Chapter 5

NEVANLINNA CHARACTERISTIC OF $f(qz + c)$ AND APPLICATIONS TO $q_c$

DIFFERENCE EQUATIONS
5.1 ON NEVANLINNA CHARACTERISTIC OF $f(qz + c)$ AND ITS APPLICATIONS TO $q_c$ DIFFERENCE EQUATIONS

5.2 Introduction, Results and Definitions

It is well-known that the following logarithmic derivative estimate

$$m \left( r, \frac{f'}{f} \right) = O(\log T(r, f) + \log r) = S(r, f)$$

holds outside of a possible small exceptional set of finite linear measure, where the notation $S(r, f)$ means that the expression is of $o(T(r, f))$ where $m(r, f)$ denotes the Nevanlinna Proximity function and $T(r, f)$ is the characteristic of a meromorphic function $f$ (see [34]). It shows that the proximity function of the Logarithmic derivative of $f(z)$ grows much slower than the Nevanlinna Characteristic function of $f(z)$. The Logarithmic derivative lemma, as it is often called, has numerous applications in Complex Differential Equations [34] and it also plays a crucial role in proving the Nevanlinna Second Fundamental theorem (see [21], [34]). It has many useful results in Nevanlinna Theory and vast number of applications in the theory of ordinary differential equations. For instance, Yosida’s generalisation of Malmquist theorem, depends heavily on the lemma of Logarithmic derivative. It is generally recognised that the Logarithmic derivative estimate (5.2.1) is amongst the deepest results in the value distribution theory. We can also even find other applications of it in (see [27], [35]).

In contrast to differential equations, non-linear differential equations often admit global meromorphic solutions (see [41]) and hence Nevanlinna’s Value distribution Theory is ap-
pliable. The foundations of the theory of complex difference equations was laid by Nolund. Julia, Brikhoff, Batchelder and others in the early part of the twentieth century. Later on, Shimomura [41], Yanagihara and Laine [34] studied non-linear complex difference equations from the viewpoint of Nevanlinna theory. Recently, there has been a renewed interested in the complex analytic properties of solutions of difference equations. Ablowitz, Halburd and Herbst [1] suggested that the growth of meromorphic solutions of difference equations could be used to identify those equations which are "Painlevé type" (see [25]).

In case of applying Nevanlinna Theory to difference equations, one of the most basic questions is the growth comparison between $T(r, f(z + 1))$ and $T(r, f(z))$. (see [15]p.66) for a general meromorphic function $f(z)$. For $\eta$ be a non-zero complex number, Chiang and Feng [11] obtained an asymptotic relation between $T(r, f(z + \eta))$ and $T(r, f(z))$ as,

$T(r, f(z)) \sim T(r, f(z + \eta))$, holds for finite order meromorphic functions.

In this chapter, we shall however, concentrate on the value distribution properties of $f(qz + c)$ for $q$ and $c$ to be any non-zero complex numbers, and their related expressions with their applications to $q$ linear difference equations. It turns out that the results we obtained in this paper also allow us to give a direct proof of Nevanalninna-type theorems as in [1]. We try to obtain the following precise relation

$$T(r, f(qz)) \sim T(r, f(qz + c)).$$

holds for finite order meromorphic functions which took the need of both the

$$N(r, f(qz + c)) \sim N(r, f(qz))$$

for finite order meromorphic functions. as well as a version of 'q' difference analogue of classical logarithmic derivative estimate to be discussed below.

For $c$ to be a fixed non-zero complex number and for $q, c$ be any non-zero complex number.
we prove the difference analogue of (5.2.1) as
\[ m\left( r, \frac{f(qz + c)}{f(qz)} \right) = S_* (r, f) \] (5.2.2)
where, \( S_* (r, f) \) means that the left hand side of (5.2.2) is of slower growth than \( T(r, f) \) in some sense. We shall also show that (5.2.2) holds for a finite order meromorphic function of \( f(z) \). More precisely, we show that if \( f(z) \) is a meromorphic function of finite order \( \sigma \), then we have.
\[ m\left( r, \frac{f(qz + c)}{f(qz)} \right) + m\left( r, \frac{f(qz)}{f(qz + c)} \right) = O(r^{\sigma-1+\epsilon}) \] (5.2.3)
for an arbitrary \( \epsilon > 0 \), holds without any exceptional set. In relation to the above result (5.2.2) we learnt that Halburd and Korhonen [25] have also obtained a same estimate, for \( f(z) \) to be any meromorphic function as,
\[ m\left( r, \frac{f(z + c)}{f(z)} \right) = o\left( \frac{T(r + |c|, f)^{1+\epsilon}}{r^\delta} \right) \] (5.2.4)
for all \( r \) outside of a possible exceptional set \( E \) with finite logarithmic measure, for \( c \in \mathbb{C}, \delta < 1, \) and \( \epsilon > 0 \). On removing \( \epsilon \) from the above estimate they also obtained the following result in case of finite order meromorphic function as
\[ m\left( r, \frac{f(z + c)}{f(z)} \right) = o\left( \frac{T(r + |c|, f)}{r^\delta} \right) \] (5.2.5)
Corresponding to the above estimate (5.2.4), in this paper we obtain a difference analogue for any meromorphic function \( f(z) \) to be.
\[ m\left( r, \frac{f(qz + c)}{f(qz)} \right) = o\left( \frac{T(|q|r + |c|, f)^{1+\epsilon}}{r^\delta} \right) \] (5.2.6)
for \( q, c \in \mathbb{C} - \{0\} \), all \( r \) outside of possible exceptional set \( E \) with finite logarithmic measure \( \int_E \frac{dr}{r} < \infty \).

Although our estimate regarding (5.2.6) is somewhat weaker than (5.2.1), we show that
it is sufficient for our applications and more importantly this difference analogue appears
to be in its most useful form when applied to study finite order meromorphic solutions
of difference equations, which is in agreement with the findings in applications include.
for instance, a difference analogue of the Clunie Lemma [10]. At last we have also
studied Gundersen's point-wise estimate for the logarithmic derivative for a meromorphic
function $f(z)$ of order $\sigma$. We need the following lemmas to prove our main results.

5.3 Lemmas

Lemma 5.3.1. ([11]p-7): Let $\alpha$ be a given constant with $0 < \alpha \leq 1$. Then there exists a
constant $C_\alpha > 0$ depending only on $\alpha$ such that

$$\log(1 + x) \leq C_\alpha x^\alpha. \quad (5.3.1)$$

holds for $x \geq 0$. In particular, $C_1 = 1$.

Lemma 5.3.2. ([11]p-7): Let $\alpha$, $0 < \alpha < 1$ be given as in lemma 5.3.1. Then for any
two complex numbers $z_1$ and $z_2$, we have the inequality

$$\left| \log \left( \frac{z_1}{z_2} \right) \right| \leq C_\alpha \left( \left| \frac{z_1 - z_2}{z_2} \right|^{\alpha} + \left| \frac{z_2 - z_1}{z_1} \right|^{\alpha} \right). \quad (5.3.2)$$

Lemma 5.3.3. Let $\alpha$, $0 \leq \delta \leq 1$ be given, then for every given complex number $c$,and
$q \in \mathbb{C}$ we have,

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{|qre^{i\theta} - |c_n||^\delta} d\theta \leq \frac{1}{(1 - \delta)r^\delta |q|^\delta}$$

Proof. We have for $0 \leq \theta \leq \frac{\pi}{2}, |qre^{i\theta} - |c_n|| > \frac{2}{\pi}r\theta$ so

$$|qre^{i\theta} - |c_n|| \geq \frac{2}{\pi}r\theta|q|.$$
Therefore

\[
\int_0^{2\pi} \frac{d\theta}{|qre^{i\theta} - |c_n||^\delta} \leq 4 \int_0^{\frac{\pi}{2}} \frac{d\theta}{|q|\delta r\delta} \\
= 4 \left( \frac{\pi}{2} \right) \delta \frac{1}{|q|\delta r\delta} \int_0^{\frac{\pi}{2}} \frac{d\theta}{\delta} \\
= 4 \left( \frac{\pi}{2} \right) \delta \frac{1}{|q|^\delta r^\delta (1 - \delta)|q|^\delta} \\
= \frac{2\pi}{r^\delta(1 - \delta)|q|^\delta}.
\]

Therefore

\[
\frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{|qre^{i\theta} - |c_n||^\delta} \leq \frac{1}{r^\delta(1 - \delta)|q|^\delta}.
\]

\[\square\]

Lemma 5.3.4. Let f be a meromorphic function such that f(0) \neq 0, \infty and let c \in \mathbb{C}. Then for all \alpha > 1, \delta < 1, q \in \mathbb{C} - \{0\} and r \geq 1

\[
m \left( r, \frac{f(qz + c)}{f(qz)} \right) + m \left( r, \frac{f(qz)}{f(qz + c)} \right) \leq \frac{K(\alpha, \delta, c, q)}{r^\delta} \left( T(\alpha(|q|r + |c|, f)) + \log^+ \frac{1}{|f(0)|} \right)
\]

where

\[
K(\alpha, \delta, c, q) = \frac{8(3\alpha + 1)|c||q|^\delta + 8\alpha(\alpha - 1)|c|^\delta}{(\alpha - 1)^2\delta(\delta - 1)|q|^\delta}
\]

holds without any exceptional set.

