APPENDIX

Case-1:

$n_i = n_3 = 3$, $n_2 = n_4 = 3$, $k_1 = k_2 = 2$.

The marginal probability distribution of $T_{1i}$ under $H_0$ can be obtained as follows. We use the following notations for c.d.f's of $\text{med}(X_1, X_2, X_3)$, $\text{med}(Z_1, Z_2, Z_3)$, $\text{med}(Y_1, Y_2, Y_3)$ and $\text{med}(W_1, W_2, W_3)$.

\begin{align*}
F_{M_1}(x) &= \text{cdf of } \text{med}(X_1, X_2, X_3) \\
F_{M_2}(x) &= \text{cdf of } \text{med}(Y_1, Y_2, Y_3) \\
F_{M_3}(x) &= \text{cdf of } \text{med}(Z_1, Z_2, Z_3) \\
F_{M_4}(x) &= \text{cdf of } \text{med}(W_1, W_2, W_3).
\end{align*}

Then

\[ P[T_{1i} = 0] = P[\text{med}(X_1, X_2, X_3) < \text{med}(Z_1, Z_2, Z_3)] \]

\[ = \int_{-\infty}^{\infty} F_{M_3}(x) dF_{M_1}(x) \]

\[ = \frac{1}{2} \]

and

\[ P[T_{1i} = 1] = P[\text{med}(X_1, X_2, X_3) > \text{med}(Z_1, Z_2, Z_3)] \]

\[ = \int_{-\infty}^{\infty} F_{M_2}(x) dF_{M_1}(x) \]

\[ = \frac{1}{2}. \]
Similarly, the marginal probability distributions of $T_{12}$, $T_{21}$ and $T_{22}$ under $H_0$ can be obtained.

Further, the joint probability distribution of $T_{11}$ and $T_{12}$ can be obtained as follows.

$$P[T_{11} = 0, T_{12} = 0] = P[\text{med}(X_1, X_2, X_3) < \text{med}(Z_1, Z_2, Z_3)]$$

$$= \int_{-\infty}^{\infty} F_{M_4}(x) F_{M_4}(x) dF_{M_4}(x) dF_{M_4}(x)$$

$$= \frac{1}{3}.$$

Similarly, we can calculate all the joint probabilities.

Next, we consider the joint probability distribution of $T_{11}$, $T_{12}$ and $T_{21}$.

For example, consider

$$P[T_{11} = 0, T_{12} = 0, T_{11} = 0]$$

$$= P[\text{med}(X_1, X_2, X_3) < \text{med}(Z_1, Z_2, Z_3), \text{med}(X_1, X_2, X_3) < \text{med}(W_1, W_2, W_3),$$

$$\text{med}(Y_1, Y_2, Y_3) < \text{med}(Z_1, Z_2, Z_3)]$$

$$= \int_{-\infty}^{\infty} F_{M_4}(x) F_{M_4}(x) dF_{M_4}(x) dF_{M_4}(x) + \int_{-\infty}^{\infty} F_{M_4}(x) F_{M_4}(x) dF_{M_4}(x) dF_{M_4}(x)$$

$$= \frac{5}{24} \text{ under } H_0.$$

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All other joint probabilities for $T_{11}$, $T_{12}$ and $T_{21}$ are worked out similarly and are given in section 5.3.

Proceeding on similar lines, we can calculate the joint distribution of $T_{11}$, $T_{21}$, $T_{22}$ and $T_{12}$, $T_{22}$ and $T_{12}$, $T_{21}$, $T_{22}$.

Now we consider the joint probability distribution of $T_{11}$, $T_{12}$, $T_{21}$ and $T_{22}$.

