Chapter 1

INTRODUCTION AND PRELIMINARIES
1.1 INTRODUCTION

1.2 A Brief Introduction to Nevanlinna Theory of Meromorphic Functions

In 1925, Nevanlinna R. [39]. made a decisive contribution to the value distribution theory, by introducing the characteristic function $T(r, f)$ as an efficient tool by proving first and second fundamental theorems. These significant contributions have provided considerable impetus over the years to the study of entire and meromorphic functions and continue to serve as a language to future work. His theory (so called Nevanlinna Theory) was widely used in the study of Uniqueness of functions, Normality criteria, Borel directions, Fix points of meromorphic functions and even in solving Complex Differential Equations. Nowadays Nevanlinna Theory is finding many applications in Difference equations also.

Let $f(z)$ be a meromorphic function (i.e regular except for poles) and not constant in the complex plane. For any complex $a$, including $\infty$, we denote by $n(r, a) = n(r, a, f)$, the number of roots with due count of multiplicity, of the equation $f(z) = a$ in $|z| < r$ and by $n(r, \infty)$ the number of poles of $f(z)$ in $|z| < r$. We write

$$N(r, a) = \int_0^r \frac{n(t, a) - n(0, a)}{t} + n(0, a) \log r. \quad (1.2.1)$$

Next we define.

$$\log^+ x = \begin{cases} \log x & \text{if } x \geq 1 \\ 0 & \text{if } 0 \leq x < 1. \end{cases}$$

The following property is obvious

$$\log x = \log^+ x - \log^+ 1.$$
The quantity $m(r, f)$ is defined as follows:

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta.$$  \hspace{1cm} (1.2.2)

The term $m(r, f)$ is called **Proximity function** of $f$ and is a sort of average magnitude of $\log |f(z)|$ on arcs of $|z| = r$ where $|f(z)|$ is large.

The basic result here is the existence of **Characteristic function** $T(r, f)$ such that, for every $a$,

$$T(r, f) = N(r, a) + m(r, a) + O(1),$$

as $r \to \infty$, where $m(r, a) \geq 0$. This is the **first fundamental theorem of Nevanlinna** see [21]. It plays an important role in the theory of meromorphic functions. It provides an upper bound to the number of roots of the equation $f(z) = a$, valid for all $r$ and $a$.

For an entire function $f$ the study of the comparative growth properties of $T(r, f)$ and $\log M(r, f)$ is a popular problem among the researchers.

In case of transcendental entire function $f$, $M(r, f)$ grows faster than any positive power of $r$. Thus in order to estimate the growth of transcendental entire functions we choose a composition function $e^{kr}$, $k > 0$ that grows more rapidly than any positive power of $r$.

More precisely $f$ is said to be a function of **finite order**, if there exists a positive constant $k$ such that $\log M(r) < r^k$ for all sufficiently large values of $r(r > r_0(k)$ say). The infimum of such $k$’s is called the **order of $f$**.

If no such $k$ exists, then $f$ is said to be of **infinite order**. For example $e^z$ has finite order, but $e^{e^z}$ has infinite order.

Let $\rho$ be the order of $f$. We now define **order of a meromorphic function** $f(z)$ as

$$\rho = \lim_{r \to \infty} \frac{\log \log M(r)}{\log r},$$

(1.2.3)
The lower order $A$ of $f$ is defined as follows

$$
A = \lim_{r \to \infty} \frac{\log \log M(r)}{\log r}.
$$

(1.2.4)

Clearly, $A \leq \rho$. In particular for a function $f$, $A = \rho$, then $f$ is said to be of **regular growth**.

Jensen's formula thus asserts that $T(r, f) = T(r, \frac{1}{f}) + \log |f(0)|$, when $\log |f(0)| \neq 0$ or $\infty$. The inequalities

$$
\log^+ \left| \prod_{\nu=1}^{p} a_{\nu} \right| \leq \sum_{\nu=1}^{p} \log^+ |a_{\nu}|.
$$

$$
\log^+ \left| \sum_{\nu=1}^{p} a_{\nu} \right| \leq \sum_{\nu=1}^{p} \log^+ |a_{\nu}| + \log p.
$$

which hold for complex numbers $a_1, a_2, \ldots, a_p$, when applied to $f_{\nu}(z)$, $\nu = 1, 2, \ldots, p$ lead to

$$
m \left( r, \sum_{\nu=1}^{p} f_{\nu}(z) \right) \leq \sum_{\nu=1}^{p} m(r, f_{\nu}(z)) + \log p
$$

$$
m \left( r, \prod_{\nu=1}^{p} f_{\nu}(z) \right) \leq \sum_{\nu=1}^{p} m(r, f_{\nu}(z)).
$$