Proof. Let \{a_n\} denote sequence of all zeros of f and \{b_m\} be a sequence of all poles of f where \{a_n\} and \{b_m\} are listed according to their multiplicites and ordered by increasing modulus. By Poisson-Jensen formula, we obtain that

\[
\log |f(qz)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(\rho e^{i\theta})| \Re \left( \frac{\rho e^{i\theta} + qz}{\rho e^{i\theta} - qz} \right) d\theta \\
+ \sum_{|a_n| < \rho} \log \left| \frac{\rho(qz - a_n)}{\rho^2 - a_n qz} \right| - \sum_{|b_m| < \rho} \log \left| \frac{\rho(qz - b_m)}{\rho^2 - b_m qz} \right|.
\]
\[
\log |f(qz + c)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(\rho e^{i\theta})| \text{Re} \left( \frac{\rho e^{i\theta} + qz + c}{\rho e^{i\theta} - (qz + c)} \right) d\theta + \sum_{|a_n| < \rho} \log \left| \frac{\rho(qz + c - a_n)}{\rho^2 - a_n(qz + c)} \right| - \sum_{|b_m| < \rho} \log \left| \frac{\rho(qz + c - b_m)}{\rho^2 - b_m(qz + c)} \right|.
\]

Subtracting (5.3.4) from (5.3.3) yields,

\[
\log \left| \frac{f(qz + c)}{f(qz)} \right| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(\rho e^{i\theta})| \text{Re} \left( \frac{\rho e^{i\theta} + qz + c}{\rho e^{i\theta} - (qz + c)} - \frac{\rho e^{i\theta} + qz}{\rho e^{i\theta} - qz} \right) d\theta + \sum_{|a_n| < \rho} \log \left| \frac{\rho(qz + c - a_n)}{\rho^2 - a_n(qz + c)} \rho - \frac{\rho^2 - a_n qz}{\rho^2 - a_n(qz + c)} \right| - \sum_{|b_m| < \rho} \log \left| \frac{\rho(qz + c - b_m)}{\rho^2 - b_m(qz + c)} \rho - \frac{\rho^2 - b_m qz}{\rho^2 - b_m(qz + c)} \right|.
\]

Therefore

\[
\left| \log \left| \frac{f(qz + c)}{f(qz)} \right| \right| = |S_1(z) + S_2(z) - S_3(z)|.
\]

We have

\[
\left| \log \left| \frac{f(qre^{i\phi} + c)}{f(qre^{i\phi})} \right| \right| = \log^+ \frac{f(qre^{i\phi} + c)}{f(qre^{i\phi})} + \log^+ \frac{1}{\left| \frac{f(qre^{i\phi} + c)}{f(qre^{i\phi})} \right|} = \log^+ \frac{f(qre^{i\phi} + c)}{f(qre^{i\phi})} + \log^+ \frac{f(qre^{i\phi})}{f(qre^{i\phi} + c)}.
\]

Multiplying (5.3.5) by \(\frac{1}{2\pi}\) and integrating on \(|z| = r\) to get the Proximity function,

\[
m(r, \frac{f(qz + c)}{f(qz)}) + m(r, \frac{f(qz)}{f(qz + c)}) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{f(qre^{i\phi} + c)}{f(qre^{i\phi})} \right| + \left| \frac{f(qre^{i\phi})}{f(qre^{i\phi} + c)} \right| d\phi \]
\[
= \frac{1}{2\pi} \int_0^{2\pi} |S_1(re^{i\phi}) + S_2(re^{i\phi}) - S_3(re^{i\phi})| d\phi.
\]
\[
m \left( r, \frac{f(qz + c)}{f(qz)} \right) + m \left( r, \frac{f(qz)}{f(qz + c)} \right) \leq \frac{1}{2\pi} \int_0^{2\pi} \left| S_1(re^{i\phi}) \right| d\phi + \int_0^{2\pi} \left| S_2(re^{i\phi}) \right| d\phi + \int_0^{2\pi} \left| S_3(re^{i\phi}) \right| d\phi.
\]

(5.3.6)

Now, we consider,

\[
|S_1(re^{i\phi})| = \left| \frac{1}{2\pi} \int_0^{2\pi} \log |f(\rho e^{i\theta})| \operatorname{Re} \left( \frac{2\rho e^{i\theta} c}{(\rho e^{i\theta} - qre^{i\theta} - c)(\rho e^{i\theta} - qre^{i\theta})} \right) d\theta \right|
\]

\[
\leq \frac{1}{2\pi} \int_0^{2\pi} \log |f(\rho e^{i\theta})| \left| \frac{2\rho e^{i\theta} c}{(\rho e^{i\theta} - qre^{i\theta} - c)(\rho e^{i\theta} - qre^{i\theta})} \right| d\theta \quad \text{as } \operatorname{Re} z \leq |z|
\]

\[
\leq \frac{2\rho|c|}{(\rho - |qr - c|)(\rho - |qr|)} \int_0^{2\pi} \left| \log |f(\rho e^{i\theta})| \right| d\theta
\]

\[
\leq \frac{2\rho|c|}{(\rho - |qr - c|)(\rho - |qr|)} \left\{ \int_0^{2\pi} \log^+ |f(\rho e^{i\theta})| d\theta + \int_0^{2\pi} \log^+ \frac{1}{|f(\rho e^{i\theta})|} d\theta \right\}
\]

\[
as \quad |\log x| \leq \log^+ x + \log^+ \frac{1}{x}
\]

\[
\begin{align*}
2\rho|c| & \leq \frac{m(\rho, f) + m\left(\rho, \frac{1}{f}\right)}{2\rho|c|} \\
& \leq \frac{2\rho|c|}{(\rho - |qr - c|)(\rho - |qr|)} \left( T(\rho, f) + T\left(\rho, \frac{1}{f}\right) \right) \\
& \leq \frac{4\rho|c|}{(\rho - |qr - c|)(\rho - |qr|)} \left( T(\rho, f) + \log^+ \frac{1}{|f(0)|} \right) \\
& \leq \frac{4\rho|c|}{(\rho - |qr - c|)^2} \left( T(\rho, f) + \log^+ \frac{1}{|f(0)|} \right)
\end{align*}
\]

Therefore,

\[
|S_1(re^{i\phi})| \leq \frac{4\rho|c|}{(\rho - |qr - c|)^2} \left( T(\rho, f) + \log^+ \frac{1}{|f(0)|} \right).
\]

(5.3.7)

Next we consider \( S_3 \) and \( S_2 \) together and by denoting \( c_n = \{a_n\} \cup \{b_m\} \) we have,

\[
|S_2(re^{i\phi})| + |S_3(re^{i\phi})| = \left| \sum_{|a_n| < \rho} \log \left| \frac{(qz + c - a_n)(\rho^2 - \bar{a}_n qz)}{\left(\rho^2 - \bar{a}_n(qz + c)\right)(qz - a_n)} \right| \right| + \left| \sum_{|b_m| < \rho} \log \left| \frac{(qz + c - b_m)(\rho^2 - \bar{b}_m qz)}{\left(\rho^2 - \bar{b}_m(qz + c)\right)(qz - b_m)} \right| \right|
\]

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\[ \frac{1}{2\pi} \int_0^{2\pi} |S_2(re^{i\theta})| + \frac{1}{2\pi} \int_0^{2\pi} |S_3(re^{i\theta})| \leq \frac{1}{2\pi} \int_0^{2\pi} \left( \sum_{|a_n|<\rho} \log \left| \frac{(qre^{i\theta} + c - a_n)}{(\rho^2 - a_n (qre^{i\theta} + c))} \right| \right) \, d\theta \]
\[ + \frac{1}{2\pi} \int_0^{2\pi} \left( \sum_{|b_m|<\rho} \log \left| \frac{(qre^{i\theta} + c - b_m)}{(\rho^2 - b_m (qre^{i\theta} + c))} \right| \right) \, d\theta \]

\[ = \left\{ \sum_{|a_n|<\rho} \frac{1}{2\pi} \int_0^{2\pi} \log \left( 1 + \frac{c}{qre^{i\theta} - a_n} \right) \left( 1 + \frac{\bar{a}_n c}{\rho^2 - a_n (qre^{i\theta} + c)} \right) \, d\theta \right\} \]
\[ + \left\{ \sum_{|b_m|<\rho} \frac{1}{2\pi} \int_0^{2\pi} \log \left( 1 + \frac{c}{qre^{i\theta} - b_m} \right) \left( 1 + \frac{\bar{b}_m c}{\rho^2 - b_m (qre^{i\theta} + c)} \right) \, d\theta \right\} \]

\[ \leq \left\{ \sum_{|a_n|<\rho} \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| 1 + \frac{c}{qre^{i\theta} - a_n} \right| \, d\theta + \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| 1 + \frac{\bar{a}_n c}{\rho^2 - a_n (qre^{i\theta} + c)} \right| \, d\theta \right\} \]
\[ + \left\{ \sum_{|b_m|<\rho} \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| 1 + \frac{c}{qre^{i\theta} - b_m} \right| \, d\theta + \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| 1 + \frac{\bar{b}_m c}{\rho^2 - b_m (qre^{i\theta} + c)} \right| \, d\theta \right\} \]

Therefore

\[ |S_2(re^{i\theta})| + |S_3(re^{i\theta})| \leq \sum_{|a_n|<\rho} \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| 1 + \frac{c}{qre^{i\theta} - a_n} \right| \, d\theta + \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| 1 - \frac{c}{qre^{i\theta} - a_n + c} \right| \, d\theta \]
\[ + \sum_{|b_m|<\rho} \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| 1 + \frac{c}{qre^{i\theta} - b_m} \right| \, d\theta + \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| 1 - \frac{\bar{b}_m c}{\rho^2 - b_m (qre^{i\theta} + c)} \right| \, d\theta. \]

(5.3.8)

We apply Lemma 5.3.1 with \( \alpha = 1 \) and Lemma 5.3.2 to the first, second, third and fourth summand in (5.3.8), this yields

\[ \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| 1 + \frac{c}{qre^{i\theta} - c_n} \right| \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{\delta} \log^+ \left| 1 + \frac{c}{qre^{i\theta} - c_n} \right|^\delta \, d\theta \]
\[ \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{\delta} \log^+ \left( 1 + \left| \frac{c}{qre^{i\theta} - c_n} \right|^\delta \right) \, d\theta \]
\[ \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{\delta} \left\{ \left| \frac{c}{qre^{i\theta} - c_n} \right|^\delta \right\} \, d\theta \]
\[ = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{\delta} \left| \frac{c}{qre^{i\theta} - c_n} \right|^\delta \, d\theta \]

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\[
\frac{1}{2\pi} \int_0^{2\pi} \log^+ \left( 1 + \frac{c}{qre^{i\theta} - c_n} \right) d\theta \leq \frac{1}{\delta} \left( \frac{|c|}{r} \right) \frac{1}{(1 - \delta)|q|^\delta}.
\]

Therefore
\[
\frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| 1 + \frac{c}{qre^{i\theta} - c_n} \right| d\theta \leq \frac{1}{\delta} \left( \frac{|c|}{r} \right) \frac{1}{(1 - \delta)|q|^\delta}.
\]