First consider,

$$P[T_{11} = 0, T_{12} = 0, T_{21} = 0, T_{22} = 0]$$

$$= P[\text{med}(X_1, X_2, X_3) < \text{med}(Z_1, Z_2, Z_3), \text{med}(X_1, X_2, X_3) < \text{med}(W_1, W_2, W_3), \text{med}(Y_1, Y_2, Y_3) < \text{med}(Z_1, Z_2, Z_3), \text{med}(Y_1, Y_2, Y_3) < \text{med}(W_1, W_2, W_3)]$$

$$= \int F_{M_1}(x) F_{M_2}(y) dF_{M_1}(x) dF_{M_2}(y) + \int F_{M_2}(x) F_{M_1}(y) dF_{M_2}(x) dF_{M_1}(y)$$

$$= \frac{1}{6}$$

This procedure is adopted for obtaining joint probability of $T_{11}$, $T_{12}$, $T_{21}$ and $T_{22}$, for all other configurations.

Case-2:

$$n_1 = 4, n_3 = 3, n_2 = 4, n_4 = 3, k_1 = k_2 = 2.$$ 

We use the following notations for $\text{med}(X_1, X_2, X_3)$, $\text{med}(X_1, X_2, X_3)$, $\text{med}(X_1, X_3, X_4)$, $\text{med}(X_2, X_3, X_4)$, $\text{med}(Y_1, Y_2, Y_3)$, $\text{med}(Y_1, Y_2, Y_4)$, $\text{med}(Y_1, Y_3, Y_4)$ and $\text{med}(Y_2, Y_3, Y_4)$.
MI(x) = med(X'X^2fX^3),
M^2(X) = med(X'X^2, X^4),
M^3(x) = med(X'X^3, X^4),
M^4(x) = med(X^2, X^3, X^4),
M^1(Y) = ... X^4
Xi Xi
Xi Xi
x3 Xi
Ms(X) M^4(X)
x3 x3
x4 x4
Xa X2
x3 X4
X4 x2
X4 X3
Xi X4
X3 X4
x3 Xa
X4 X4
Xi x3

The computation of marginal and joint probability distributions in this case is not as simple as in the earlier case: There are 4! permutations of orderings of X_1, X_2, X_3, X_4. The 24 permutations and the corresponding values of M_1(X), M_2(X), M_3(X) and M_4(X) are given below.

<table>
<thead>
<tr>
<th>Configuration</th>
<th>M_1(X)</th>
<th>M_2(X)</th>
<th>M_3(X)</th>
<th>M_4(X)</th>
</tr>
</thead>
<tbody>
<tr>
<td>X_1&lt;X_2&lt;X_3&lt;X_4</td>
<td>X_2</td>
<td>X_2</td>
<td>X_3</td>
<td>X_3</td>
</tr>
<tr>
<td>X_1&lt;X_2&lt;X_4&lt;X_3</td>
<td>X_2</td>
<td>X_2</td>
<td>X_4</td>
<td>X_4</td>
</tr>
<tr>
<td>X_1&lt;X_3&lt;X_2&lt;X_4</td>
<td>X_3</td>
<td>X_2</td>
<td>X_3</td>
<td>X_2</td>
</tr>
<tr>
<td>X_1&lt;X_3&lt;X_4&lt;X_2</td>
<td>X_3</td>
<td>X_4</td>
<td>X_3</td>
<td>X_4</td>
</tr>
<tr>
<td>X_1&lt;X_4&lt;X_2&lt;X_3</td>
<td>X_2</td>
<td>X_4</td>
<td>X_4</td>
<td>X_2</td>
</tr>
<tr>
<td>X_1&lt;X_4&lt;X_3&lt;X_2</td>
<td>X_3</td>
<td>X_4</td>
<td>X_4</td>
<td>X_3</td>
</tr>
<tr>
<td>X_2&lt;X_4&lt;X_1&lt;X_3</td>
<td>X_1</td>
<td>X_4</td>
<td>X_1</td>
<td>X_4</td>
</tr>
<tr>
<td>X_2&lt;X_4&lt;X_3&lt;X_1</td>
<td>X_3</td>
<td>X_4</td>
<td>X_3</td>
<td>X_4</td>
</tr>
<tr>
<td>X_2&lt;X_1&lt;X_3&lt;X_4</td>
<td>X_1</td>
<td>X_1</td>
<td>X_3</td>
<td>X_3</td>
</tr>
<tr>
<td>X_2&lt;X_1&lt;X_4&lt;X_3</td>
<td>X_1</td>
<td>X_1</td>
<td>X_4</td>
<td>X_4</td>
</tr>
<tr>
<td>X_2&lt;X_3&lt;X_1&lt;X_4</td>
<td>X_3</td>
<td>X_1</td>
<td>X_1</td>
<td>X_3</td>
</tr>
</tbody>
</table>
Similarly, we can write all the 24 permutations of orderings of $Y_1, Y_2, Y_3, Y_4$. Now, we consider the computation of the marginal probability distribution of $T_{11}$.

