.4.12: Converse of the Theorem 3.4.11 need not be true.

Example 3.4.13: Let \( X = Y \) \{a, b, c, d\}; \( I = [0, 1] \) and the functions \( \alpha, \beta, \gamma, \delta, \mu, \eta, \nu: X \to I \) be defined as,

\[
a(x) = \begin{cases} 1, & \text{if } x = a \\ 0, & \text{otherwise} \end{cases} \quad \beta(x) = \begin{cases} 1, & \text{if } x = b \\ 0, & \text{otherwise} \end{cases}
\]

\[
\delta(x) = \begin{cases} 1, & \text{if } x = a, d \\ 0, & \text{otherwise} \end{cases} \quad \gamma(x) = \begin{cases} 1, & \text{if } x = a, b \\ 0, & \text{otherwise} \end{cases}
\]

\[
\eta(x) = \begin{cases} 1, & \text{if } x = a, b, c \\ 0, & \text{otherwise} \end{cases} \quad \mu(x) = \begin{cases} 1, & \text{if } x = b, c \\ 0, & \text{otherwise} \end{cases}
\]

\[
\nu(x) = \begin{cases} 1, & \text{if } x = a, b, d \\ 0, & \text{otherwise} \end{cases}
\]

Consider \( T = \{0, 1, \alpha, \delta, \mu, \eta\} \) and \( S = \{0, 1, \alpha, \beta, \gamma, \delta, \eta, \nu\} \). Here \( (X, T) \) and \( (Y, S) \) are fuzzy topological spaces. Let the mapping \( f: X \to Y \), from a
fts X into a fts Y be defined by \( f(a) = a, f(b) = b, f(c) = c \) and \( f(d) = d \). Then \( f \) is fuzzy minimal open map but not fuzzy strongly minimal open map, as the image of the fuzzy minimal open set \( \mu \) in \((X, T)\) is \( \lambda : Y \to I \) defined by,

\[
\lambda(x) = \begin{cases} 
1, & \text{if } x = b, c \\
0, & \text{otherwise}
\end{cases}
\]

is not a fuzzy minimal open set in \((Y, S)\).

**Theorem 3.4.14:** If \( f : X \to Y \), from a fts X into a fts Y is any fuzzy mapping, then the following statements are equivalent.

i) \( f \) is fuzzy strongly minimal open map.

ii) For any fuzzy subset \( \alpha \) of \( Y \) and any fuzzy maximal closed set \( \beta \) in \( X \) containing \( f^{-1}(\alpha) \), there exists a fuzzy maximal closed set \( \delta \) in \( Y \) containing \( \alpha \) such that \( f^{-1}(\delta) < \beta \).

**Proof:** (i) \( \Rightarrow \) (ii). Let \( f : X \to Y \), from a fts X into a fts Y be fuzzy minimal open map and let \( \alpha \) be any fuzzy subset of \( Y \) and \( \beta \) be any fuzzy maximal closed set in \( X \) containing \( f^{-1}(\alpha) \). i.e. \( f^{-1}(\alpha) < \beta \). Now \( \beta \) is any fuzzy maximal closed set means \( 1 - \beta \) is a fuzzy minimal open set in \( X \). Since \( f \) is fuzzy minimal open map, \( f(1 - \beta) \) is a fuzzy open set in \( Y \). Therefore \( [1 - f(1 - \beta)] \) is a fuzzy maximal closed set in \( Y \).

Let us take \( \delta = [1 - f(1 - \beta)] \).

To prove \( \alpha < \delta \): We have \( f^{-1}(\alpha) < \beta \) which implies \( 1 - \beta < 1 - f^{-1}(\alpha) \).

\( f^{-1}(1 - \alpha) \) implies \( f(1 - \beta) < 1 - \alpha \) implies \( \alpha < 1 - f(1 - \beta) = \delta \).

Therefore \( \alpha < \delta \).

To prove \( f^{-1}(\delta) < \beta \): Now \( f^{-1}(\delta) = f^{-1}(1 - f(1 - \beta)) = 1 - f^{-1} [f(1 - \beta)] < \beta \).