Similarly,
\[
\frac{1}{2\pi} \int_0^{2\pi} \log^+ \left( 1 + \frac{c c_n}{\rho^2 - c_n(qre^{i\theta} + c)} \right) d\theta \leq \frac{1}{\rho - |q| r - |c|}.
\]

Similarly,
\[
\frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| 1 - \frac{c c_n}{\rho^2 - c_n(qre^{i\theta} + c)} \right| d\theta \leq \frac{|c|}{\rho - |q| r - |c|}.
\]
Substituting all (5.3.9), (5.3.10), (5.3.11), and (5.3.12) in (5.3.8), we get that

\[
\frac{1}{2\pi} \int_0^{2\pi} |S_2(re^{i\phi})|d\phi + \frac{1}{2\pi} \int_0^{2\pi} |S_3(re^{i\theta})|d\theta \leq \sum_{k=1}^{\rho} \frac{1}{\delta(1-\delta)|q|^\delta} \left( \frac{|c|}{r} \right)^\delta + \sum_{k=1}^{\rho} \frac{|c|}{(\rho - |q|r - |c|)} + \frac{|c|}{(\rho - |q|r - |c|)}
\]

\[
\leq \left( n(\rho, f) + n(\rho, \frac{1}{f}) \right) \left( \frac{2}{\delta(1-\delta)|q|^\delta} \left( \frac{|c|}{r} \right)^\delta + \frac{2|c|}{(\rho - |q|r - |c|)} \right) \quad (F)
\]

Using (E) and (F) in (5.3.6), we get

\[
m \left( \frac{r, f(qz + c)}{f(qz)} \right) + m \left( \frac{r, f(qz)}{f(qz + c)} \right)
\]

\[
\leq \frac{2\rho|c|}{(\rho - |q|r - |c|)^2} \left( m(\rho, f) + m(\rho, 1/f) + \log^+ \frac{1}{|f(0)|} \right) + \left( n(\rho, f) + n(\rho, \frac{1}{f}) \right) \left( \frac{2}{\delta(1-\delta)|q|^\delta} \left( \frac{|c|}{r} \right)^\delta + \frac{2|c|}{(\rho - |q|r - |c|)} \right) \quad (5.3.13)
\]

Using that [21][p.37]

\[
\left( n(\rho, f) + n(\rho, \frac{1}{f}) \right) \leq \frac{4\alpha}{\alpha - 1} \left( T(\alpha(|q|r + |c|), f) + \log^+ \frac{1}{|f(0)|} \right)
\]

and using

\[
\rho = \frac{\alpha + 1}{2} (|q|r + |c|).
\]

we get that

\[
m \left( \frac{r, f(qz + c)}{f(qz)} \right) + m \left( \frac{r, f(qz)}{f(qz + c)} \right) \leq \frac{4\alpha}{\alpha - 1} \left( T(\alpha(|q|r + |c|), f) + \log^+ \frac{1}{|f(0)|} \right)
\]

\[
+ \frac{\alpha + 1}{2} \left( \frac{2|c|}{(\alpha - 1)^2(|q|r + |c|) - (|q|r + |c|)^2} \left( \frac{|c|}{r} \right)^\delta + \frac{2}{\delta(1-\delta)} \left( \frac{|c|}{r} \right)^\delta \right)
\]

\[
= \left\{ \frac{8\alpha}{(\alpha - 1)^2(1-\delta)} \left( \frac{|c|}{|q|r} \right)^\delta + \frac{16\alpha}{(\alpha - 1)^2(|q|r + |c|) + (\rho - |q|r - |c|)} \left( \frac{|c|}{r} \right)^\delta + \frac{8(\alpha + 1)|c|}{(\alpha - 1)^2(|q|r + |c|)} \right\}
\]

\[
\left( \frac{8}{(\alpha - 1)^2(1-\delta)} \left( \frac{|c|}{|q|r} \right)^\delta + \frac{8|c|}{(\alpha - 1)^2(|q|r + |c|)} \left( 2\alpha + \alpha + 1 \right) \right)
\]

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\[
\left( T(\alpha(|q| r + |c|), f) + \log^+ \frac{1}{|f(0)|} \right) \\
\leq \left\{ \frac{8|c|(3\alpha + 1)}{(\alpha - 1)^2(|q| r + |c|)} + \frac{8\alpha}{\alpha - 1} \left( \frac{|c|}{r} \right)^{\delta} \frac{1}{\delta(1 - \delta)} \right\} \\
\left( T(\alpha(|q| r + |c|), f) + \log^+ \frac{1}{|f(0)|} \right).
\]

Therefore
\[
m \left( r, \frac{f(qz + c)}{f(qz)} \right) + m \left( r, \frac{f(qz)}{f(qz + c)} \right) \leq \frac{8|c|(3\alpha + 1)r^{\delta}(1 - \delta) + 8\alpha(\alpha - 1)(|q| r + |c|)|c|^\delta}{r^\delta(1 - \delta)(\alpha - 1)^2(|q| r + |c|)} \\
\left( T(\alpha(|q| r + |c|), f) + \log^+ \frac{1}{|f(0)|} \right),
\]
from which follows the assertion of the lemma. □

The following are the immediate consequences of the Lemma 5.3.4.

**Corollary 5.3.1.** If \( \rho, \delta, \rho' \) are real numbers such that \( \delta < 1, \rho > 0 \), such that \( \max\{1, |q| r + |c|\} < \rho < \rho' \) for \( c \in \mathbb{C} - \{0\} \), then we have,

\[
m \left( r, \frac{f(qz + c)}{f(qz)} \right) + m \left( r, \frac{f(qz)}{f(qz + c)} \right) \leq \frac{2|c|\rho}{(\rho - |q| r - |c|)^2} (m(\rho, f) + m(\rho, 1/f)) + \frac{2\rho'}{\rho' - \rho} \left( \frac{|c|}{\rho - |q| r - |c|} + \frac{|c|^\delta}{\delta(1 - \delta)r^\delta} \right) (N(\rho', f) + N(\rho', 1/f)).
\]

**Proof.** From (5.3.13) we have,

\[
m \left( r, \frac{f(qz + c)}{f(qz)} \right) + m \left( r, \frac{f(qz)}{f(qz + c)} \right) \leq \frac{2|c|\rho}{(\rho - |q| r - |c|)^2} (m(\rho, f) + m(\rho, 1/f)) \\
+ \left( n(\rho, f) + n(\rho, \frac{1}{f}) \right) + 2 \left( \frac{|c|^\delta}{\delta(1 - \delta)r^\delta|q|} + \frac{|c|}{(\rho - |q| r - |c|)} \right).
\]

Since \( \rho > \rho' > 1 \) we deduce,

\[
N(\rho', f) \geq \int_{\rho}^{\rho'} \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log \rho' \\
\geq n(\rho, f) \int_{\rho}^{\rho'} \frac{dt}{t} - n(0, f) \int_{\rho}^{\rho'} \frac{dt}{t} + n(0, f) \log \rho' \\
\geq n(\rho, f) \frac{\rho' - \rho}{\rho'}.
\]

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Hence

\[
n(p.f) \leq \frac{\rho}{\rho' - \rho} N(p'.f) \]

\[
n(p.1/f) \leq \frac{\rho}{\rho' - \rho} N(p'.1/f). \]

By using these we have the required assertion. □

**Corollary 5.3.2.** Let \( f(z) \) be a meromorphic function of finite order \( \sigma \) and let \( c, q \) be two non-zero complex numbers. Then for each \( \epsilon > 0 \), we have

\[
m\left( r, \frac{f(qz + c)}{f(qz)} \right) + m\left( r, \frac{f(qz)}{f(qz + c)} \right) = O(r^{\sigma - 1 + \epsilon})
\]

(5.3.14)

**Proof.** Since \( f(z) \) has finite order, \( \sigma < +\infty \), so given \( \epsilon > 0 \), \( 0 < \epsilon < 2 \), we have

\[
T(r, f) = O(r^{\sigma + \epsilon/2})
\]

for all \( r \). By choosing \( \alpha = 1 - \epsilon/2 \), \( \rho = 2r \), \( \rho' = 3r \), and \( r > \max \{|c|, 1/2\} \) in Corollary 5.3.1 we get the required assertion. □

**Corollary 5.3.3.** Let \( c_1, c_2 \) and \( q \) be non-zero complex numbers such that \( c_1 \neq c_2 \) and let \( f(z) \) be a finite order meromorphic function. If \( \sigma \) is the order of \( f(z) \), then for each \( \epsilon > 0 \), we have

\[
m\left( r, \frac{f(qz + c_1)}{f(qz + c_2)} \right) = O(r^{\sigma - 1 + \epsilon}).
\]

We also deduce Corollary 5.3.3 form the Corollary 5.3.1.

**Lemma 5.3.5.** (see[15]): Let \( z_1, z_2, z_3, \ldots, z_p \) be any finite collection of complex numbers, and let \( B > 0 \) be any given positive number. Then there exists a finite collection of closed disks \( D_1, D_2, \ldots, D_q \) with corresponding radii \( r_1, r_2, \ldots, r_q \) that satisfy

\[
r_1 + r_2 + \ldots + r_q = 2B.
\]
such that if \( z \notin D_j \) for \( j = 1, 2, \ldots, q \), then there is a permutation of the points \( z_1, z_2, \ldots, z_p \) say \( z_1, z_2, \ldots, z_p \) that satisfies

\[
|z - z_i| > B \frac{l}{p}
\]

for \( l = 1, 2, \ldots, p \), where the permutation may depend on \( z \).

**Lemma 5.3.6 (37).** \([34]\): Let \( f(z) \) be a meromorphic function. Then for all irreducible rational functions in \( f(z) \),

\[
R(z, f) = \frac{P(z, f)}{Q(z, f)} = \frac{\sum_{i=0}^{p} a_i(z) f^i}{\sum_{j=0}^{p} b_j(z) f^j}
\]

such that the meromorphic function coefficients \( a_i(z), b_j(z) \) satisfying \( T(r, a_i) = S(r, f) \) for \( i = 0, 1, \ldots, p \); \( T(r, b_j) = S(r, f) \) for \( j = 0, 1, \ldots, q \). Then we have

\[
T(r, R(z, f)) = \max\{p, q\} T(r, f) + S(r, f).
\]

**Lemma 5.3.7.** ([1], p. 902-905): Given \( \epsilon > 0 \) and a meromorphic function \( f(z) \), the Nevanlinna characteristic function \( T \) satisfies

\[
T(r|q|, f(z + c)) \leq (1 + \epsilon) T(|q| r + |c|, f) + k
\]

for \( q, c \in \mathbb{C} \) and for all \( r \geq 1/\epsilon \), for some constant \( k \).