For example,

$$P[T_{11} = 0] = P[M_1(X) < \text{med}(Z_1, Z_2, Z_3), M_2(X) < \text{med}(Z_1, Z_2, Z_3), M_3(X) < \text{med}(Z_1, Z_2, Z_3), M_4(X) < \text{med}(Z_1, Z_2, Z_3)]$$

<table>
<thead>
<tr>
<th>X_2&lt;X_3&lt;X_4&lt;X_1</th>
<th>X_3</th>
<th>X_4</th>
<th>X_4</th>
<th>X_3</th>
</tr>
</thead>
<tbody>
<tr>
<td>X_3&lt;X_1&lt;X_2&lt;X_4</td>
<td>X_1</td>
<td>X_2</td>
<td>X_1</td>
<td>X_2</td>
</tr>
<tr>
<td>X_3&lt;X_1&lt;X_4&lt;X_2</td>
<td>X_1</td>
<td>X_4</td>
<td>X_1</td>
<td>X_4</td>
</tr>
<tr>
<td>X_3&lt;X_4&lt;X_1&lt;X_2</td>
<td>X_1</td>
<td>X_1</td>
<td>X_4</td>
<td>X_4</td>
</tr>
<tr>
<td>X_3&lt;X_4&lt;X_2&lt;X_1</td>
<td>X_2</td>
<td>X_2</td>
<td>X_4</td>
<td>X_4</td>
</tr>
<tr>
<td>X_3&lt;X_2&lt;X_1&lt;X_4</td>
<td>X_2</td>
<td>X_1</td>
<td>X_1</td>
<td>X_2</td>
</tr>
<tr>
<td>X_3&lt;X_2&lt;X_4&lt;X_1</td>
<td>X_2</td>
<td>X_4</td>
<td>X_4</td>
<td>X_2</td>
</tr>
<tr>
<td>X_4&lt;X_2&lt;X_1&lt;X_3</td>
<td>X_1</td>
<td>X_2</td>
<td>X_1</td>
<td>X_2</td>
</tr>
<tr>
<td>X_4&lt;X_2&lt;X_3&lt;X_1</td>
<td>X_3</td>
<td>X_2</td>
<td>X_3</td>
<td>X_2</td>
</tr>
<tr>
<td>X_4&lt;X_3&lt;X_1&lt;X_2</td>
<td>X_1</td>
<td>X_1</td>
<td>X_3</td>
<td>X_3</td>
</tr>
<tr>
<td>X_4&lt;X_3&lt;X_2&lt;X_1</td>
<td>X_2</td>
<td>X_2</td>
<td>X_3</td>
<td>X_3</td>
</tr>
<tr>
<td>X_4&lt;X_1&lt;X_2&lt;X_3</td>
<td>X_2</td>
<td>X_1</td>
<td>X_1</td>
<td>X_2</td>
</tr>
<tr>
<td>X_4&lt;X_1&lt;X_3&lt;X_2</td>
<td>X_3</td>
<td>X_1</td>
<td>X_1</td>
<td>X_3</td>
</tr>
</tbody>
</table>
Now consider the case $X_1 < X_2 < X_3 < X_4$. In this case,