We deduce that

$$
T(r, \sum_{\nu=1}^{p} f_{\nu}(z)) \leq \sum_{\nu=1}^{p} T(r, f_{\nu}(z)) + \log p
$$

$$
T(r, \prod_{\nu=1}^{p} f_{\nu}(z)) \leq \sum_{\nu=1}^{p} T(r, f_{\nu}(z)).
$$

Thus

$$
T(r, f + a) \leq T(r, f) + T(r, a) + \log 2 \tag{1.2.5}
$$

$$
= T(r, f) + \log^+ |a| + \log 2. \tag{1.2.6}
$$

writing $f$ for $f + a$ and $f - a$ we get

$$
T(r, f) \leq T(r, f - a) + \log^+ |a| + \log 2 \tag{1.2.7}
$$
And replacing $a$ by $-a$ (1.2.6) leads to

$$T(r, f - a) \leq T(r, f) + \log^+ |a| + \log 2. \quad (1.2.8)$$

(1.2.7) and (1.2.8) together imply

$$|T(r, f) - T(r, f - a)| \leq \log^+ |a| + \log 2. \quad (1.2.9)$$

Jensen’s formula together with (1.2.9) leads to the first fundamental theorem of Nevanlinna. We also note that since $m(r, a) \geq 0$, the theorem gives an upper bound for $N(r, a)$.

1.3 The Second Fundamental Theorem

The first fundamental theorem gave us an upper bound for $N(r, a)$ and hence the number of zeros of $f(z) - a$. The more complicated question of lower bounds is answered by the second fundamental theorem. It shows that the term $N(r, a)$ is, in general, dominant in the sum $m + N$ and moreover in $N(r, a)$ the sum is not much decreased if multiple roots are counted simply.

**Theorem 1.3.1.** Suppose that $f$ is a non constant meromorphic function in $|z| \leq r$. Let $a_1, a_2, ..., a_q (q \geq 2)$ be distinct finite complex numbers, $\delta > 0$ and suppose that $|a_\mu - a_\nu| \geq \delta$ for $1 \leq \mu < \nu \leq q$. Then

$$m(r, \infty) + \sum_{\nu=1}^{q} m(r, a_\nu) \leq 2T(r, f) - N_1(r) + S(r, f) \quad (1.3.1)$$

where $N_1(r)$ is positive and is given by

$$N_1(r) = N \left( r, \frac{1}{f'} \right) + 2N(r, f) - N(r, f') \quad (1.3.2)$$
and

\[ S(r, f) = m \left( r, \frac{f}{f'} \right) + m \left( r, \sum_{\nu=1}^{q} \frac{f'}{f - a_{\nu}} \right) + q \log^+ \left( \frac{3q}{8} \right) + \log 2 + \log \frac{1}{|f'(0)|} \]

with modifications if \( f(0) = 0 \), or \( f(0) = \infty \), and \( f'(0) = 0 \). The quantity \( S(r, f) \) will in general play the role of an unimportant error term. \( N_1(r) \) measures the multiple roots of \( f \) measuring each \( k \) fold point exactly \((k - 1)\) times.

The proof of second fundamental theorem is not difficult but estimation \( S(r) \) is however complicated.

The famous Picard's theorem states that a trancendental meromorphic function \( f(z) \) must assume all the values in the complex plane with atmost two exceptions. The second fundamental theorem permits a significant extension of Picard's theorem. It shows that in general the term \( m(r, a) \) is small compared with \( T(r, f) \) and so \( N(r, a) \) comes near to \( T(r, f) \), the maximum possible growth allowed by the first fundamental theorem. More precisely, we define the deficiency,

\[ \delta(a, f) = \limsup_{r \to \infty} \frac{m(r, a)}{T(r, f)} = 1 - \limsup_{r \to \infty} \frac{N(r, a)}{T(r, f)}. \]