The following are our main important results.

### 5.4 Statement and Proofs of Main Theorems

**Theorem 5.4.1.** : Let \( f(z) \) be a meromorphic function with exponent of convergence of poles \( \lambda(\frac{1}{f}) = \lambda < +\infty \), \( c \neq 0 \) and \( q \) be any two fixed complex numbers, then for each \( \epsilon > 0 \),

\[
N(r, f(qz + c)) = N(r, f(qz)) + O(r^{\lambda - 1 + \epsilon}) + O(\log r). \quad (5.4.1)
\]
Proof. Let \( \{a_n\} \) be a sequence of poles of \( f(z) \) with due count of multiplicity. Then \( \{ \frac{a_n}{q} \} \) for \( q \in \mathbb{C} \setminus \{0\} \) is a sequence of poles of \( f(qz) \) and \( \{ \frac{a_n - c}{q} \} \) is the sequence of poles of \( f(qz + c) \). Thus by the definition of \( N(r, f) \) we deduce.

\[
|N(r, f(qz + c)) - N(r, f(qz))| \leq |c| \left( \sum_{0 < |a_n - c| < |q|r} \frac{1}{|a_n|} + \sum_{0 < |a_n - c| < |q|r} \frac{1}{|a_n - c|} \right)
\]

Applying Lemma 5.3.2 with \( \alpha = 1 \), to the first summand of (5.4.2), we deduce that

\[
|\log \left( \frac{a_n}{a_n - c} \right)| \leq \left| \frac{a_n - (a_n - c)}{a_n - c} \right| + \left| \frac{(a_n - c) - a_n}{a_n} \right| = \left| \frac{c}{a_n - c} \right| + \left| \frac{c}{a_n} \right|.
\]

Applying Lemma 2.7.2 with inequality \( 0 < |a_n - c| < |q|r \) and \( |a_n| \geq |q|r \) to the second summand of (5.4.2) we obtain that

\[
\log \left( \frac{|q|r}{|a_n - c|} \right) = \log \left( \frac{|q|r - |a_n - c|}{|a_n - c|} + 1 \right) \leq \frac{|q|r - |a_n - c|}{|a_n - c|} \leq \frac{|q|r + |c| - |a_n|}{|a_n - c|} \leq \frac{|c|}{a_n - c}.
\]

Applying Lemma 5.3.1 with \( |a_n - c| \geq |q|r |a_n| < |q|r \) to the third summand of (5.4.2)

\[
\log \left( \frac{|q|r}{|a_n|} \right) = \log \left( \frac{|q|r - |a_n|}{|a_n|} + 1 \right) \leq \frac{|q|r - |a_n|}{|a_n|} \leq \frac{|c|}{a_n}.
\]

Combining (5.4.3), (5.4.4), and (5.4.5) in the inequality (5.4.2), we deduce

\[
|N(r, f(qz + c)) - N(r, f(qz))| \leq |c| \left( \sum_{0 < |a_n - c| < |q|r} \frac{1}{|a_n|} + \sum_{0 < |a_n - c| < |q|r} \frac{1}{|a_n - c|} \right)
\]
We first compute the first summand in (5.4.6). We divide the range $0 < |a_n - c| < |q|r$ into two ranges. $0 < |a - c| < |q|r$ and $|c| < |a - c| < |q|r$.

When $|a_n - c| > |c|$, then

$$\frac{1}{|a_n - c|} = \frac{1}{|a_n|} \left| 1 + \frac{c}{a_n - c} \right| \leq \frac{1}{|a_n|} \left( 1 + \frac{|c|}{|a_n - c|} \right) < \frac{2}{|a_n|}. $$

Thus, for $|c| < |q|r$

$$\sum_{0 < |a_n - c| < |q|r} \frac{1}{|a_n - c|} = \sum_{0 < |a_n - c| < |c|} \frac{1}{|a_n - c|} + \sum_{|c| < |a_n - c| < |q|r} \frac{1}{|a_n - c|}$$

$$\leq \sum_{0 < |a_n - c| < |c|} \frac{1}{|a_n - c|} + \sum_{|c| < |a_n - c| < |q|r} \frac{2}{a_n}$$

$$= 2 \sum_{|c| < |a_n - c| < |q|r} \frac{1}{a_n} + O(1)$$

$$\leq 2 \left( \sum_{0 < |a_n| < |q|r + |c|} \frac{1}{a_n} \right) + O(1). \quad (G)$$

With (G),(5.4.6) becomes

$$|N(r, f(qz + c)) - N(r, f(qz))| \leq |c| \left\{ 2 \sum_{0 < |a_n| < |q|r + |c|} \frac{1}{|a_n|} + \sum_{0 < |a_n| < |q|r + |c|} \frac{1}{|a_n|} \right\} + O(\log r)$$

$$\leq 3|c| \left( \sum_{0 < |a_n| < |q|r + |c|} \frac{1}{|a_n|} \right) + O(\log r). \quad (5.4.7)$$

We have the following cases:

**Case(1):** $\lambda \geq 1$. By Hölder’s inequality, we have for any $\varepsilon > 0$.

$$\sum_{0 < |a_n| < |q|r + |c|} \frac{1}{|a_n|} \leq \left( \sum_{0 < |a_n| < |q|r + |c|} \frac{1}{|a_n|^{\lambda + \varepsilon}} \right)^{\frac{1}{\lambda + \varepsilon}} + \left( \sum_{0 < |a_n| < |q|r + |c|} \frac{1}{|a_n|^{\frac{\lambda + \varepsilon}{1 + \varepsilon}}} \right)^{\frac{1 + \varepsilon}{\lambda + \varepsilon}}$$

$$\leq O(1). n(|q|r + |c|, f)^{\frac{\lambda + \varepsilon}{1 + \varepsilon}}$$

But $n(|q|r + |c|, f) = O((|q|r + |c|, f)^{\lambda + \varepsilon}) = O(r^{\lambda + \varepsilon}).$

$$\sum_{0 < |a_n| < |q|r + |c|} \frac{1}{|a_n|} = O(r^{\lambda - 1 + \varepsilon}) \quad (5.4.8)$$
Case(2): \( \lambda < 1 \). We have by definition of exponent of convergence.

\[
\sum_{0<|a_n|<|q| r^{|c|}} \frac{1}{|a_n|} = O(1). \tag{5.4.9}
\]

We obtain the required assertion by combining (5.4.7), (5.4.8), and (5.4.9) i.e.

\[
|N(r, f(qz + c)) - N(r, f(qz))| = O(r^{\lambda-1}) + O(\log r)
\]

\[\square\]

**Theorem 5.4.2.** Let \( f(z) \) be a meromorphic function with order \( \sigma = \sigma(f) \), \( \sigma < +\infty \), and let \( c,q \) be two non-zero complex number, then for each \( \epsilon > 0 \), we have

\[
T(r, f(qz + c)) = T(r, f(qz)) + O(r^{\sigma-1+\epsilon}) + O(\log r). \tag{5.4.10}
\]

**Proof.** Since \( f(z) \) has finite order, \( \sigma \) so that \( \lambda(\frac{1}{f}) \leq \sigma < \infty \). From Theorem 5.4.1 we have that

\[
|N(r, f(qz + c)) - N(r, f(qz))| = O(r^{\lambda-1+\epsilon}) + O(\log r) \tag{H}
\]

which implies that

\[
N(r, f(qz + c)) = O(r^{\lambda-1+\epsilon}) + N(r, f(qz)) + O(\log r)
\]

holds for function \( f(z) \). Hence, by using (H) and Corollary 5.3.2 yields that

\[
T(r, f(qz + c)) = m(r, f(qz + c)) + N(r, f(qz + c)) \leq m(r, f(qz)) + m \left( r, \frac{f(qz + c)}{f(qz)} \right) + O(r^{\sigma-1+\epsilon}) + N(r, f(qz)) + O(\log r)
\]

\[
T(r, f(qz + c)) \leq T(r, f(qz)) + O(r^{\sigma-1+\epsilon}) + O(\log r). \tag{5.4.11}
\]
Similarly, we deduce

\[
T(r, f(qz)) = m(r, f(qz)) + N(r, f(qz))
\]

\[
\leq m(r, f(qz + c)) + m \left( r, \frac{f(qz)}{f(qz + c)} \right) + O(r^{\sigma + 1}) + N(r, f(qz + c))
\]

\[
= m(r, f(qz + c)) + N(r, f(qz + c)) + m \left( r, \frac{f(qz)}{f(qz + c)} \right) - N(r, f(qz + c))
\]

\[
+ N(r, f(qz + c)) + O(r^{\sigma + 1}) + O(\log r)
\]

\[
= T(r, f(qz + c)) + m \left( r, \frac{f(qz)}{f(qz + c)} \right) + O(r^{\sigma + 1}) + O(\log r)
\]

\[
T(r, f(qz)) \leq T(r, f(qz + c)) + O(r^{\sigma + 1}) + O(\log r).
\]

From (5.4.11) and (5.4.12),

\[
T(r, f(qz + c)) = T(r, f(qz)) + O(r^{\sigma + 1}) + O(\log r).
\]

\[
\square
\]

### 5.5 Pointwise estimate of Logarithmic Derivative

The pointwise logarithmic derivative estimate of finite order meromorphic functions play an important role in complex differential equations. In particular, the following estimate of Gundersen (see [16] Corollary 2), gives an upper bound of logarithmic derivative, which states as follows: If \( f(z) \) is a meromorphic function, let \( k \geq 1 \) be an integer, \( \sigma > 1, \epsilon > 0 \) are real numbers then there exists a set \( E \subset (1, \infty) \) of finite logarithmic measure, \( (a) \) and a constant \( A > 0 \), depends only on \( \sigma \), such that for all \( z \notin E \cup [0, 1] \), we have

\[
\left| \frac{f'(z)}{f(z)} \right| \leq A \left( \frac{T(\arg, f)}{r} + \frac{n(\arg)}{r} \log^+ \log^r n(\arg) \right)
\]

where \( n(t) = n(t, f) + n(t, \frac{1}{f}) \).

