Chapter 3

VALUE DISTRIBUTION OF MEROMORPHIC FUNCTIONS SATISFYING GENERALISED MODIFIED THIRD PAINLEVÉ DIFFERENTIAL EQUATION
3.1 ON GENERALISED MODIFIED THIRD PAINLEVÉ DIFFERENTIAL EQUATION

3.2 Introduction, Results and Definitions

We start our considerations of the Painlevé transcendents by examining the following three differential equations:

\[
\begin{align*}
    f'' &= 6f^2 + z \\ 
    f'' &= 2f^3 + zw + \alpha_2 \\ 
    ff'' &= \frac{1}{2}(f')^2 + \frac{3}{2}f^4 + 4zf^3 + 2(z^2 - \alpha_4)f^2 + \beta_4
\end{align*}
\]  

(3.2.1) (3.2.2) (3.2.3)

where \( \alpha_j, \beta_j \in \mathbb{C} \). These equations, which are called usually as the first, the second and the fourth Painlevé Differential equations, see e.g. [19] p-157, have been recently under careful study: Hinkkanen and Laine [24] proved that all the solutions of (3.2.1) and (3.2.2) are meromorphic, and later on using different techniques, N. Steinmetz showed that all the solutions of (3.2.3) are meromorphic. Furthermore S. Shimomura proved a number of results about the value distribution of the solutions of equations (3.2.1), (3.2.2), (3.2.3).

Now we consider the following two types of differential equations: commonly known as third and fifth Painlevé Differential Equations:

\[
\begin{align*}
    f'' &= \left(\frac{(f')^2}{f} - \frac{f'}{z} + \frac{\alpha_3 f^2 + \beta_3}{z} + \gamma_3 f^3 + \frac{\delta_3}{f}\right) \\
    f'' &= \left(1 + \frac{1}{f - 1}\right) \left(\frac{(f')^2}{f} - \frac{f'}{z} + \frac{(f - 1)^2}{z^2 f} (\alpha_5 f^2 + \beta_5) + \frac{\gamma_5 f}{z} + \frac{\delta_5 f (f + 1)}{f - 1}\right)
\end{align*}
\]  

(3.2.4) (3.2.5)

where \( \alpha_j, \beta_j, \gamma_j, \delta_j \in \mathbb{C} \) for \( j = 3.5 \). They are usually called as third and fifth Painlevé Differential Equations. see e.g. ([24] pp-157).
Unlike the cases of first, the second, and the fourth Painlevé differential equations, it is immediately seen that not all the solutions of (3.2.4) are meromorphic (see [31]). As an example, take \( f(z) = cz^{1/3} \) with \( c^3 = -\beta_3/\alpha_3 \) in the case \( \beta_3 = \gamma_3 = 0 \) and \( \alpha_3 \delta_3 \neq 0 \). However, by making a transformation \( \Psi(\xi) = z f(z) = e^{i\xi/2} f(e^{i\xi/2}) \). \( z = e^{i\xi/2} \), and replacing \( (\alpha_3, \beta_3, \gamma_3, \delta_3) \) with \( (4\alpha_3, 4\beta_3, 4\gamma_3, 4\delta_3) \), we have an equation of the form

\[
ff'' = (f')^2 + \alpha_3 f^3 + \beta_3 f^4 + \gamma_3 e^2 f + \delta_3 e^{2z}
\]

and this is called the modified third Painlevé Equation.

Hinkkanen and Laine proved in [23] that all solutions of this modified third Painlevé equation are meromorphic.

Similarly, not all solutions of (3.2.5) are meromorphic (see [22]). But by making transformation \( \Psi(\xi) = f(z) = f(e^\xi) \). \( z = e^\xi \), we have an equation of the form

\[
f'' = \left(\frac{1}{2f} + \frac{1}{f-1}\right) (f')^2 + (f-1)^2 (\alpha_3 f + \frac{\beta_3}{f}) + \gamma_3 e^2 f + \frac{\delta_3 e^{2z} f(f+1)}{f-1}
\]

and this is called the modified fifth Painlevé Equation. And it has been proved that all the solutions of modified fifth Painlevé equation are meromorphic (see [22]).

We can also observe that when \( \gamma_3 \neq 0 \), all the poles of solutions of (3.2.6) are simple. When \( \gamma_3 = 0 \), by assuming that \( \alpha_3 \neq 0 \), all the poles of solutions of (3.2.6) are double in this case (see [31]).