Since \( (1 - \beta) < f^{-1} [f(1 - \beta)] \Rightarrow 1 - f^{-1} [f(1 - \beta)] < \beta \).

Therefore \( f^{-1}(\delta) < \beta \).
Let $\lambda$ be any fuzzy minimal open set in $X$, so that $1 - \lambda$ is a fuzzy maximal closed set in $X$. Then $f^{-1}[1 - f(\lambda)] < 1 - \lambda$. Here $\alpha = 1 - f(\lambda)$. From (ii) there exists a fuzzy maximal closed set $\delta$ in $Y$ containing $[1 - f(\lambda)]$ such that $f^{-1}(\delta) < 1 - \lambda = \beta$ implies $\lambda < 1 - f^{-1}(\delta)$. Now $[1 - f(\lambda)] < \delta$ implies that $1 - \delta < f(\lambda) < f[1 - f^{-1}(\delta)] = 1 - f[f^{-1}(\delta)] < 1 - \delta$. Therefore $f(\lambda) = 1 - \delta$, where $\delta$ is a fuzzy maximal closed set in $Y$. Then $1 - \delta$ is a fuzzy minimal open set in $Y$. Therefore $f: X \to Y$ is fuzzy strongly minimal open map.

**Theorem 3.4.15:** If $f: X \to Y$, from a fts $X$ into a fts $Y$ and $h: Y \to Z$, from a fts $Y$ into a fts $Z$ are fuzzy strongly minimal open mappings, then $hof: X \to Z$ is a fuzzy strongly minimal open map.

**Proof:** Let $f: X \to Y$, from a fts $X$ into a fts $Y$ be fuzzy minimal open map and $\beta$ be any fuzzy minimal open set in $X$. Then $f(\beta)$ is a fuzzy minimal open set in $Y$. But $h: Y \to Z$ is fuzzy strongly minimal open map. Therefore $h(f(\beta))$ is a fuzzy minimal open set in $Z$. That is $[hof](\beta)$ is a fuzzy minimal open set in $Z$. Hence $hof: X \to Z$ is fuzzy strongly minimal open map.

**Theorem 3.4.16:** If $f: X \to Y$, from a fts $X$ into a fts $Y$ and $h: Y \to Z$, from a fts $Y$ into a fts $Z$ are any two mappings and $hof: X \to Z$, from a fts $X$ into a fts $Z$ is fuzzy strongly minimal open map. Then,

(i) $h$ is fuzzy strongly minimal open map if $f$ is surjective and fuzzy minimal irresolute map.

(ii) $f$ is fuzzy strongly minimal open map if $h$ is injective and fuzzy minimal irresolute map.
Proof: (i) Let $\alpha$ be any fuzzy minimal open set in $Y$. Since $f$ is fuzzy minimal irresolute map, $f^{-1}(\alpha)$ is a fuzzy minimal open set in $X$. But $hof: X \to Z$ is fuzzy strongly minimal open map. Then $hof[f^{-1}(\alpha)]$ is fuzzy minimal open set in $Z$. Since $f$ is surjective, $hof[f^{-1}(\alpha)] = h[f(f^{-1}(\alpha))] = h(\alpha)$ is a fuzzy minimal open set in $Z$. Hence $h$ is fuzzy strongly minimal open map.

(ii) Let $\beta$ be any fuzzy minimal open set in $X$. Since $hof: X \to Z$ is fuzzy strongly minimal open map, $hof(\beta)$ is a fuzzy minimal open set in $Z$. But $h$ is fuzzy minimal irresolute and injective map. Therefore $hr^{-1}[hof(\beta)] = (hr^{-1}h)f(\beta) = f(\beta)$ is a fuzzy minimal open set in $Y$. Hence $f$ is fuzzy strongly minimal open map.

Definition 3.4.17: A mapping $f: X \to Y$, from a fts $X$ into a fts $Y$ is said to be fuzzy maximal open (briefly f-max open) map if the image of every fuzzy maximal open set in $X$ is a fuzzy open set in $Y$.

Theorem 3.4.18: Every fuzzy open map is fuzzy maximal open map.