$$P[T_{11} = 0] = P[X_2 < \text{med}(Z_1, Z_2, Z_3), X_2 < \text{med}(Z_1, Z_2, Z_3), X_3 < \text{med}(Z_1, Z_2, Z_3), X_1 < X_2 < X_3 < X_4]$$

$$= \int_{-\infty}^{\infty} P[X_1 < X_2 < X_3 < \text{min}(z, X_4)] dF_{M_3}(z)$$

$$= \frac{13}{840}.$$  

Similarly, we can obtain the probabilities for different orderings of $X_1, X_2, X_3,$ and $X_4$. Combining the probabilities for the different configurations, we have

$$P[T_{11} = 0] = \frac{13}{35}.$$  

The marginal distribution of $T_{11}$ obtained in this way is given in table 5.5.

Further the joint probability distribution of $T_{11}$ and $T_{12}$ can be obtained in a similar manner by considering different cases.

Case i: $X_1 < X_2 < X_3 < X_4$

$$P[T_{11} = 0, T_{12} = 0] = P[M_1(X) < \text{med}(Z_1, Z_2, Z_3), M_2(X) < \text{med}(Z_1, Z_2, Z_3),$$

$$M_3(X) < \text{med}(Z_1, Z_2, Z_3), M_4(X) < \text{med}(Z_1, Z_2, Z_3),$$

$$M_1(X) < \text{med}(W_1, W_2, W_3), M_2(X) < \text{med}(W_1, W_2, W_3),$$

$$M_3(X) < \text{med}(W_1, W_2, W_3), M_4(X) < \text{med}(W_1, W_2, W_3),$$

$$X_1 < X_2 < X_3 < X_4]$$
Evaluating the probabilities in all the twenty four cases and adding, we have

\[ P[T_{11} = 0, T_{12} = 0] = \frac{43}{210}. \]

The joint probability distribution of \( T_{11} \) and \( T_{12} \) obtained in this way is presented in table 5.6.

It may be noted that the probability that \( T_{11} \) and \( T_{12} \) taking the value 1 or 3 is zero.

Further, we obtain the joint probability distribution of \( T_{11}, T_{12} \) and \( T_{21} \). For example,
\[ P[T_{11} = 0, T_{12} = 0, T_{21} = 0] \]

\[ = P[M_1(X) < \text{med}(Z_1, Z_2, Z_3), M_2(X) < \text{med}(Z_1, Z_2, Z_3), \]

\[ M_3(X) < \text{med}(Z_1, Z_2, Z_3), M_4(X) < \text{med}(Z_1, Z_2, Z_3), \]

\[ M_1(Y) < \text{med}(Z_1, Z_2, Z_3), M_2(Y) < \text{med}(Z_1, Z_2, Z_3), \]

\[ M_3(Y) < \text{med}(Z_1, Z_2, Z_3), M_4(Y) < \text{med}(Z_1, Z_2, Z_3) ] \]

We first consider the case \( X_1 < X_2 < X_3 < X_4, Y_1 < Y_2 < Y_3 < Y_4 \). In this case,

\[ P[T_{11} = 0, T_{12} = 0, T_{21} = 0] \]

\[ = P \left[ X_2 < \text{med}(Z_1, Z_2, Z_3), X_3 < \text{med}(Z_1, Z_2, Z_3), \right. \]

\[ X_2 < \text{med}(W_1, W_2, W_3), X_3 < \text{med}(W_1, W_2, W_3), \]

\[ Y_2 < \text{med}(Z_1, Z_2, Z_3), Y_3 < \text{med}(Z_1, Z_2, Z_3), \]

\[ X_1 < X_2 < X_3 < X_4, Y_1 < Y_2 < Y_3 < Y_4 \] \]

\[ = \int \int P[X_1 < X_2 < X_3 < \min(z, w, X_4)] \ P[Y_1 < Y_2 < Y_3 < \min(z, Y_4)] \]

\[ 36 F(z) \bar{F}(z) F(w) \bar{F}(w) \ dF(z) \ dF(w). \]

\[ \int \int P[X_1 < X_2 < X_3 < \min(z, w, X_4)] P[Y_1 < Y_2 < Y_3 < \min(z, Y_4)] \]

\[ 36 F(z) \bar{F}(z) F(w) \bar{F}(w) \ dF(z) \ dF(w) \]

\[ + \int \int P[X_1 < X_2 < X_3 < \min(z, w, X_4)] P[Y_1 < Y_2 < Y_3 < \min(z, Y_4)] \]

\[ 36 F(z) \bar{F}(z) F(w) \bar{F}(w) \ dF(z) \ dF(w) \]