Throughout this chapter we use the standard notations of Nevanlinna Theory, i.e \( m(r, f) \), \( n(r, f) \), \( N(r, f) \) and \( T(r, f) \) denote the proximity function, the non-integrated counting function, the counting function and characteristic function of \( f \) and the following:

\[
N_1(r, f) := 2N(r, f) - N(r, f')
\]

\[
N_B(r, f) := N\left(r, \frac{1}{f'}\right) + 2N(r, f) - N(r, f')
\]
Here in this chapter we generalise the modified third Painlevé Differential equation and study the value distributions of its meromorphic solutions. And the Generalised Modified Third Painlevé Equation is as follows:

\[ f^{(2m)} = \sum_{i=1}^{m} (-1)^{i+1} f^{(2m-i)} f^{(i)} + \sum_{i=\min(4,m+2)}^{2m+2} \alpha_i f^i + \beta e^z f + \gamma e^{2z} \quad (3.2.8) \]

where \( \alpha_i, \beta, \gamma \in \mathbb{C} \) for \( m \geq 1 \). At \( z = z_0 \), let \( f \) has a pole of order \( p \), then

\[ f(z) = \frac{\Psi(z)}{(z - z_0)^p}, \]

where, \( \Psi(z) \neq 0 \) and \( \Psi(z) \) is analytic in the neighbourhood of \( z_0 \), and

\[ f^{(j)} = \frac{\Psi_j(z)}{(z - z_0)^{p+j}} \]

where, \( \Psi_j(z_0) \neq 0 \) and \( \Psi_j(z) \) is analytic in the neighbourhood of \( z_0 \), for \( j = 1, 2, \ldots p \).

Substituting in (3.2.8), we get the two cases as follows:

**Case(a):** For \( m \geq 2 \), \( \alpha_{2m+2} \neq 0 \) we get

\[
\frac{\Psi(z)}{(z - z_0)^p (z - z_0)^{p+2m}} = \frac{\Psi_{2m}(z)}{(z - z_0)^{p+2m-1}} + \frac{\Psi_{2m-1}(z)}{(z - z_0)^{p+2m-1}} + \frac{\Psi_{2m-1}(z)}{(z - z_0)^{p+2m-1}} + \alpha_{2m+1}(z - z_0)^{2pm+p} + \ldots
\]

\[
+ \frac{\Psi(z)^{2m+2}}{(z - z_0)^{2m+2+p}} + \beta e^z \frac{\Psi(z)}{(z - z_0)^{2m+2+p}} + \gamma e^{2z}
\]

\[ 2p + 2m = 2mp + 2p \]

hence \( p = 1 \).

Therefore the poles of \( f \) are simple if \( \alpha_{2m+2} \neq 0 \).

**Case(b):** For \( m = 1 \), \( \alpha_4 = 0 \), assuming \( \alpha_3 \neq 0 \). then

\[
\frac{\Psi(z)}{(z - z_0)^p (z - z_0)^{p+2}} = \frac{\Psi_1(z)}{(z - z_0)^{p+1}} + \frac{\Psi_1(z)}{(z - z_0)^{p+1}} + \alpha_3 (z - z_0)^{3p} + \ldots
\]
\[
\psi(z) + \alpha_1 \frac{\psi(z)}{(z - z_0)} + \beta e^z \frac{\psi(z)}{(z - z_0)} + \gamma e^{2z} = 2p + 2 = 3p
\]

\[\text{hence } p = 2.\]

Therefore the poles of \(f\) are double if \(\alpha_1 = 0, \alpha_3 \neq 0\). And hence in this case if we assume that the solutions are transcendental meromorphic as in (see\([5]\)), then for \(m \geq 1\), if \(\alpha_{2m+2} \neq 0\) the poles of solutions of (3.2.8) are simple and when \(\alpha_{2m+2} = 0\), assuming \(\alpha_{2m+1} \neq 0\) the poles of solutions of (3.2.8) are double.

Now, in this chapter we shall study the value distribution of solutions of (3.2.8). The following are our main results. We need the following important lemmas to prove our results.