Proof: Let $f: X \to Y$, from a fts $X$ into a fts $Y$ be fuzzy open map and let $\alpha$ be any fuzzy maximal open set in $X$. As every fuzzy maximal open set is a fuzzy open set, $\alpha$ is fuzzy open set. Since $f$ is fuzzy open map, $f(\alpha)$ is fuzzy open set in $Y$. Therefore $f$ is fuzzy maximal open map.

Remark 3.4.19: Converse of the Theorem 3.4.18 need not be true.

Example 3.4.20: Let $X = Y \{a, b, c\}; I = [0, 1]$ and the functions $\alpha, \beta, \gamma, \delta: X \to I$ be defined as,

$$
\alpha(x) = \begin{cases} 1, & \text{if } x = a \\ 0, & \text{otherwise} \end{cases} \quad \beta(x) = \begin{cases} 1, & \text{if } x = b \\ 0, & \text{otherwise} \end{cases}
$$
Consider \( T = \{0, 1, \alpha, \beta, \gamma\} \) and \( S = \{0, 1, \alpha, \gamma, \delta\} \). Here \((X, T)\) and \((Y, S)\) are fuzzy topological spaces. Let the mapping \( f: X \rightarrow Y \), from a fts \( X \) into a fts \( Y \) be defined by \( f(a) = a, f(b) = b, f(c) = c \). Then \( f \) is fuzzy maximal open map but not fuzzy open map, as the image of a fuzzy open set \( P \) in \((X, T)\) is not a fuzzy open in \((Y, S)\).

Theorem 3.4.21: If \( f: X \rightarrow Y \), from a fts \( X \) into a fts \( Y \) is any fuzzy mapping, then the following statements are equivalent.

i) \( f \) is fuzzy maximal open map.

ii) For any fuzzy subset \( \alpha \) of \( Y \) and any fuzzy minimal closed set \( \beta \) in \( X \) containing \( f^{-1}(\alpha) \), there exists a fuzzy closed set \( \delta \) in \( Y \) containing \( \alpha \) such that \( f^{-1}(\delta) < \beta \).

Proof: (i) \( \Rightarrow \) (ii). Let \( f: X \rightarrow Y \), from a fts \( X \) into a fts \( Y \) be fuzzy maximal open map and let \( \alpha \) be any fuzzy subset of \( Y \) and \( \beta \) be any fuzzy minimal closed set in \( X \) containing \( f^{-1}(\alpha) \). i.e. \( f^{-1}(\alpha) < \beta \). Now \( \beta \) is any fuzzy minimal closed set means \( 1 - \beta \) is a fuzzy maximal open set in \( X \). Since \( f \) is fuzzy maximal open map, \( f(1 - \beta) \) is a fuzzy open set in \( Y \). Therefore \( [1 - f(1 - \beta)] \) is a fuzzy closed in \( Y \). Let us take \( \delta = [1 - f(1 - \beta)] \).

To prove \( \alpha < \delta \): We have \( f^{-1}(\alpha) < \beta \) which implies \( 1 - \beta < 1 - f^{-1}(\alpha) = f^{-1}(1 - \alpha) f(1 - \beta) < 1 - \alpha \) implies \( \alpha < 1 - f(1 - \beta) = \delta \). Therefore \( \alpha < \delta \).

To prove \( f^{-1}(\delta) < \beta \): Now \( f^{-1}(\delta) = f^{-1}(1 - f(1 - \beta)) = 1 - f^{-1}[f(1 - \beta)] < \beta \), since \( (1 - \beta) < f^{-1}[f(1 - \beta)] \Rightarrow 1 - f^{-1}[f(1 - \beta)] < \beta \).
Therefore $f^{-1}(\delta) < \beta$.

(ii) $\Rightarrow$ (i). Let $\lambda$ be any fuzzy maximal open set in $X$, so $1 - \lambda$ is a fuzzy minimal closed set in $X$. Then $f^{-1}[1 - f(\lambda)] < 1 - \lambda$. Here $\alpha = 1 - f(\lambda)$.