\( (b) \) if \( f(z) \) has finite order \( \sigma \), such that for all \( z \notin E \cup [0, 1] \), we have

\[
\left| \frac{f'(z)}{f(z)} \right| \leq |z|^{\sigma + 1 + \epsilon}
\]
Now we obtain a difference analogue of Gundersen’s logarithmic derivative estimate as follows:

**Theorem 5.5.1.** Let \( f(z) \) be a meromorphic function \( c.q \in \mathbb{C} - \{0\} \) let \( \gamma > 1 \) and \( \epsilon > 0 \) be real numbers. Then there exist a subset \( E \subset (1, \infty) \) of finite logarithmic measure. a) a constant \( A \) depends only on \( \gamma \) on \( c.q \) such that for all \( |z| \notin E \cup [0,1] \), we have

\[
\left| \log \left( \frac{f(qz + c)}{f(qz)} \right) \right| \leq A \left( \frac{T(\gamma r, f)}{r} + \frac{n(\gamma r)}{r} \log^2 r \log^+ n(\gamma r) \right) \tag{5.5.1}
\]

b) if that \( f(z) \) is of finite order, \( \sigma \) such that for all \( |z| = r \notin E \cup [0,1] \), we have

\[
e^{-e^{\sigma - 1 + \epsilon}} \leq \left| \frac{f(qz + c)}{f(qz)} \right| \leq e^{-e^{\sigma - 1 + \epsilon}} \tag{5.5.2}
\]

**Proof.** Let \( z \) be such that \( |z| = r < \rho - |c| \). Let \( \beta > 1 \) and \( \rho = \beta r + |c| \). We choose \( \rho \) such that \( |c| < \beta(\beta - 1)r \) for \( r > \rho \). We apply Lemma 5.3.2 with \( c_0 = 1 \) for \( \alpha = 1 \).

With this now, proof of Lemma 5.3.4 becomes

\[
\left| \log \left( \frac{f(qz + c)}{f(qz)} \right) \right| \leq \frac{1}{2\pi} \int_0^{2\pi} \left| \log |f(p e^{i\theta})| \right| \left( \frac{2\rho|c|}{(\rho - |q||r - |c||)^2} \right) d\theta + \sum_{|\alpha_n| < \rho} \left| \log \left( \frac{\rho^2 - \tilde{a}_n(qz + c)}{\rho^2 - \tilde{a}_n qz} \right) \right|
\]

\[
+ \sum_{|b_m| < \rho} \left| \log \left( \frac{\rho^2 - \tilde{b}_m(qz + c)}{\rho^2 - \tilde{b}_m qz} \right) \right| + \sum_{|\alpha_n| < \rho} \left| \log \left( \frac{qz + c - \tilde{a}_n}{qz - \tilde{a}_n} \right) \right| + \sum_{|b_m| < \rho} \left| \log \left( \frac{qz + c - \tilde{b}_m}{qz - \tilde{b}_m} \right) \right| \tag{5.5.3}
\]

We apply Lemma 5.3.2 with \( \alpha = 1 \) to the second and third summands of (5.5.3) for

\[
|a_n| < \rho, \text{ we obtain}
\]

\[
\left| \log \left( \frac{\rho^2 - \tilde{a}_n(qz + c)}{\rho^2 - \tilde{a}_n qz} \right) \right| \leq \left| \frac{\tilde{a}_n c}{\rho^2 - \tilde{a}_n qz} \right| + \left| \frac{\tilde{a}_n c}{\rho^2 - \tilde{a}_n(qz + c)} \right| \leq \frac{|c|}{\rho - |q||z|} + \frac{|c|}{\rho - |q||z| - |c|}
\]

Similarly,

\[
\left| \log \left( \frac{\rho^2 - \tilde{b}_m(qz + c)}{\rho^2 - \tilde{b}_m qz} \right) \right| \leq \frac{2|c|}{\rho - |q||z| - |c|}
\]
Also we apply Lemma 5.3.2 with $c_\alpha = 1$. for $\alpha = 1$. to the fourth and fifth summand yields

$$\left| \log \frac{qz + c - a_n}{qz - a_n} \right| \leq |c| \left( \frac{1}{|qz - a_n|} + \frac{1}{|qz - c - a_n|} \right)$$

$$\left| \log \frac{qz + c - b_m}{qz - b_m} \right| \leq |c| \left( \frac{1}{|qz - b_m|} + \frac{1}{|qz - c - b_m|} \right)$$

Combining these equations in (5.5.3) we obtain

$$\left| \log \frac{f(qz + c)}{f(qz)} \right| \leq \frac{2|c|}{(\rho - |q|r - |c|)^2} \left( T(\rho, f) + \log + \frac{1}{f(0)} \right)$$

$$+ \frac{2|c|}{(\rho - |q|r - |c|)^2} \left( n(\rho, f) + n(\rho, \frac{1}{f}) \right)$$

$$+ |c| \sum_{|a_n| < \rho} \left( \frac{1}{|qz - a_n|} + \frac{1}{|qz + c - a_n|} \right)$$

$$+ |c| \sum_{|b_m| < \rho} \left( \frac{1}{|qz - b_m|} + \frac{1}{|qz + c - b_m|} \right)$$

$$\leq \frac{4|c|\rho}{(\rho - |q|r - |c|)^2} T(\rho, f) + |c| \sum_{|c_k| < |q| + |c|} \left( \frac{1}{|qz - a_n|} + \frac{1}{|qz + c - a_n|} \right)$$

$$\leq \frac{|c|\beta^2}{(\beta - |q|)^2 r^2} T(\beta^2 r, f) + |c| \sum_{|\beta^2 r|} \left( \frac{1}{|qz - a_n|} + \frac{1}{|qz + c - a_n|} \right)$$

$$= 4|c| \left( \frac{\beta}{\beta - |q|} \right)^2 T(\beta^2 r, f) + |c| \sum \frac{1}{|qz - d_k|}.$$  (5.5.4)

where $\{c_k\}_{k \in \mathbb{N}} = \{a_n\}_{n \in \mathbb{N}} \cup \{b_m\}_{m \in \mathbb{N}}$ and $\{d_k\}_{k \in \mathbb{N}} = \{c_k\}_{k \in \mathbb{N}} \cup \{c_k - c\}_{k \in \mathbb{N}}$ and the sequence $d_k$ is listed according to their multiplicities and ordered by increasing modulus.

Now let $\gamma = \beta^2$, and applying Lemma 5.3.5 to the second summand of the above equation with $|d_k| < \rho = \gamma r$ so that we deduce for all $|z| \notin E \cup [0, 1]$, where the set $E$ has finite logarithmic measure,

$$\sum_{|d_k| < \gamma r} \frac{1}{|qz - d_k|} \leq \gamma^2 n(\gamma^2 r) r \log^2 r \log n(\gamma^2 r).$$

Combining (5.5.4) with above equation we get that

$$\left| \log \frac{f(qz + c)}{f(qz)} \right| \leq 4|c| \left( \frac{\beta}{\beta - |q|} \right)^2 T(\gamma r, f) + |c| \left( \gamma^2 n(\gamma^2 r) r \log^2 r \log n(\gamma^2 r) \right)$$

$$\leq |c| \left[ 4|c| \left( \frac{\beta}{\beta - |q|} \right)^2 T(\gamma r, f) + |c| \left( \gamma^2 n(\gamma^2 r) r \log^2 r \log n(\gamma^2 r) \right) \right]$$
which gives (5.5.1) if $\gamma^2$ replaced by $\gamma$.

(ii) If $f(z)$ is of finite order, then given $\epsilon > 0$, it is easy to deduce the estimate (5.5.2) from the estimate (5.5.1).

We obtain the following result.

**Corollary 5.5.1.** If $c_1$, $c_2$ and $q$ be any arbitrary complex numbers, and let $f(z)$ be a meromorphic function of finite order $\sigma$. Let $\epsilon > 0$ be given, then there exists a subset $E \subset \mathbb{R}$, with finite logarithmic measure such that for all $r \in E \cup [0, 1]$, we have

$$\exp(-r^{\sigma - 1 + \epsilon}) \leq \left| \frac{f(qz + c_1)}{f(qz + c_2)} \right| \leq \exp(r^{\sigma - 1 + \epsilon}).$$

### 5.6 Applications to Difference Equations:

We apply Theorem 5.4.2 to give the proof of the following theorem which were the main objective in Ablowitz, Halburd, Herbst (see[1]).

**Theorem 5.6.1.** If $c_1 , c_2 , \ldots , c_n$ and $q$ are any fixed non-zero complex numbers. If the difference equation

$$\sum_{j=1}^{n} y(qz + c_n) = \frac{a_0(z) + a_1(z)y(qz) + \ldots + a_p y(qz)^p}{b_0(z) + b_1(z)y(qz) + \ldots + b_m y(qz)^m} \tag{5.6.1}$$

with polynomial co-efficients $a_i , b_j$ admit a finite order meromorphic solution $f(qz)$, then we have $\max \{p, m\} \leq n$.

**Proof.** Without loss of generality, we assume that $f(qz)$ to be a finite order transcendental meromorphic solution to (5.6.1). Now by applying Lemma 5.3.5 on the right side of the...
above equation with Theorem 5.4.1. we get that

\[
\max \{p, m\} T(r, f(qz)) = T(r, R(z, f(qz))) + S(r, f)
\]
\[
\leq T(r, \sum_{j=1}^{n} y(qz + c_n)) + S(r, f)
\]
\[
\leq nT(r, f(qz)) + O(r^{\sigma - 1 + \varepsilon}) + O(\log r) + S(r, f)
\] (5.6.2)

since (5.6.2) is independent of \(c_j\). This yields the required assertion. \(\square\)

**Remark 5.6.1.** : This theorem was first verified with \(n = 2\) and was written in above generalised form. With respect to \(n = 2\) the above theorem reduces to the following theorem.