### 3.3 Lemmas

**Lemma 3.3.1. (Clunie Lemma) ([31]P-17):** Let \(f(z)\) be a transcendental meromorphic solution of finite order of the differential equation

\[f^n P(z, f) = Q(z, f),\]

where \(P(z, f)\) and \(Q(z, f)\) are two differential polynomials in \(f\) and its derivatives with rational co-efficients. If the total degree of \(Q(z, f)\) as a polynomial in \(f\) and its derivatives is \(\leq n\), then

\[m(r, P(f)) = O(\log r).\]

**Lemma 3.3.2. (Logarithmic Derivative Lemma) ([31], P-56):** Let \(f(z)\) be a transcendental meromorphic solution of differential equation (3.2.8)) and, \(k \geq 1\), be an integer.
Then,

\[ m\left(r, \frac{F^{(k)}}{F}\right) = O(r). \]

where \( F = f + g \), for some rational function \( g \).

Similar to the Lemma 3.3.1, we state the following finite order version of Mohon'ko's lemma.

**Lemma 3.3.3. (Mohon'ko Lemma)** Let \( P(z, f, f', f'', \ldots, f^{(n)}) = 0 \) be an algebraic differential equation i.e. \( P(z, u, u', u'', \ldots, u^{(n)}) = 0 \) is a polynomial in \( u \) and its coefficients are belonging to the field \( \{ h : T(r, h) = O(r) \} \) and let \( f \) be a transcendental meromorphic solution of the above equation and if a constant \( a \in \mathbb{C} \) does not solve the equation, then

\[ m\left(r, \frac{1}{f - a}\right) = O(\log r). \]

**Remark 3.3.1.** If we suppose that all the assumptions of the Lemma 3.3.1 and Lemma 3.3.3 hold, except that the coefficients of \( P \) belong to the field \( \{ h : T(r, h) = O(r) \} \), instead of being rational, and that \( T(r, f) = O(e^{\Gamma r}) \) for some \( \Gamma > 0 \), instead of \( f \) being of finite order, we have

\[ m(r, 1/f) = O(r) \]

The following are our main results.

### 3.4 Statement and Proof of Main Theorems

**Theorem 3.4.1.** Let \( f \) be an arbitrary solution of (3.2.8), then

(i) \( m(r, f) = O(r) \).
(ii) If \( c = 0 \in \mathbb{C} \) and if \( \gamma \neq 0 \), then \( m \left( r, \frac{1}{f} \right) = O(r) \).

If \( c \neq 0 \in \mathbb{C} \) and if \( \gamma = 0 \), then \( m \left( r, \frac{1}{f-c} \right) = O(r) \).

(iii) \( m \left( r, \frac{1}{f} \right) \leq \frac{2m-1}{2m} T(r, f) + O(r) \).

**Proof.** (i) (3.2.8) can be written as

\[
\sum_{i=1}^{2m+2} \alpha_i f_i = f f^{(2m)} - \sum_{i=1}^{m} (-1)^{i+1} f^{(2m-i)} f^{(i)} - \beta e^z f - \gamma e^{2z} + \sigma_{2m+2} \alpha_{2m+2} f^{2m+2} = P (z, f, f', f'' ..., f^{(2m)}, f^{(2m+1)})
\]

or

\[
\sigma_{2m+2} f^{2m+2} = P (z, f, f', f'', ..., f^{(2m)}, f^{(2m+1)})
\]

where, \( P(z, f, f', f'', ..., f^{(2m)}, f^{(2m+1)}) \) differential polynomials in \( f \) and its derivatives with the coefficients are belonging to the field \( \{ h; T(r, h) = O(r) \} \) such that the total degree of \( P(z, f, f', f'', ..., f^{(2m)}, f^{(2m+1)}) \) is atmost \( 2m + 1 \), and hence by using Clunie lemma we have

\( m(r, f) = O(r) \)

(ii)

If \( f = c \in \mathbb{C} \), then

\[
0 = \sum_{i=\min(4m+2)}^{2m+2} \alpha_i c^i + \beta e^z c + \gamma e^{2z}.
\]

For \( c = 0 \in \mathbb{C} \), if \( \gamma \neq 0 \), then \( c = 0 \in \mathbb{C} \) does not solve (3.4.1), and hence Mohon’ko Lemma

\( m \left( r, \frac{1}{f} \right) = O(r) \).