From (ii) there exists a fuzzy closed set $\delta$ in $Y$ containing $[1 - f(\lambda)]$ such that $f^{-1}(\delta) < 1 - \lambda$. Here $f^{-1}(\delta) < 1 - \lambda$ implies $\lambda < 1 - f^{-1}(\delta)$. Now $[1 - f(\lambda)] < \delta$ implies $1 - \delta < f(\lambda) < f[1 - f^{-1}(\delta)] = 1 - f(f^{-1}(\delta)) < 1 - \delta$.

Therefore $f(\lambda) = 1 - \delta$, where $\delta$ is a fuzzy closed set in $Y$. Then $1 - \delta$ is a fuzzy open set in $Y$. i.e., $f(\lambda)$ is a fuzzy open set in $Y$. Therefore $f$ is fuzzy maximal open map.

**Remark 3.4.22:** Composition of fuzzy maximal open mappings need not be fuzzy maximal open map.

**Example 3.4.23:** Let $X = Y = Z = \{a, b, c\}$, $I = [0, 1]$ and the functions $\alpha, \beta, \gamma, \delta: X \rightarrow I$ be defined as,

$$
\alpha(x) = \begin{cases} 1, & \text{if } x = a \\ 0, & \text{otherwise} \end{cases} \quad \beta(x) = \begin{cases} 1, & \text{if } x = b \\ 0, & \text{otherwise} \end{cases}
$$

$$
\gamma(x) = \begin{cases} 1, & \text{if } x = a, b \\ 0, & \text{otherwise} \end{cases} \quad \delta(x) = \begin{cases} 1, & \text{if } x = a, c \\ 0, & \text{otherwise} \end{cases}
$$

Consider $T = \{0, 1, \alpha, \beta, \gamma\}$, $S = \{0, 1, \alpha, \gamma\}$ and $P = \{0, 1, \alpha, \gamma, \delta\}$. Here $(X, T)$, $(Y, S)$ and $(Z, P)$ are fuzzy topological spaces. Let the mappings $f: X \rightarrow Y$, from a fts $X$ into a fts $Y$ be defined by $f(a) = f(b) = a$, $f(c) = c$ and $h: Y \rightarrow Z$, from a fts $Y$ into a fts $Z$ be defined by $h(a) = b$, $h(b) = a$ and $h(c) = c$.

Clearly $f$ and $h$ are fuzzy maximal open maps but $hof: X \rightarrow Z$ is not a fuzzy maximal open map. Since the image of a fuzzy maximal open set $\gamma$ in $(X, T)$ is $\lambda: Y \rightarrow I$, defined by
Again the image of a fuzzy set $\lambda(x)$ in $(Y, S)$ is $\mu: Z \to I$, defined by

$$\mu(x) = \begin{cases} 1, & \text{if } x = b \\ 0, & \text{otherwise} \end{cases}$$

is not a fuzzy open set in $(Z, P)$.

Therefore $hof: X \to Z$ is not a fuzzy maximal open map.

**Theorem 3.4.24:** If $f: X \to Y$, from a fts $X$ into a fts $Y$ is fuzzy maximal open map and $h: Y \to Z$, from a fts $Y$ into a fts $Z$ is fuzzy open map, then $hof: X \to Z$ is fuzzy maximal open map.

**Proof:** Let $f: X \to Y$, from a fts $X$ into a fts $Y$ be fuzzy maximal open map and $\beta$ be any fuzzy maximal open set in $X$. Then $f(\beta)$ is a fuzzy open set in $Y$. But $h: Y \to Z$ is fuzzy open map. Therefore $h[f(\beta)]$ is a fuzzy open set in $Z$. That is $[h \circ f](\beta)$ is a fuzzy open set in $Z$. Hence $hof: X \to Z$ is fuzzy maximal open map.

**Theorem 3.4.25:** If $f: X \to Y$, from a fts $X$ into a fts $Y$ and $h: Y \to Z$, from a fts $Y$ into a fts $Z$ are any two mappings and $hof: X \to Z$, from a fts $X$ into a fts $Z$ is fuzzy maximal open map. Then,

(i) $h$ is fuzzy maximal open map if $f$ is surjective and fuzzy maximal irresolute map.

(ii) $f$ is fuzzy maximal open map if $h$ is injective and fuzzy continuous map.