**Theorem 5.6.2.** : If \(c_1, c_2\) and \(q\) are any non-zero complex numbers. If the difference equation

\[
y(qz + c_1) + y(qz + c_2) = a_0(z) + a_1(z)y(qz) + \ldots + a_p y(qz)^p
\]
\[
+ b_0(z) + b_1(z)y(qz) + \ldots + b_m y(qz)^m
\]

with polynomial co-efficients \(a_j, b_j\) admits a finite order meomorphic solution \(f(qz)\), then we have \(\max \{p, m\} \leq 2\).

As a another remark we can even replace the \(\sum_{j=1}^{n} y(qz + c_n)\) in Theorem 5.6.1 by \(\prod_{j=1}^{n} y(qz + c_n)\) and the conclusion of the theorem still remains the same.

**Theorem 5.6.3.** : Let \(A_0(z), A_1(z), \ldots, A_n(z)\) be entire functions such that there exists an integer \(l, 0 \leq l \leq n\), such that

\[
\sigma(A_l) > \max_{0 \leq j \leq n, j \neq l} \{\sigma(A_j)\}.
\] (5.6.3)

If \(f(qz)\) is a meromorphic solution to

\[
A_n(z)y(qz + n) + \ldots + A_1(z)y(qz + 1) + A_0(z)y(qz) = 0.
\] (5.6.4)

then we have \(\sigma(f) \geq \sigma(A_l) + 1\).
Proof. Let us choose $\sigma$ in relation to (5.6.3) so that

$$\max_{0 \leq j < n, j \neq l} \{\sigma(A_j)\} < \sigma < \sigma(A_l). \quad (5.6.5)$$

holds. Let us suppose that $f(qz)$ is a finite order meromorphic solution to (5.6.5) such that

$$\sigma(f(z)) < \sigma(A_l) + 1 \quad (5.6.6)$$

Now we divide (5.6.4) by $f(qz + l)$ to get

$$A_n(z) \frac{f(qz + n)}{f(qz + l)} + \ldots + A_1(z) + \ldots A_0(z) \frac{f(z)}{f(qz + l)} = 0.$$ 

Since (5.6.5) and (5.6.6) holds together, so we choose $\epsilon > 0$, such that the inequalities

$$\sigma(f(z)) + 2\epsilon < \sigma(A_l) + 1, \quad \text{and} \quad \sigma + 2\epsilon < \sigma(A_l),$$

hold simultaneously. With $\epsilon < 0$ using the Corollary 5.3.3, we have

$$m \left( r, \frac{f(qz + j)}{f(qz + l)} \right) \leq O(r^{\sigma(f)-1+\epsilon}). \quad (5.6.7)$$

Then we deduce by (5.6.7) that

$$m(r, A_l(z)) \leq \max_{0 \leq j < n, j \neq l} m \left( r, \frac{f(qz + j)}{f(qz + l)} \right) + \sum_{j \neq l} m(r, A_j),$$

$$\leq O(r^{\sigma(f)-1+\epsilon}) + O(r^{\sigma+\epsilon}).$$

which is a contradiction.

We next apply the theorem to solve a problem of Whittaker [46] concerning the $q$ linear difference equation.

**Problem:** Let $\sigma$ be real number, let $\Psi(z)$ be a given entire function with order $\sigma(\Psi) = \sigma$.

Then the equation

$$F(qz + c) = \Psi(z)F(qz) \quad (5.6.8)$$

admits a meromorphic solution of order $\sigma(F) = \sigma + 1$. \qed

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Proof. Whittaker constructed a meromorphic solution $F(z)$ to (5.6.8) such that

$$\sigma(F) \leq \sigma(\Psi) + 1$$

Since $\Psi$ is entire, it satisfies the assumption (5.6.3) and leads to the conclusion that

$$\sigma(F) \geq \sigma(\Psi) + 1$$

this completes the proof. □

Theorem 5.6.4. : Let $P_0(z), P_1(z), \ldots, P_n(z)$ be polynomials such that there exists an integer, $l$, $0 \leq l \leq n$ so that

$$\deg(P_l) > \max_{0 \leq i \leq n, j \neq l} \{\deg(P_i)\}$$

holds. Suppose that $f(z)$ is a meromorphic solution to

$$P_n(z)y(qz + n) + \ldots + P_1(z)y(qz + 1) + P_0y(qz) = 0, \quad (5.6.10)$$

then we have $\sigma(f(z)) \geq 1$

Proof. We assume that the (5.6.10) admits a meromorphic solution $f(z)$ with $\sigma(f(z)) < 1$.

We now divide through (5.6.10) by $f(qz + l)$ to obtain

$$P_n(z)\frac{f(qz + n)}{f(qz + l)} + \ldots + P_1(z)\frac{f(qz)}{f(qz + l)} + P_0 = 0. \quad (5.6.11)$$

Since $\sigma(f(z)) < 1$, so let us choose an $\epsilon > 0$, so that $\epsilon < 1 - \sigma(f(z))$ and Corollary 5.5.1. implies that $0 \leq j \leq l$ or $1 < l \leq n$, then

$$\left|\frac{f(qz + j)}{f(qz + l)}\right| \leq \exp(r^\sigma - 1 + \epsilon) = \exp(o(1)) \quad (5.6.12)$$

holds outside a possible set $r$ of finite logarithmic measure. We deduce that (5.10.2) is bounded outside of a possible set $r$ of finite logarithmic measure. We now apply (5.10.2) to (5.6.11), which gives

$$|P_l(z)| \leq \max_{0 \leq i \leq n} |P_i(z)| \left|\frac{f(qz + j)}{f(qz + l)}\right| \leq O(1) \max_{0 \leq i \leq n} |P_j(z)|$$
as \(|z| \to \infty\). outside a possible set \(r\) of finite logarithmic measure. Which is a contradiction to (5.6.9). □

5.7 Conclusion

(1) The proof the Lemma 5.3.4 can be viewed as discrete analogue of Lemma of Logarithmic derivative, given by Nevanlinna which has many applications not only in proof of classsical Nevanlinna second fundamental theorem but also in the proof of Yosida's generalization of Malmquist theorems. And it has also vast number of applications in the theory of ordinary differential equations.

(2) The above mentioned special properties of finite order meromorphic functions distinguish themselves from general meromorphic functions and they are in strong agreement with integrability detector of difference equation as proposed in [1].

(3) Our first investigations leads up to give an answer to the Whittakar's problem via the Corollary 5.3.2 and Corollary 5.3.3 of Lemma 5.3.4 which is amongst most basic results of first order \(q_c\) difference equations from the view point of Nevanlinna Theory.
5.8 ON THE QC DIFFERENCE ANALOGUE OF LEMMA OF THE LOGARITHMIC DERIVATIVE AND ITS APPLICATIONS

The Lemma on the Logarithmic Derivative states that outside of a possible small exceptional set

\[ m \left( r, \frac{f'}{f} \right) = O(\log T(r, f) + \log r) \]  \hspace{1cm} (5.8.1)

where \( m(r, f) \) denotes the Nevanlinna Proximity function and \( T(r, f) \) is the characteristic of a meromorphic function \( f \) (see [21]). This is undoubtedly one of the most useful results of Nevanlinna Theory, having a vast number of applications in the theory of meromorphic functions and on the theory of ordinary differential equations.

The purpose of this section is to prove a difference analogue of the Lemma on the Logarithmic Derivative, and to apply it to study meromorphic solutions of large classes of difference equations. The difference analogue appears to be in its most useful form when applied to study finite order meromorphic solutions of difference equations, which is in agreement with the findings in applications include, for instance, a difference analogue of the Clunie Lemma [10]. The original lemma has proved to be an invaluable tool in the study of non-linear differential equations. The difference analogue gives similar information about the finite order meromorphic solutions of non-linear difference equations.

We need the following important lemma to prove our main result.

**Lemma 5.8.1.** Let \( f \) be a meromorphic function such that \( f(0) \neq 0, \infty \) and let \( c \in \mathbb{C} \). Then for all \( \alpha > 1, \delta < 1, q \in \mathbb{C} \setminus \{0\} \) and \( r \geq 1 \)

\[ m \left( r, \frac{f(qz + c)}{f(qz)} \right) \leq \frac{K(\alpha, \delta, c, q)}{r^\delta} \left( T(\alpha(|q|r + |c|, f)) + \log^+ \frac{1}{|f(0)|} \right) \]
where
\[
k(a, \delta, c, q) = \frac{8(3\alpha + 1)|c||q|^\delta + 8\alpha(\alpha - 1)|c|^\delta}{(\alpha - 1)^2\delta(\delta - 1)|q|^\delta}.
\]

\textbf{Proof.} Proof of the Lemma 5.8.1 follows as the proof of the Lemma 5.3.4 by defining the exceptional set
\[
E = \left\{ \phi \in [0, 2\pi]; \frac{|f(qre^{i\phi} + c)|}{f(qre^{i\phi})} \geq 1 \right\}
\]
this implies that
\[
\left| \log \left( \frac{f(qre^{i\phi} + c)}{f(qre^{i\phi})} \right) \right| = \log^+ \left( \frac{f(qre^{i\phi} + c)}{f(qre^{i\phi})} \right) + \log^+ \frac{1}{\left( \frac{f(qre^{i\phi} + c)}{f(qre^{i\phi})} \right)}
\]
\[
= \log^+ \frac{f(qre^{i\phi} + c)}{f(qre^{i\phi})}.
\]

Multiplying (5.3.5) by $\frac{1}{2\pi}$ and integrating on $|z| = r$ to get the Proximity function,
\[
m \left( r, \frac{f(qz + c)}{f(qz)} \right) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left( \frac{f(qre^{i\phi} + c)}{f(qre^{i\phi})} \right) d\phi
\]
\[
= \frac{1}{2\pi} \int_0^{2\pi} \left| S_1(re^{i\phi}) + S_2(re^{i\phi}) - S_3(re^{i\phi}) \right| d\phi
\]
\[
m \left( r, \frac{f(qz + c)}{f(qz)} \right) \leq \frac{1}{2\pi} \int_0^{2\pi} \left| S_1(re^{i\phi}) \right| d\phi + \int_0^{2\pi} \left| S_2(re^{i\phi}) \right| d\phi + \int_0^{2\pi} \left| S_3(re^{i\phi}) \right| d\phi
\]
(5.8.2)