For \( c \neq 0 \in \mathbb{C} \), if \( \gamma = 0 \) does not solve (3.4.1), again by Mohon’ko Lemma

\( m \left( r, \frac{1}{f-c} \right) = O(r) \).
(iii) Let

\[ U(z) = f^{(2m)} - \sum_{i=\min(4,m+2)}^{2m+2} \alpha_i f_i^{-1} \]

\[ Uf = ff^{(2m)} - \sum_{i=\min(4,m+2)}^{2m+2} \alpha_i f_i^{-1} \]

\[
= \sum_{i=1}^{m} (-1)^{i+1} f^{(2m-i)} f^{(i)} + \sum_{i=\min(4,m+2)}^{2m+2} \alpha_i f_i^{(i)} + \beta e^z f + \gamma e^{2z} - \sum_{i=\min(1,m+2)}^{2m+2} \alpha_i i^{-1} f_i^{(i)}
\]

\[ Uf = \sum_{i=1}^{m} f^{(2m-i)} f^{(i)} - \sum_{i=\min(4,m+2)}^{2m+2} \alpha_i i^{-1} f_i^{(i)} + \beta e^z f + \gamma e^{2z}. \]

Now, differentiating the above equation we get,

\[ U'f + Uf' = \sum_{i=1}^{m} (-1)^{i+1} \left( f^{(2m-i)} f^{(i+1)} + f^{(2m+1-i)} f^{(i)} \right) - \sum_{i=\min(4,m+2)}^{2m+2} \alpha_i i^{-1} f_i^{(i)} + \beta e^z (f + f') + 2\gamma e^{2z}
\]

\[ U'f = \sum_{i=1}^{m} (-1)^{i+1} \left( f^{(2m-i)} f^{(i+1)} + f^{(2m+1-i)} f^{(i)} \right) - \sum_{i=\min(4,m+2)}^{2m+2} \alpha_i i^{-1} f_i^{(i)} + \beta e^z (f + f') + 2\gamma e^{2z} - f f^{(2m)} + \sum_{i=\min(4,m+2)}^{2m+2} \alpha_i i^{-1} f_i^{(i)}
\]

\[ U'f = \sum_{i=1}^{m} (-1)^{i+1} \left( f^{(2m-i)} f^{(i+1)} + f^{(2m+1-i)} f^{(i)} \right) + \beta e^z (f + f') + 2\gamma e^{2z} - f f^{(2m)}
\]

\[ U' = (-1)^{m+1} \frac{f^{(m)}}{f} f^{(m+1)} + \beta e^z \frac{f + f'}{f} + 2\gamma e^{2z} \frac{f}{f}. \quad (3.4.2)
\]

We can also show that,

\[ U(z) = \frac{A}{z - z_0} + \text{analytic function}, \]

where \( A \) is some constant. Now, put

\[ g = \frac{U}{f} \]

\[ g = \sum_{i=1}^{m} (-1)^{i+1} \frac{f^{(2m-i)} f^{(i)}}{f^2} - \sum_{i=\min(4,m+2)}^{2m+2} \alpha_i i^{-2} f_i^{(i)} + \beta \frac{e^z}{f} + \gamma \frac{e^{2z}}{f^2}
\]

\[ \text{hence} \quad m(r, g) = O(r) \]
since \( \gamma \neq 0 \). using the part(i). (ii) of Theorem 3.4.1 with the lemma of Logarithmic Derivatives. Now, let

\[
f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n
\]

\[
f(z) = \sum_{n=0}^{\infty} a_n \beta^n \quad \text{where} \quad \beta = z - z_0
\]

\[
f'(z) = \sum_{n=0}^{\infty} a_n \delta_n \beta^{n-1}
\]

\[
f^{(2m)}(z) = \sum_{n=0}^{\infty} a_n \delta_n (\delta_n - 1) \cdots (\delta_n - 2m + 1) \beta^{n-2m}.
\]

Substituting these in (3.2.8) we get.

\[
ff^{(2m)}(z) = \left( \sum_{n=0}^{\infty} a_n \delta_n \beta^n \right) \left( \sum_{n=0}^{\infty} a_n \delta_n (\delta_n - 1) \cdots (\delta_n - 2m + 1) \beta^{n-2m} \right)
\]

\[
= (a_0 \beta^0 + a_1 \beta^1 + \cdots + a_n \beta^n + \cdots)(a_0 \delta_0 (\delta_0 - 1) \cdots (\delta_0 - 2m + 1) \beta^{2m-2m} + \cdots)
\]

\[
= a_0^2 \delta_0 (\delta_0 - 1) \cdots (\delta_0 - 2m + 1) \beta^{2m-2m} + \cdots
\]