**Proof:** Follows from the Theorem 3.4.9.

**Definition 3.4.26:** A mapping $f: X \to Y$, from a fts $X$ into a fts $Y$ is said to be a fuzzy strongly maximal open (briefly f- s-m open) map if the image of every fuzzy maximal open set in $X$ is a fuzzy maximal open set in $Y$. 

\[\lambda(x) = \begin{cases} 1, & \text{if } x = a \\ 0, & \text{otherwise} \end{cases}\]
Theorem 3.4.27: Every fuzzy strongly maximal open map is fuzzy maximal open map.

Proof: Follows from the Definition 3.4.26 and the Theorem 3.4.18.

Remark 3.4.28: Converse of the Theorem 3.4.27 need not be true.

Example 3.4.29: Let $X = \{a, b, c, d\}$; $I = [0, 1]$ and the functions $\alpha, \beta, \gamma, \delta: X \to I$ be defined as,

\[
\alpha(x) = \begin{cases} 
1, & \text{if } x = a \\
0, & \text{otherwise}
\end{cases} \\
\beta(x) = \begin{cases} 
1, & \text{if } x = b \\
0, & \text{otherwise}
\end{cases} \\
\gamma(x) = \begin{cases} 
1, & \text{if } x = a, b \\
0, & \text{otherwise}
\end{cases} \\
\delta(x) = \begin{cases} 
1, & \text{if } x = a, b, c \\
0, & \text{otherwise}
\end{cases}
\]

Consider $T = \{0, 1, \alpha, \beta, \gamma\}$ and $S = \{0, 1, \alpha, \beta, \gamma, \delta\}$. Here $(X, T)$ and $(Y, S)$ are fuzzy topological spaces. Let the mapping $f: X \to Y$, from a fts $X$ into a fts $Y$ be defined by $f(a) = a$, $f(b) = b$, $f(c) = c$ and $f(d) = d$. Then $f$ is fuzzy maximal open map but not fuzzy strongly maximal open map, as the image of the fuzzy maximal open set $\gamma$ in $(X, T)$ is $\lambda: Y \to I$ defined by,

\[
\lambda(x) = \begin{cases} 
1, & \text{if } x = a, b \\
0, & \text{otherwise}
\end{cases}
\]

is not a fuzzy maximal open set in $(Y, S)$.

Theorem 3.4.30: If $f: X \to Y$, from a fts $X$ into a fts $Y$ is any fuzzy mapping, then the following statements are equivalent.

i) $f$ is fuzzy strongly maximal open map.

ii) For any fuzzy subset $\alpha$ of $Y$ and any fuzzy minimal closed set $\beta$ in $X$ containing $f^{-1}(\alpha)$, there exists a fuzzy minimal closed set $\delta$ in $Y$ containing $\alpha$ such that $f^{-1}(\delta) < \beta$. 

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Proof: (i) ⇒ (ii). Let \( f: X \to Y \), from a fts \( X \) into a fts \( Y \) be fuzzy maximal open map and let \( \alpha \) be any fuzzy subset of \( Y \) and \( \beta \) be any fuzzy minimal closed set in \( X \) containing \( f^{-1}(\alpha) \). i.e. \( f^{-1}(\alpha) < \beta \). Now \( \beta \) is any fuzzy minimal closed set means \( 1 - \beta \) is a fuzzy maximal open set in \( X \). Since \( f \) is fuzzy maximal open map, \( f(1-\beta) \) is a fuzzy maximal open set in \( Y \). Therefore \([1 - f(1 - \beta)] \) is a fuzzy minimal closed set in \( Y \).

Let us take \( \delta = [1 - f(1 - \beta)] \).

To prove \( \alpha < \delta \): We have \( f^{-1}(\alpha) < \delta \) which implies \( 1 - \beta < 1 - f^{-1}(\alpha) = f^{-1}(1 - \alpha) \) implies \( f(1 - \beta) < 1 - \alpha \) implies \( \alpha < 1 - f(1 - \beta) = \delta \).

Therefore \( \alpha < \delta \).

To prove \( f^{-1}(\delta) < \beta \): Now \( f^{-1}(\delta) = f^{-1}(1 - f(1 - \beta)) = 1 - f^{-1}[f(1 - \beta)] < \beta \), since \( (1 - \beta) f^{-1}[f(1 - \beta)] \Rightarrow 1 - f^{-1}[f(1 - \beta)] < \beta \). Therefore \( f^{-1}(\delta) < \beta \).