We may regard \( \delta(a, f) \) loosely as the proposition by which the number of roots of the equation \( f(z) = a \) less than maximum permitted number. With this definition we have as an easy consequence of deficiency relation. This states that set of deficient values i.e. for which \( \delta(a) > 0 \), is countable and that

\[ \sum \delta(a) \leq 2. \quad (1.3.3) \]

where the sum is taken over all deficient values. This result still holds if multiple roots of \( f(z) = a \) are counted only once in the definitions of \( n(r, a) \), \( N(r, a) \) and \( \delta(a, f) \). Now we may set

\[ \Theta(a) = \Theta(a, f) = 1 - \lim_{r \to \infty} \frac{N(r, a)}{T(r, f)}. \]
where $N(r, a; f) = N(r, a)$ is the counting function for distinct $a$-points

$$\theta(a) = \theta(a, f) = \lim_{r \to \infty} \inf \frac{N(r, a) - \overline{N}(r, a)}{T(r, f)}.$$ 

Evidently, given $\epsilon > 0$, we have for sufficiently large values of $r$, so that

$$\Theta(a) \geq \delta(a) + \theta(a).$$

The quantity $\delta(a)$ is known as the **deficiency of the values** $a$ and $\theta(a)$ is called the **index of multiplicity**. Evidently $\delta(a)$ is positive only if there are relatively few roots of equation $f(z) = a$, while $\theta(a)$ is positive if there are relatively many multiple roots. The following fundamental theorem is the **Nevanlinna’s theorem on deficient values** which also plays an important role.

**Theorem 1.3.2.** [21 pp.43]: Let $f$ be a non-constant meromorphic function defined on the plane. Then the set of values $a$ for which $\Theta(a) > 0$ is countable and we have, on summing over all such values $a$

$$\sum_{a} \{\delta(a) + \theta(a)\} \leq \sum_{a} \Theta(a) \leq 2.$$ 

The main objective of this thesis is to study some applications of Nevanlinna theory to various results of the above mentioned type. This thesis is organized into five chapters.

**Chapter 1** describes a brief introduction to Nevanlinna Theory of Meromorphic Functions and few of the preliminaries to be required in the whole thesis.

**Chapter 2** is devoted to a brief collection of results on uniqueness of functions, sharing values which have significant importance in deciding the uniqueness of functions. Nevanlinna who proved two meromorphic functions are identically equal when they share five values ignoring multiplicities. A considerable amount of research work was done in this
direction and most of these results are based on the criteria that two functions and their
derivatives share some points either counting multiplicities or ignoring multiplicities.
But, recently, Fang and Hong [13] showed that two entire functions \( f \) and \( g \) are iden-
tically equal, when their differential polynomials \( f''f'(f - 1) \) and \( g''g'(g - 1) \) share 1
counting multiplicities (CM).

In Section 1, we prove similar results for meromorphic functions when the differential
polynomials \( f'(1 - af^n) \) and \( g'(1 - ag^n) \) share a polynomial \( P(z) \).

In Section 2, by introducing the notion of order of multiplicity, we shall study mero-
morphic functions that share 1 CM. The result improves the theorem due to Puranik,
Kulkarni [40],[32]. As a particular case, we get many interesting results.

In many cases, it is impossible to find an explicit solution for a given differential equation.
Solving techniques usually work in some limited special cases and numerical methods
yield only approximations.

Nevanlinna theory offers an efficient way to bypass this problems, since it allows the
study of the properties of a solution without knowing its explicit form. The requirement
needed, in order to apply this method it is necessary that the solution must be mer-
morphic, either in the whole complex plane, or in a smaller domain where the growth of
the solution is sufficiently fast, near the boundary of the domain.

In Chapter 3, we have found an explicit form for the generalised modified III Painlevé
Differential Equation. Applying Nevanlinna Theory we study the value distribution
properties of transcendental meromorphic solutions of such equations.

W. Wittich, whose research in the area of differential equation began in 1942, was the
first to make systematic studies in the applications of Nevanlinna Theory to complex
differential equations. Later due to the efforts of several mathematicians including Bank
and Laine[9] Nevanlinna theory became the leading tool in analyzing solutions of com-
Like differential equations the theory of $q$ difference equation has a wide range of applications in both Mathematics and Physics. This, in many occasions, leads to results in the theory of difference equations, which are natural discrete analogues of corresponding results of differential equations.

In Chapter 4, we have proved some such consequences of $q$ difference analogue of lemma of logarithmic derivative for $q$ difference equations.

In Chapter 5, Section 1, we investigate the growth of Nevanlinna characteristic of $f(qz + c)$ for fixed $q, c \in \mathbb{C}$. In particular, we obtain a precise asymptotic relation between $T(r, f(qz + c))$ and $T(r, f)$ which is only true for finite order meromorphic function. We apply these results to give new growth estimates of meromorphic solutions to higher order linear $'q_c$ difference equations'. This also allows us to solve the questions posed by Ablowitz, Halburd and Hebrst concerning integrable $q_c$ difference equations. We have also obtained the pointwise logarithmic derivative estimates of finite order meromorphic functions.