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Working out the probabilities for all the 24 cases of X configuration and 24 cases of Y configuration, we have

\[ P[T_{11} = 0, T_{12} = 0, T_{21} = 0] = \frac{5308}{47775}. \]

Similarly, we can compute the probabilities for all possible values of \( T_{11}, T_{12} \) and \( T_{21} \). The joint probability distribution of \( T_{11}, T_{12} \) and \( T_{21} \) and other three at a time joint probability distributions are given in table 5.7. Further consider the joint probability distribution of \( T_{11}, T_{12}, T_{21} \) and \( T_{22} \). For example,

\[ P[T_{11} = 0, T_{12} = 0, T_{21} = 0, T_{22} = 0] \]

\[ = P[M_1(X) < \text{med}(Z_1, Z_2, Z_3), M_2(X) < \text{med}(Z_1, Z_2, Z_3), M_3(X) < \text{med}(Z_1, Z_2, Z_3), M_4(X) < \text{med}(Z_1, Z_2, Z_3)] \]

\[ M_3(Y) < \text{med}(Z_1, Z_2, Z_3), M_4(Y) < \text{med}(Z_1, Z_2, Z_3), M_1(Y) < \text{med}(W_1, W_2, W_3), M_2(Y) < \text{med}(W_1, W_2, W_3), M_3(Y) < \text{med}(W_1, W_2, W_3)] \]

As before, we consider the case \( X_1 < X_2 < X_3 < X_4, Y_1 < Y_2 < Y_3 < Y_4 \) first.
In this case

\[
P[T_{11} = 0, T_{12} = 0, T_{21} = 0, T_{22} = 0]
\]

\[
= P[X_2 < \text{med}(Z_1, Z_2, Z_3), X_3 < \text{med}(Z_1, Z_2, Z_3),
\]

\[
X_2 < \text{med}(W_1, W_2, W_3), X_3 < \text{med}(W_1, W_2, W_3),
\]

\[
Y_2 < \text{med}(Z_1, Z_2, Z_3), Y_3 < \text{med}(Z_1, Z_2, Z_3),
\]

\[
Y_2 < \text{med}(W_1, W_2, W_3), Y_3 < \text{med}(W_1, W_2, W_3),
\]

\[
X_1 < X_2 < X_3 < X_4, Y_1 < Y_2 < Y_3 < Y_4].
\]

\[
= \int P[X_1 < X_2 < X_3 < \text{min}(z, w, X_4)] P[Y_1 < Y_2 < Y_3 < \text{min}(z, w, Y_4)]
\]

\[
36 F(z) \bar{F}(z) F(w) \bar{F}(w) dF(z) dF(w)
\]

\[
= \int \int A_1 dF(z) dF(w) + \int \int A_1 dF(z) dF(w)
\]

where

\[
A_1 = P[X_1 < X_2 < X_3 < \text{min}(z, w, X_4)] P[Y_1 < Y_2 < Y_3 < \text{min}(z, w, Y_4)]
\]

\[
36 F(z) \bar{F}(z) F(w) \bar{F}(w)
\]

Thus evaluating the probability for all 24 configurations \( X \) and 24 configurations of \( Y \) we have,

\[
P[T_{11} = 0, T_{12} = 0, T_{21} = 0, T_{22} = 0] = \frac{23}{273}.
\]

In a similar manner, one can compute the joint probabilities for other possible values of \( T_{11}, T_{12}, T_{21} \) and \( T_{22} \).