Consider.

\[
a_{2m+2}f^{2m+2} = a_{2m+2} \left( \sum_{n=0}^{\infty} \delta_n \beta^n \right)^{2m+2}
\]

\[
= a_{2m+2} \left( \sum_{n=0}^{\infty} \delta_n \beta^{0} + \delta_1 \beta^{1} + \cdots + \delta_n \beta^n + \cdots \right)^{2m+2}
\]

\[
= a_{2m+2} \left[ \delta_0^{2m+2} \beta^{2m\delta_0 + 2\delta_0} + (\delta_1 \beta^{1} + \cdots + \delta_n \beta^n + \cdots)^{2m+2} \right].
\]

Equating the powers of \( \beta \) both from \( ff^{(2m)} \) and \( a_{2m+2}f^{2m+2} \) we get.

\[
2m\delta_0 + 2\delta_0 = 2\delta_0 - 2m
\]

\[
-2m = 2m\delta_0
\]

\[
hence \quad \delta_0 = -1
\]

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then.

\[
\alpha_{2m+2}\alpha_0^{2m+2} = \alpha_0^2(-1)(-2)...(-2m)
\]

or

\[
\alpha_{2m+2}\alpha_0^{2m} = (-1)^{2m}(-2m)!
\]

or

\[
\alpha_0 = \frac{(-1)^{2m}2m!}{\alpha_{2m+2}}
\]

or

\[
\alpha_0 = \left(\frac{-1)^{2m}2m!}{\alpha_{2m+2}}\right)^{1/2m}
\]

or

\[
\alpha_0 = \frac{2m!(1)^{1/2m}}{\alpha_{2m+2}}
\]

or

\[
= c_1 e^{\frac{k1}{m}} \text{ where } c_1 = \frac{2m!}{\alpha_{2m+2}}
\]

or

\[
= h \text{ for some constant } h.
\]

Therefore

\[
f(z) = \frac{\alpha_0}{(z-z_0)} + a_1(z-z_0)^{a_1} + a_2(z-z_0)^{a_2} + \ldots
\]

or

\[
f(z) = \frac{h}{(z-z_0)} + \text{analytic function}
\]

since,

\[
U(z) = \frac{A}{(z-z_0)} + \text{analytic function}
\]

therefore

\[
g = \frac{U}{f} + \text{analytic function}
\]

or

\[
g = ce^{\frac{k1}{m}} + \text{analytic function}
\]

therefore at a pole of \(f\), \(g\) is equal to \(ce^{\frac{k1}{m}}\) for some constant \(c\).

Hence every pole of \(f\) is a zero of \(g^{2m} - c\) and so

\[
N(r,f) = N\left(\frac{1}{g^{2m}-c}\right)
\]

\[
\leq 2mT(r,g) + O(1)
\]

\[
= 2mN(r,g) + O(r) \quad (as \quad m(r,g) = O(r)).
\]

As poles of \(g\) occurs only at zeros of \(f\), we get

\[
N(r,g) \leq N\left(\frac{1}{f}\right). \quad (3.4.3)
\]
Using the above two equations we have

\[ N(r, f) \leq 2mN \left( r, \frac{1}{f} \right) + O(r). \]

\[ m \left( r, \frac{1}{f} \right) = T(r, f) - N \left( r, \frac{1}{f} \right) + O(1) \]

\[ \leq T(r, f) - \frac{1}{2m} N(r, f) + O(r) \]

\[ \leq T(r, f) - \frac{1}{2m} (T(r, f) - \text{m}(r, f)) + O(r) \]

\[ \leq \frac{2m - 1}{2m} T(r, f) + O(r). \]

\[ \square \]

With the help of the above theorem we prove the following version of Second Fundamental Theorem for the generalised modified third Painlevé differential Equation.