In Section 2 we apply Nevanlinna theory to obtain some information about solutions of algebraic difference-differential equations. Among other things, we obtain a theorem which can be described as an analogue of the lemma of logarithmic derivative for $'q_c$ difference equations'. This theorem is further applied to prove a Lemma of Clunie type, Lemma of Mohon'ko for these equations.
1.4 PRELIMINARIES

Throughout the thesis we denote by $\mathbb{C}$, the set of complex numbers. Our meromorphic functions are non-constant meromorphic functions defined in the whole complex plane $\mathbb{C}$. Following are some of the basic definitions and lemmas that are needed further in our thesis.

**Definition 1.4.1.** For any non-constant meromorphic function $f(z)$ we denote by $S(r,f)$ any function satisfying

$$S(r,f) = o(T(r,f))$$

as $r \to \infty$, except possibly outside a set of $r$ of finite linear measure.

**Definition 1.4.2.** A meromorphic function $a$ is said to be a small function of $f$ provided that $T(r,a) = S(r,f)$, that is $T(r,a) = o(T(r,f))$ as $r \to \infty$ outside of a possible exceptional set of finite linear measure. For any non-constant meromorphic function $f$, we denote by $S(f)$, the family of all meromorphic functions $g$ (including constants and the infinite constant $\infty$) satisfying $T(r,g) = o(T(r,f))$.

**Definition 1.4.3.** Let $f$ be a non-constant meromorphic function defined in the open complex plane $\mathbb{C} \cup \infty$. Let $k$ be a positive integer or $\infty$ and $a \in \overline{\mathbb{C}}$. We denote by

(i) $N(r,a,f)$, the counting function of distinct zeros of $f - a$.

(ii) $N_k(r,a,f)$, the counting function of distinct zeros of $f - a$ with order of multiplicity less than or equal to $k$.

(iii) $N_k(r,a,f)$ the counting function of zeros of $f - a$ with order of multiplicity less than or equal to $k$, each zero being counted according to its multiplicity.

(iv) $N_k(r,a,f)$ the counting function of zeros of $f - a$ wherein a zero of $f - a$ with order of multiplicity less than or equal to $k$ is counted according to its multiplicity.
and a zero of \( f - a \) with order of multiplicity > \( k \) is counted exactly \( k \) times.

**Definition 1.4.4.** Let \( f \) be a non-constant meromorphic function in the complex plane and \( a \in \overline{\mathbb{C}} \). The **deficiency of** \( a \) **with respect to** \( f \) is defined by

\[
\delta(a, f) = \lim_{r \to \infty} \frac{m\left(r, \frac{1}{f-a}\right)}{T(r, f)} = 1 - \lim_{r \to \infty} \frac{N(r, a, f)}{T(r, f)}
\]

Further, the **ramification index** of \( a \in \overline{\mathbb{C}} \) with respect to \( f \) is defined as

\[
\Theta(a, f) = 1 - \lim_{r \to \infty} \frac{N(r, a, f)}{T(r, f)}
\]

It is obvious that \( 0 \leq \delta(a, f) \leq 1 \) and \( 0 \leq \Theta(a, f) \leq 1 \).

### 1.5 Exceptional Values

**Definition 1.5.1.** Let \( f(z) \) be a meromorphic function and \( 'a' \) be any complex number. \( 'a' \) is called a **Picard's exceptional value or exceptional value in the sense of Picard or evP of** \( f \), if \( n(r, a, f) = O(1) \), i.e., if \( f(z) - a \) has finitely many zeros.

We now state the famous Picard theorem which is a simple application of the second fundamental theorem.

**Theorem 1.5.1 (Picard theorem).** Any transcendental meromorphic function in the complex plane has at most two Picard exceptional values.

**Definition 1.5.2.** Let \( f \) be a transcendental meromorphic function in \( \mathbb{C} \) with the order \( \lambda \). A complex number \( a \) is said to be a **Borel exceptional value or exceptional value in the sense of Borel or evB of** \( f \) if

\[
\lim_{r \to \infty} \log^+ \frac{N\left(r, \frac{1}{f-a}\right)}{\log r} < \lambda
\]

Clearly, every Picard exceptional value is a Borel exceptional value.