**Case-3:**

When \( n_1 = n_3 = 4, n_2 = n_4 = 4, k_1 = k_2 = 2 \), one can obtain the marginal and joint probability distributions of \( T_{11}, T_{12}, T_{21} \) and
Appendix

T_{22} by considering the 4! orderings of X observations, 4! orderings of Y observations, 4! orderings of Z observations and 4! orderings of W observations.

We use the following notations for med(Z_1, Z_2, Z_3), med(Z_1, Z_2, Z_4), med(Z_2, Z_3, Z_4), med(W_1, W_2, W_3), med(W_1, W_2, W_4), med(W_1, W_3, W_4) and med(W_2, W_3, W_4).

\[ M_1(Z) = \text{med}(Z_1, Z_2, Z_3), \quad M_1(W) = \text{med}(W_1, W_2, W_3), \]
\[ M_2(Z) = \text{med}(Z_1, Z_2, Z_4), \quad M_2(W) = \text{med}(W_1, W_2, W_4), \]
\[ M_3(Z) = \text{med}(Z_1, Z_3, Z_4), \quad M_3(W) = \text{med}(W_1, W_3, W_4) \]
\[ M_4(Z) = \text{med}(Z_2, Z_3, Z_4), \quad M_4(W) = \text{med}(W_2, W_3, W_4). \]

The marginal distribution of \( T_{11} \) can be obtained as follows.

Consider first

\[ P[T_{11} = 0] \]
\[ = P[M_1(X) < M_1(Z), M_1(X) < M_2(Z), M_1(X) < M_3(Z), M_1(X) < M_4(Z), \]
\[ M_2(X) < M_1(Z), M_2(X) < M_2(Z), M_2(X) < M_3(Z), M_2(X) < M_4(Z), \]
\[ M_3(X) < M_1(Z), M_3(X) < M_2(Z), M_3(X) < M_3(Z), M_3(X) < M_4(Z), \]
\[ M_4(X) < M_1(Z), M_4(X) < M_2(Z), M_4(X) < M_3(Z), M_4(X) < M_4(Z)]. \]

When \( X_1 < X_2 < X_3 < X_4, Z_1 < Z_2 < Z_3 < Z_4 \) we have,

\[ P[T_{11} = 0] \]
\[ = P[X_2 < Z_2, X_3 < Z_3, X_3 < Z_2, X_3 < Z_3, X_1 < X_2 < X_3 < X_4, \]
\[ Z_1 < Z_2 < Z_3 < Z_4] \]
Similarly evaluating the probabilities for other configurations of
X and Z observations, we have

\[ p[T_{11} = 0] = \frac{17}{40320}. \]

Similarly, one can obtain the probabilities for other possible
values of \( T_{11} \).

Further consider the joint probability distribution of \( T_{11} \) and \( T_{12} \).

We have

\[
P[T_{11} = 0, T_{12} = 0] = P[M_1(X) < M_1(Z), M_1(X) < M_2(Z), M_2(X) < M_2(Z), M_2(X) < M_3(Z), M_3(X) < M_4(Z),
\]

\[
M_3(X) < M_2(Z), M_3(X) < M_3(Z), M_3(X) < M_4(Z),
\]

\[
M_4(X) < M_1(Z), M_4(X) < M_2(Z), M_4(X) < M_3(Z), M_4(X) < M_4(Z),
\]

\[
M_1(X) < M_1(W), M_1(X) < M_2(W), M_1(X) < M_3(W), M_1(X) < M_4(W),
\]

\[
M_2(X) < M_1(W), M_2(X) < M_2(W), M_2(X) < M_3(W), M_2(X) < M_4(W),
\]

\[
M_3(X) < M_1(W), M_3(X) < M_2(W), M_3(X) < M_3(W), M_3(X) < M_4(W),
\]

\[
M_4(X) < M_1(W), M_4(X) < M_2(W), M_4(X) < M_3(W), M_4(X) < M_4(W)
\]

Consider the configuration

\[ X_1 < X_2 < X_3 < X_4, Z_1 < Z_2 < Z_3 < Z_4, W_1 < W_2 < W_3 < W_4. \]
When all the 24 configurations of $X$, $Z$ and $W$ observations are taken into account, we have

$$P[T_{11} = 0, T_{12} = 0] = \frac{19}{2280960}.$$  

Similarly all other joint probabilities of $T_{11}$ and $T_{12}$ can be worked out.