(ii) ⇒ (i). Let \( \lambda \) be any fuzzy maximal open set in \( X \), so \( 1 - \lambda \) is a fuzzy minimal closed set in \( X \). Then \( f^{-1}[1 - f(\lambda)] < 1 - \lambda \). Here \( \alpha = 1 - f(\lambda) \).

From (ii) there exists a fuzzy minimal closed set \( \delta \) in \( Y \) containing \([1 - f(\lambda)]\) such that \( f^{-1}(\delta) < 1 - \lambda = \beta \). Here \( f^{-1}(\delta) < 1 - \lambda \) implies \( \lambda < 1 - f^{-1}(\delta) \).

Now \([1 - f(\lambda)] < \delta \) implies that \( 1 - \delta < f(\lambda) < f[1 - f^{-1}(\delta)] = 1 - f[f^{-1}(\delta)] < 1 - \delta \).

Therefore \( f(\lambda) = 1 - \delta \), where \( \delta \) is a fuzzy minimal closed set in \( Y \). Then \( 1 - \delta \) is a fuzzy maximal open set in \( Y \). i.e. \( f(\lambda) \) is a fuzzy maximal open set in \( Y \). Therefore \( f \) is fuzzy maximal open map.

Theorem 3.4.31: If \( f: X \to Y \), from a fts \( X \) into a fts \( Y \) and \( h: Y \to Z \), from a fts \( Y \) into a fts \( Z \) are fuzzy strongly maximal open mappings, then \( hof: X \to Z \), from a fts \( X \) into a fts \( Z \) is fuzzy strongly maximal open map.

Proof: Let \( f: X \to Y \), from a fts \( X \) into a fts \( Y \) be fuzzy strongly maximal open map and \( \beta \) be any fuzzy maximal open set in \( X \). Then \( f(\beta) \) is a fuzzy
maximal open set in Y. But \( h: Y \to Z \) is fuzzy strongly maximal open map. Therefore \( h [ f (\beta)] \) is a fuzzy maximal open set in Z. That is \( [h \circ f](\beta) \) is a fuzzy maximal open set in Z. Hence \( h \circ f: X \to Z \) is fuzzy strongly maximal open map.

**Theorem 3.4.32:** If \( f: X \to Y \), from a fts X into a fts Y and \( h: Y \to Z \), from a fts Y into a fts Z are any two fuzzy mappings and \( h \circ f: X \to Z \), from a fts X into a fts Z is fuzzy strongly maximal open map. Then,

(i) \( h \) is fuzzy strongly maximal open map if \( f \) is surjective and fuzzy maximal irresolute map.

(ii) \( f \) is fuzzy strongly maximal open map if \( h \) is injective and fuzzy maximal irresolute map.

**Proof:** (i) Let \( \alpha \) be any fuzzy maximal open set in Y. Since \( f \) is fuzzy maximal irresolute map, \( f^{-1}(\alpha) \) is a fuzzy maximal open set in X. But \( h \circ f: X \to Z \) is fuzzy strongly maximal open map. Then \( h \circ f[f^{-1}(\alpha)] \) is a fuzzy maximal open set in Z. Since \( f \) is surjective, \( h \circ f[f^{-1}(\alpha)] = h[f(f^{-1}(\alpha))] = h(\alpha) \) is a fuzzy maximal open set in Z. Hence \( h \) is fuzzy strongly maximal open map.

(ii) Let \( \beta \) be any fuzzy maximal open set in X. Since \( h \circ f: X \to Z \) is fuzzy strongly maximal open map, \( h \circ f(\beta) \) is a fuzzy maximal open set in Z. But \( h \) is fuzzy maximal irresolute and injective map. Therefore \( h^{-1} [h \circ f(\beta)] = (h^{-1} h) f (\beta) = f (\beta) \) is a fuzzy maximal open set in Y. Hence \( f \) is fuzzy strongly maximal open map.

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Remark 3.4.33: From the above results we have the following implications.