Now right hand inequality of the above can be solved as in Lemma 5.3.4 to get the required assertion. \(\square\)

In this paper we prove the following main result.
5.9 Difference analogue of the Lemma on the Logarithmic Derivative:

Theorem 5.9.1. Let $f$ be a non-constant meromorphic function, $c \in \mathbb{C}$, $\delta < 1$, $\epsilon > 0$ and $q \in \mathbb{C} - \{0\}$. Then

$$m \left( r, \frac{f(qz + c)}{f(qz)} \right) = o \left( \frac{T(|q|r + |c|.f)^{1+\epsilon}}{r^{\delta}} \right)$$

(5.9.1)

for all $r$ outside of a possible exceptional set $E$ with finite logarithmic measure

$$\int_{E} \frac{dr}{r} < \infty.$$

Proof. Let $\xi(x)$ and $\phi(r)$ be positive, nondecreasing, continuous functions defined for $e \leq x \leq \infty$ and $r_0 \leq r < \infty$, respectively, where $r_0$ is such that $T(|q|r + |c|.f) \geq e$ for all $r \geq r_0$. Then by Borel's Lemma [12][Lemma 3.31]

$$T \left( |q|r + |c| + \frac{\phi(r)}{\xi(T(|q|r + |c|.f))} \cdot f \right) \leq 2T(|q|r + |c|.f)$$

for all $r$ outside of a set $E$ satisfying

$$\int_{E \cap [r_0, R]} \frac{dr}{\phi(r)} \leq \frac{1}{\xi(e)} + \frac{1}{\log 2} \int_{e}^{T(|q|r + |c|.f)} \frac{dx}{x\xi(x)}$$

where $R < \infty$. Therefore, by choosing $\phi(r) = r$ and $\xi(x) = x^{\epsilon/2}$ with $\epsilon > 0$, and defining

$$\alpha = 1 + \frac{r}{(|q|r + |c|)T(|q|r + |c|.f)^{\epsilon/2}}$$

(5.9.2)

we have

$$T(\alpha(|q|r + |c|).f) = T \left( |q|r + |c| + \frac{\phi(r)}{\xi(T(|q|r + |c|.f))} \cdot f \right) \leq 2T(|q|r + |c|.f)$$

(5.9.3)

for all $r$ outside of a set $E$ with finite logarithmic measure. Hence, if $f(0) \neq 0, \infty$, the assertion follows by combining (5.9.2), (5.9.3) with the Lemma 5.8.1. Otherwise we
apply Lemma 5.8.1 with the function \( g(z) = z^p f(z) \), where \( p \in \mathbb{Z} \) is chosen such that \( g(0) \neq 0, \infty \).

\[ \square \]

When \( f \) is of finite order, the right hand side of (5.9.1) is small compared to \( T(r, f) \) and therefore relation (5.9.1) is a natural analogue of the lemma on the Logarithmic Derivative (5.8.1).

If \( f \) is of infinite order, the quantity \( T(|q|r + |c|. f) r^{-\delta} \) may be comparable to \( T(r, f) \). In the finite-order we can also remove the \( \epsilon \) in the Theorem 5.9.1.

**Corollary 5.9.1.** Let \( f \) be a non-constant meromorphic function of finite order, \( c \in \mathbb{C} \) and \( \delta < 1 \), and \( q \in \mathbb{C} \setminus \{0\} \). Then

\[
m \left( r, \frac{f(qz + c)}{f(qz)} \right) = o \left( \frac{T(|q|r + |c|. f)}{r^\delta} \right)
\]

for all \( r \) outside of a possible exceptional set with finite logarithmic measure.

**Proof.** Choose any \( \delta < 1 \) and denote \( \delta' = (1 + \delta)/2 \leq 1 \). Since \( f \) is of finite order, we have \( T(|q|r + |c|. f) \leq r^\rho \) for some \( \rho > 0 \) and for all \( r \) sufficiently large. Therefore by Theorem 5.9.1

\[
m \left( r, \frac{f(qz + c)}{f(qz)} \right) = o \left( \frac{T(|q|r + |c|. f)}{r^{\delta'-\epsilon}} \right)
\]

where \( \epsilon > 0 \). The assertion follows by choosing \( \epsilon = (\delta' - \delta)/\rho \). \( \square \)

Note that by replacing \( z \) by \( z + h \), where \( h \in \mathbb{C} \) and \( c \) by \( c - h \) in Corollary 5.9.1 and using the inequality \( T(|q|r. f(z + h)) \leq (1 + \epsilon)T(|q|r + |h|. f(z)) \), \( \epsilon > 0 \), \( r > r_0 \), we immediately have

\[
m \left( r, \frac{f(q(z + h) + c - h)}{f(q(z + h))} \right) = o \left( \frac{T(|q|r + |c - h| + |h|. f(z + h))}{r^\delta} \right)
\]
\[
\begin{align*}
&= o \left( \frac{(1 + \varepsilon)T(|q|r + |c - h| + |h|. f(z))}{r^\delta} \right) \\
&= o \left( \frac{T(|q|r + |c - h| + |h|. f(z))}{r^\delta} \right)
\end{align*}
\]

\[
m \left( r, \frac{f(q(z + h) + c - h)}{f(q(z))} \right) = o \left( \frac{T(|q|r + |c - h| + |h|. f(z))}{r^\delta} \right)
\]

for all \( \delta < 1 \) outside of a possible exceptional set with finite logarithmic measure. \( \square \)

### 5.10 Difference Analogue of the Clunie and Mohon'ko lemmas:

The Lemma on the Logarithmic Derivative is an integral part of the proof of the Second Main Theorem, one of the deepest results of Nevanlinna theory. In addition, logarithmic derivative estimates are crucial for applications to complex differential equations. Similarly, Theorem 5.9.1 enables an efficient study of complex analytic properties of finite order meromorphic solutions of difference equations. We are concerned with functions which are polynomials in \( f(qz + c_j) \), where \( c_j \in \mathbb{C} \), \( q \in \mathbb{C} - \{0\} \) with coefficients \( a_\lambda(z) \) such that

\[
T(r, a_\lambda(z)) = o(T(r, f))
\]

except possibly for a set of \( r \) having finite logarithmic measure. Such functions will be called **difference polynomials in** \( f(z) \). If \( P(z, f) \) is difference polynomial in \( f(z) \) of degree at most \( n \) then \( P(z, f) \) is of the form

\[
a_\lambda(z)f(qz)^{a_0}f(qz + c_1)^{a_1}...f(qz + c_n)^{a_n}
\]

The following theorem is analogous to the Clunie Lemma, which has numerous applications in the study of complex differential equations and beyond.
Theorem 5.10.1. Let \( f(z) \) be a nonconstant meromorphic solution of
\[
(f(qz))^n P(z, f) = Q(z, f).
\]
where \( P(z, f) \) and \( Q(z, f) \) are difference polynomials in \( f(z) \), and let \( \delta < 1 \) and \( \epsilon > 0 \).

If the degree of \( Q(z, f) \) as a polynomial in \( f(z) \) and its shifts is at most \( n \), then
\[
m(r, P(z, f)) = o\left(\frac{T(|q| r + |c| f)^{1+\epsilon}}{r^\delta}\right) + o(T(r, f))
\]
for all \( r \) outside of a possible exceptional set with finite logarithmic measure.

Proof. In calculating the proximity function, we split the region of integration into two parts. By defining
\[
E_1 = \{ \theta \in [0, 2\pi] : |f(qre^{i\theta})| < 1 \}
\]
and \( E_2 \) be the complement of \( E_1 \).

\[
m(r, P(z, f)) = \frac{1}{2\pi} \int_{E_1} \log^+ |P(re^{i\theta}, f)| d\theta + \frac{1}{2\pi} \int_{E_2} \log^+ |P(re^{i\theta}, f)| d\theta
\]

Now, we have each term of \( P(z, f) \) is of the form
\[
a_\lambda(z) f(qz)^{\lambda_0} f(qz + c_1)^{\lambda_1} \ldots f(qz + c_\nu)^{\lambda_\nu}
\]
so writing \( \lambda = (\lambda_0, \lambda_1, \ldots, \lambda_\nu) \)

\[
P(z, f) = \sum_{\lambda \in I} P_\lambda(z, f)
\]

\[
= \sum_{\lambda \in I} a_\lambda(z) f(qz)^{\lambda_0} f(qz + c_1)^{\lambda_1} \ldots f(qz + c_\nu)^{\lambda_\nu}
\]

for each \( \lambda \), by using (5.10.1) we have
\[
|P_\lambda(z, f)| = |a_\lambda(z) f(qz)^{\lambda_0} f(qz + c_1)^{\lambda_1} \ldots f(qz + c_\nu)^{\lambda_\nu}|
\]

\[
\leq |a_\lambda(z)| \left| \frac{f(qre^{i\theta} + c_1)}{f(qre^{i\theta})} \right|^{\lambda_1} \left| \frac{f(qre^{i\theta} + c_2)}{f(qre^{i\theta})} \right|^{\lambda_2} \ldots \left| \frac{f(qre^{i\theta} + c_\nu)}{f(qre^{i\theta})} \right|^{\lambda_\nu}.
\]
Therefore for each \( \lambda \), we obtain

\[
\frac{1}{2\pi} \int_{E_1} \log^+ |P_\lambda(r e^{i\theta}, f)| \, d\theta \leq m(r, a_\lambda(z)) + O \left( \sum_{j=1}^{\nu} m \left( \frac{r}{f(q^z + c_j)} \right) \right)
\]

By using the Theorem 5.9.1 and our assumption of \( a_\lambda \) implies that

\[
\frac{1}{2\pi} \int_{E_1} \log^+ |P_\lambda(r e^{i\theta}, f)| \, d\theta = o(T(r, f)) + o \left( \frac{T(|q| r + |c|) f^{1+\epsilon}}{r^2} \right)
\]

(5.10.3)
on a set of logarithmic density 1. Similarly, on

\[
E_2 = \{ \theta \in [0, 2\pi] : |f(qr e^{i\theta})| \geq 1 \}.
\]