Further the joint probability distribution of $T_{11}$, $T_{12}$ and $T_{21}$ can be obtained as follows.
Consider

\[ P[T_{11} = 0, T_{12} = 0, T_{21} = 0] \]

\[ = P[M_1(X) < M_1(Z), M_1(X) < M_2(Z), M_1(X) < M_3(Z), M_1(X) < M_4(Z), \]
\[ M_2(X) < M_1(Z), M_2(X) < M_2(Z), M_2(X) < M_3(Z), M_2(X) < M_4(Z), \]
\[ M_3(X) < M_1(Z), M_3(X) < M_2(Z), M_3(X) < M_3(Z), M_3(X) < M_4(Z), \]
\[ M_4(X) < M_1(Z), M_4(X) < M_2(Z), M_4(X) < M_3(Z), M_4(X) < M_4(Z), \]
\[ M_1(X) < M_1(W), M_1(X) < M_2(W), M_1(X) < M_3(W), M_1(X) < M_4(W), \]
\[ M_2(X) < M_1(W), M_2(X) < M_2(W), M_2(X) < M_3(W), M_2(X) < M_4(W), \]
\[ M_3(X) < M_1(W), M_3(X) < M_2(W), M_3(X) < M_3(W), M_3(X) < M_4(W), \]
\[ M_4(X) < M_1(W), M_4(X) < M_2(W), M_4(X) < M_3(W), M_4(X) < M_4(W), \]
\[ M_1(Y) < M_1(Z), M_1(Y) < M_2(Z), M_1(Y) < M_3(Z), M_1(Y) < M_4(Z), \]
\[ M_2(Y) < M_1(Z), M_2(Y) < M_2(Z), M_2(Y) < M_3(Z), M_2(Y) < M_4(Z), \]
\[ M_3(Y) < M_1(Z), M_3(Y) < M_2(Z), M_3(Y) < M_3(Z), M_3(Y) < M_4(Z), \]
\[ M_4(Y) < M_1(Z), M_4(Y) < M_2(Z), M_4(Y) < M_3(Z), M_4(Y) < M_4(Z)] \].

In the case

\[ X_1 < X_2 < X_3 < X_4, \quad Y_1 < Y_2 < Y_3 < Y_4, \quad Z_1 < Z_2 < Z_3 < Z_4, \quad W_1 < W_2 < W_3 < W_4. \]

We have

\[ P[T_{11} = 0, T_{12} = 0, T_{21} = 0] \]

\[ = P[X_2 < Z_2, X_2 < Z_3, X_3 < Z_2, X_3 < Z_3, X_2 < W_2, X_2 < W_3, X_3 < W_2, \]
\[ X_3 < W_3, Y_2 < Z_2, Y_2 < Z_3, Y_3 < Z_2, Y_3 < Z_3, X_1 < X_2 < X_3 < X_4, \]
\[ Z_1 < Z_2 < Z_3 < Z_4, \quad Y_1 < Y_2 < Y_3 < Y_4, \quad W_1 < W_2 < W_3 < W_4]. \]
Appendix

\[ \begin{align*}
&= \iint P[X_1 < X_2 < X_3 < \min(z_2, w_2, X_4)] P[Y_1 < Y_2 < Y_3 < \min(z_2, Y_4)] \\
&P[Z_1 < z_2 < Z_3 < Z_4] P[W_1 < w_2 < W_3 < W_4] dF(z_2) dF(w_2) \\
&= \iint B_1 dF(z_2) dF(w_2) + \iint B_1 dF(z_2) dF(w_2)
\end{align*} \]

where

\[ B_1 = P[X_1 < X_2 < X_3 < \min(z_2, w_2, X_4)] P[Y_1 < Y_2 < Y_3 < \min(z_2, Y_4)] \\
P[Z_1 < z_2 < Z_3 < Z_4] P[W_1 < w_2 < W_3 < W_4]. \]