**Theorem 3.4.2.** Let \( f \) be an arbitrary solution of (3.2.8). Let \( c_1, 0, c_2, c_3, \ldots, c_p \in \mathbb{C} \) be distinct complex numbers. Then

\[ \sum_{j=1}^{p} m \left( r, \frac{1}{f - c_j} \right) + N \left( r, \frac{1}{f} \right) + N(r, f) - \overline{N}(r, f) = 2T(r, f) + O(r). \]

**Proof.** We have the following two cases.

**Case (1):** If \( \alpha_{2m+2} \neq 0 \) all the poles of \( f \) are simple for \( m \geq 1 \). So \( N(r, f) = \overline{N}(r, f) \) and hence by Second Fundamental Theorem, we have

\[ \sum_{j=1}^{p} m \left( r, \frac{1}{f - c_j} \right) + N \left( r, \frac{1}{f} \right) \leq 2T(r, f) + O(r) \]  \hspace{1cm} (3.4.4)

or \( m \left( r, \frac{1}{f} \right) + N \left( r, \frac{1}{f} \right) \leq 2T(r, f) + O(r). \) \hspace{1cm} (3.4.5)

Since all the poles of \( f \) are simple.

\[ m \left( r, \frac{1}{f} \right) + N \left( r, \frac{1}{f} \right) \leq 2N(r, f) + O(r) = N(r, f') + O(r) = T(r, f') + O(r). \]
We now prove
\[ T(r, f') \leq m \left( r, \frac{1}{f} \right) + N \left( r, \frac{1}{f'} \right). \]

For this it is enough to prove that
\[ m \left( r, \frac{1}{f} \right) \leq m \left( r, \frac{1}{f} \right) + O(r) \]

or
\[ m \left( r, \frac{1}{f} \right) \geq m \left( r, \frac{1}{f} \right) + O(r). \]  \hfill (3.4.6)

Differentiating (3.2.8), we get
\[
ff^{(2m+1)} + f^{(2m)} f' = \sum_{i=1}^{m} (-1)^{i+1} \left( f^{(2m-i+1)} f^{(i)} + f^{(i+1)} f^{(2m-i)} \right)
+ \sum_{i=\min(4,2m+2)}^{2m+2} (i-1) f^{(i-1)} \alpha_i f' + \beta e^z (f + f') + 2\gamma e^{2z}
\]
\[
f^{(2m+1)} = \sum_{i=2}^{m} \left( f^{(2m-i+1)} f^{(i)} + \sum_{i=1}^{m} (-1)^{i+1} f^{(2m-i)} f^{(i+1)} \right)
+ \sum_{i=\min(4,2m+2)}^{2m+2} (i-1) f^{(i-1)} \alpha_i f' + \beta e^z (f + f') + 2\gamma e^{2z}
\]
\[
f^{(2m+1)} = (-1)^{m+1} f^{(m)} f^{(m+1)} + \sum_{i=\min(4,2m+2)}^{2m+2} (i-1) f^{(i-1)} \alpha_i f' + \beta e^z (f + f') + 2\gamma e^{2z}
\]

Now, there can be two subcases:

**Subcase(1.1):** If \( \gamma \neq 0 \), divide the above equation by \( ff' \), we get
\[
f^{(2m+1)} = (-1)^{m+1} \frac{f^{(m)} f^{(m+1)}}{f f'} + \sum_{i=\min(4,2m+2)}^{2m+2} (i-1) \frac{f^{(i-1)} \alpha_i}{f} + \beta e^z \frac{f + f'}{ff'} + 2\gamma e^{2z} \frac{f}{ff'}
\]
Using the (ii) of Theorem 3.4.1 and lemma of Logarithmic Derivatives, we get
\[ m \left( r, \frac{1}{f} \right) = O(r), \]
and (3.4.6) is trivially true.

**Subcase(1.2):** If \( \gamma = 0 \), then divide (3.2.8) by \( f' \), we get
\[
\frac{f^{(2m)}}{f'} = \sum_{i=1}^{m} (-1)^{i+1} \frac{f^{(2m-i)} f^{(i)}}{f f'} + \sum_{i=\min(4,2m+2)}^{2m+2} \alpha_i \frac{f^i}{f'} + \beta e^z \frac{f}{f'}.
\]

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simplifying and using lemma of Logarithmic Derivative for \( i \geq 1 \) we get.

\[
m\left( r, \frac{f}{f'} \right) = O(r) \quad \text{if} \quad \gamma = 0
\]

we have

\[
m\left( r, \frac{1}{f'} \right) \leq m\left( r, \frac{1}{f'} \right) + m\left( r, \frac{1}{f} \right)
\]

\[
m\left( r, \frac{1}{f'} \right) \leq m\left( r, \frac{1}{f} \right) + O(r).
\]

Thus (3.4.6) is true.