We note that,

\[
Q(z, f) = \sum_{\lambda \in I} |b_\lambda(z)| \left| \frac{f(q(r e^{i\theta}) + c_1)}{f(q r e^{i\theta})} \right|^{l_1} \left| \frac{f(q(r e^{i\theta}) + c_2)}{f(q r e^{i\theta})} \right|^{l_2} \cdots \left| \frac{f(q(r e^{i\theta}) + c_\nu)}{f(q r e^{i\theta})} \right|^{l_\nu}
\]

where \( l_0 + l_1 + \ldots + l_\nu \leq n \) for all \( \lambda = (l_0, l_1, \ldots, l_\nu) \in I \). Hence

\[
|P(z, f)| = \left| \frac{Q(z, f)}{f(q z)^n} \right|
\]

\[
\leq \left| \frac{1}{f(q r e^{i\theta})^n} \right| \sum_{\lambda \in I} |b_\lambda(z)| \left| \frac{f(q(r e^{i\theta}) + c_0)}{f(q r e^{i\theta})} \right|^{l_0} \left| \frac{f(q(r e^{i\theta}) + c_1)}{f(q r e^{i\theta})} \right|^{l_1} \cdots \left| \frac{f(q(r e^{i\theta}) + c_\nu)}{f(q r e^{i\theta})} \right|^{l_\nu}
\]

(5.10.4)

By using the Theorem 5.9.1 again we get that

\[
\frac{1}{2\pi} \int_{E_1} \log^+ |P_\lambda(r e^{i\theta}, f)| \, d\theta = o(T(r, f)) + o \left( \frac{T(|q| r + |c|) f^{1+\epsilon}}{r^2} \right)
\]

on a set of logarithmic density 1. The assertion of the theorem follows by combining

(5.10.2) .(5.10.3) and (5.10.4).
Similarly the above theorem can be used to obtain information about the pole
distribution of meromorphic solutions of difference equations. The next result is concerned
with distribution of \textit{slowly moving targets} \( a \) \textbf{s}uch that \( T(r, a) = o(T(r, f)) \) outside of
a possible exceptional set of finite logarithmic measure. In particular, constant functions
are always slowly moving. The following theorem is an analogues of a results due to A.Z.
Mohon'ko and V.D. Mohon'ko on differential equations.

Theorem 5.10.2. : \textit{Let} \( f(z) \text{ be a nonconstant meromorphic solution of }

\[ P(z, f) = 0 \quad (5.10.5) \]

\textit{where} \( P(z, f) \text{ is a difference polynomial in } f(z), \text{ and let } \delta < 1 \text{ and } \epsilon > 0. \text{ If } P(z, a) \neq 0 \)
\textit{for a slowly moving target } \( a \), \textit{then}

\[ m \left( r, \frac{1}{f - a} \right) = o \left( \frac{T(|g|, r + |c|, f)^{1+\epsilon}}{r^\delta} \right) + o(T(r, f)) \]

\textit{for all } \( r \) \textit{outside of a possible exceptional set with finite logarithmic measure.}

\textbf{Proof.} By substituting \( f = g + a \) in (5.10.3) we get

\[ P(z, f) = P(z, g + a) = Q(z, g) + D(a(z)) = 0 \quad (5.10.6) \]

\textit{where} \( Q(z, g) = \sum_{\nu \in I} b_\nu(z)g(qz)^{l_\nu}g(qz + c_1)^{l_1}...g(qz + c_\mu)^{l_\mu} \text{ is a difference polynomial}
\textit{in } g(z) \textit{ such that } T(r, b_\nu(z)) = o(T(r, g)) \textit{ for } \nu = (l_\nu, l_1,...l_\mu) \in I \textit{ and } T(r, D(z)) = o(T(r, g)). \textit{outside a set of logarithmic density 1. Also all the terms of } Q(z, g) \textit{ are at}
\textit{least of degree 1. Also } D \neq 0 \textit{ as it does not satisfy (5.10.4).}

Next, we compute \( m \left( r, \frac{1}{g} \right) \). By using (5.10.6)

\[ m \left( r, \frac{1}{g} \right) \leq m \left( r, \frac{D}{g} \right) + m \left( r, \frac{1}{D} \right) \]

\[ = m \left( r, \frac{Q(z, g)}{g} \right) + m \left( r, \frac{1}{D} \right) \]

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We now consider two cases. In first case we consider, if

$$|g| > 1 \Rightarrow \frac{1}{|g|} < 1$$

$$\Rightarrow \log^+ \frac{1}{|g|} = 0.$$ 

Therefore

$$m \left( r, \frac{1}{g} \right) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{|g|} d\theta = 0.$$ 

Secondly, we consider the case where

$$|g| \leq 1 \Rightarrow \frac{1}{|g|} \geq 1$$

then,

$$\left| \frac{Q(z, f)}{g(z)} \right| \leq \frac{1}{|g(z)|} \left| \sum_{\nu \in I} b_\nu g(qz)^{b_\nu} g(qz + c_1)^{b_1} \cdots g(qz + c_\mu)^{b_\mu} \right|$$

$$\leq \sum_{\nu \in I} |b_\nu(g)| \left| g(q(re^{i\theta}) + c_0) \right|^{b_\nu} \left| g(q(re^{i\theta}) + c_1) \right|^{b_1} \cdots \left| g(q(re^{i\theta}) + c_\mu) \right|^{b_\mu}$$

and

$$m(r, b_\nu) = o(T(r, g))$$

on a set of logarithmic density 1 for all $\nu \in I$. Also by using Theorem 5.9.1 for $g$ and using $\sum_{i=0}^{\nu} b_i \geq 1$, we get that

$$m \left( r, \frac{Q(z, g)}{g(z)} \right) \leq m(r, b_\lambda(z)) + O \left( \sum_{j=1}^{\nu} m \left( r, \frac{g(qz + c_j)}{g(qz)} \right) \right)$$

$$m \left( r, \frac{Q(z, g)}{g(z)} \right) = o(T(r, g)) + o \left( T(|q| r + |c|, g)^{1+\varepsilon} \right)$$

$$m \left( r, \frac{1}{g} \right) = o(T(r, g)) + o \left( T(|q| r + |c|, g)^{1+\varepsilon} \right).$$

But $g = f - a$ which implies that

$$m \left( r, \frac{1}{f - a} \right) = o(T(r, g))$$

outside of a set of $r$ values with at most finite logarithmic measure. Since $g = f - a$ the assertion follows.
Theorem 5.10.1 and Theorem 5.10.2, like Theorem 5.9.1 are particularly useful when applied to functions of finite order. The following two corollaries on the Nevanlinna deficiency illustrate this fact.

**Corollary 5.10.1.** Let $f(z)$ be a nonconstant finite order meromorphic solution of

$$f(qz)^nP(z, f) = Q(z, f),$$

where $P(z, f)$ and $Q(z, f)$ are difference polynomials in $f(z)$, and let $\delta < 1$. If the degree of $Q(z, f)$ as a polynomial in $f(z)$ and its shifts is at most $n$, then

$$m(r, P(z, f)) = o\left(\frac{T(|qr| + |cl|, f)}{r^\delta}\right) + o(T(r, f))$$

(5.10.7)

for all $r$ outside of a possible exceptional set with finite logarithmic measure. Moreover, the Nevanlinna deficiency satisfies

$$\delta(\infty, P) := \lim_{r \to \infty} \frac{m(r, P)}{T(r, P)} = 0$$

(5.10.8)

**Proof.** (5.10.7) follows by combining the proof of Theorem 5.9.1 with Corollary 5.9.1, and so we are left with (5.10.8). By a well known result due to Valiron [44] and Mohon'ko [37], we have

$$T(r, P) = \deg(P)T(r, f) + o(T(r, f))$$

(5.10.9)

outside of a possible exceptional set of finite logarithmic measure. In addition [see [34], Lemma 1.1.2] yields that if $T(r, g) = o(T(r, f))$ outside of an exceptional set of finite logarithmic measure, then

$$T(r, g) = o(T(r^{1+\epsilon}, f)).$$

for any $\epsilon > 0$, and for all $r$ sufficiently large. Thus by applying (5.10.7) together with (5.10.9) and [see [34], Lemma 1.1.2] we have,

$$m(r, P) = o\left(\frac{T(r^{1+\epsilon}, P)}{r^\delta}\right) + o(T(r^{1+\epsilon}, P))$$

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for all sufficiently large $r$. Therefore, since $P$ is of finite order,

$$m(r, P) \leq r^\rho(1+2e)^{-\delta}.$$  \hspace{1cm} (5.10.10)

where $\rho$ is the order of $P$ and $\delta < 1$. Also, there is a sequence $r_n \to \infty$ as $n \to \infty$, such that

$$T(r_n, P) \geq r_n^{\rho-\epsilon}$$ \hspace{1cm} (5.10.11)

for all $r_n$ large enough. The assertion follows by combining (5.10.10) and (5.10.11) where $\epsilon$ and $\delta$ are chosen such that $\epsilon(2\rho + 1) < \delta < 1$, and by letting $n \to \infty$. \hfill \square

**Corollary 5.10.2.** Let $f(z)$ be a non-constant finite order meromorphic solution of

$$P(z, f) = 0$$

where $P(z, f)$ is difference polynomial in $f(z)$, and let $\delta < 1$. If $P(z, a) \not= 0$ for a slowly moving target $a$, then

$$m\left(r, \frac{1}{f - a}\right) = o\left(\frac{T(|q|r + |e|, f)}{r^\delta}\right) + o(T(r, f))$$

for all $r$ outside of a possible exceptional set with finite logarithmic measure. Moreover, the Nevanlinna deficiency satisfies

$$\delta(a, f) := \lim_{r \to \infty} \frac{m(r, \frac{1}{f - a})}{T(r, f)} = 0$$

We omit the proof since it would be almost identical to that of Corollary 5.10.1.
5.11 Conclusion

In the above section we have presented a difference analogue of the Lemma on the Logarithmic Derivative. This result has potentially large number of applications in the study of difference equations. Many ideas and methods from the theory of differential equations may now be utilized together with Theorem 5.9.1 to obtain information about meromorphic solutions of difference equations. And the analogues of the Clunie Lemma may be used to ensure that finite order meromorphic solutions of certain non-linear difference equations have a large number of poles. Similarly analogue of Mohon'ko Lemma provides an easy way of telling when a finite order meromorphic solution of difference equation does not have any deficient values.