Then

\[ P[T_{11} = 0, T_{12} = 0, T_{21} = 0] = \frac{192091}{1394852659200}. \]

Further evaluating the probabilities for all the cases we have

\[ P[T_{11} = 0, T_{12} = 0, T_{21} = 0] = \frac{192091}{4204200}. \]

Similarly one can calculate the probabilities for different values assumed by \( T_{11}, T_{12} \) and \( T_{21} \) and are given in table 5.11.

Now consider the joint probability distribution of \( T_{11}, T_{12}, T_{21} \) and \( T_{22} \).

Consider

\[ P[T_{11} = 0, T_{12} = 0, T_{21} = 0, T_{22} = 0] \]

\[ = P[M_1(X) < M_1(Z), M_1(X) < M_2(Z), M_1(X) < M_3(Z), M_1(X) < M_4(Z), \]
\[ M_2(X) < M_1(Z), M_2(X) < M_2(Z), M_2(X) < M_3(Z), M_2(X) < M_4(Z), \]
\[ M_3(X) < M_1(Z), M_3(X) < M_2(Z), M_3(X) < M_3(Z), M_3(X) < M_4(Z), \]
\[ M_4(X) < M_1(Z), M_4(X) < M_2(Z), M_4(X) < M_3(Z), M_4(X) < M_4(Z). \]
We shall evaluate this probability for the case
\(X_1 < X_2 < X_3 < X_4, Y_1 < Y_2 < Y_3 < Y_4, Z_1 < Z_2 < Z_3 < Z_4, W_1 < W_2 < W_3 < W_4.\)

In this case, we have

\[
P[T_{11} = 0, T_{12} = 0, T_{21} = 0, T_{22} = 0] = P[X_2 < Z_2, X_2 < Z_3, X_3 < Z_2, X_3 < Z_3, X_2 < W_2, X_2 < W_3, X_3 < W_2, \\
X_3 < W_3, Y_2 < Z_2, Y_2 < Z_3, Y_3 < Z_2, Y_3 < Z_3, Y_2 < W_2, Y_2 < W_3, \\
Y_3 < W_3, Y_3 < W_3, X_1 < X_2 < X_3 < X_4, \\
Z_1 < Z_2 < Z_3 < Z_4, Y_1 < Y_2 < Y_3 < Y_4, W_1 < W_2 < W_3 < W_4.]
\]
\[
= \int \int P[X_1 < X_2 < X_3 < \min(z_2, w_2, X_4)]P[Y_1 < Y_2 < Y_3 < \min(z_2, w_2, Y_4)]
\]

\[
P[Z_1 < z_2 < Z_3 < Z_4]P[W_1 < w_2 < W_3 < W_4]dF(z_2)dF(w_2)
\]

\[
= \int \int C_1 dF(z_2)dF(w_2) + \int \int C_1 dF(z_2)dF(w_2)
\]

= \frac{139}{1423319040}.

where

\[
C_1= P[X_1 < X_2 < X_3 < \min(z_2, w_2, X_4)]P[Y_1 < Y_2 < Y_3 < \min(z_2, w_2, Y_4)]
\]

In this manner we can evaluate the joint probability in other cases. Combining all these cases we have

\[
P[T_{11} = 0, T_{12} = 0, T_{21} = 0, T_{22} = 0] = \frac{139}{4290}.
\]

Similar procedure applies to the computation of joint probabilities of \( T_{11}, T_{12}, T_{21} \) and \( T_{22} \), when \( T_{11}, T_{12}, T_{21} \) and \( T_{22} \) can assume other values.