**Case(2):** If \( \alpha_4 = 0 \) and assuming that \( \alpha_3 \neq 0 \), then all the poles of \( f \) are double for \( m = 1 \).

So \( N(r, f) = \frac{1}{2} N(r, f) \) and the theorem can be proved as in Case(1).

\[\square\]

**Theorem 3.4.3.** If \( f \) is transcendental meromorphic solution of (3.2.8), then

(i) \( \Phi(f) = 2 \)

(ii) If \( \alpha_{2m+2} \neq 0 \) for \( m \geq 1 \), then \( \vartheta(\infty, f) = \frac{1}{2} \)

If \( \alpha_4 = 0, (\alpha_3 \neq 0 \text{ by assumption}) \) for \( m = 1 \), then \( \vartheta(\infty, f) = 0 \)

(iii) If \( \alpha_{2m+2} \neq 0 \) for \( m \geq 1 \), then \( \Phi_e(f) = 2 \)

If \( \alpha_4 = 0, (\alpha_3 \neq 0 \text{ for } m = 1 \text{ by assumption}) \) then, \( \Phi_e(f) = \frac{3}{2} \).

**Proof.** By the First Fundamental Theorem and using Theorem 3.4.1 we have

(i) \( N_B(r, f) = N\left( r, \frac{1}{f'} \right) + 2N(r, f) - N(r, f') \)

\[
= T\left( r, \frac{1}{f'} \right) + 2T(r, f) - T(r, f') - m\left( r, \frac{1}{f'} \right) - 2m(r, f) + m(r, f')
\]

\[= 2T(r, f) + O(r).\]
Thus.

\[ \Phi(f) = \lim_{r \to \infty} \frac{N_B(r, f)}{T(r, f)} = \lim_{r \to \infty} \frac{2T(r, f) + O(r)}{T(r, f)} = 2 \]

(ii) If \( a_{2m+2} \neq 0 \), then all the poles of \( f \) are simple. Hence we have

\[ N_1(r, f) = 2N(r, f) - N(r, f') = 0. \]

Therefore,

\[ \psi(\infty, f) = \lim_{r \to \infty} \frac{N_1(r, f)}{T(r, f)} = 0. \]

If \( \alpha_4 = 0 \) assuming that \( \alpha_3 \neq 0 \), then all the poles of \( f \) are double. Hence,

\[ N_1(r, f) = 2N(r, f) - N(r, f') \]

\[ = N(r, f) - N(r, f) \]

\[ = \frac{1}{2} N(r, f) \]

\[ = \frac{1}{2} T(r, f) + O(r) \]

and hence \( \psi(\infty, f) = \frac{1}{2} \)

(iii) If \( a_{2m+2} \neq 0 \), we have

\[ N(r, f') = 2N(r, f) = 2T(r, f) + O(r). \]

Hence, using Theorem 3.4.1, we get

\[ N \left( r, \frac{1}{f'} \right) = T \left( r, \frac{1}{f'} \right) - m \left( r, \frac{1}{f'} \right) \]

\[ = T(r, f') + O(r) \]

\[ = N(r, f') + O(r) \]

\[ = 2T(r, f) + O(r). \]
So, we get

\[ \Phi_{\varepsilon}(f) = \lim_{r \to \infty} \frac{N \left( r, \frac{1}{f} \right)}{T(r, f)} \]
\[ = \lim_{r \to \infty} \frac{2T(r, f) + O(r)}{T(r, f)} \]
\[ = 2. \]

If \( \alpha_1 = 0, (\alpha_3 \neq 0) \), by assumption we get

\[ N(r, f') = N(r, f) + N(r, f) \]
\[ = N(r, f) + \frac{1}{2} N(r, f) \]
\[ = \frac{3}{2} N(r, f) + O(r) \]
\[ = \frac{3}{2} T(r, f) + O(r). \]

Thus,

\[ \Phi_{\varepsilon}(f) = \frac{3}{2}. \]

The above theorem proves that the value distribution of the meromorphic function satisfying the generalised modified Painlevé Differential Equation is extremely regular as in the case of first, second and fourth generalised Painlevé Differential Equation.