CHAPTER IV

MINIMAL OPEN SETS AND MAPS IN BITOPOLOGICAL SPACES

4.1 Introduction and preliminaries.

J. C. Kelly [11], in the year 1963, first initiated the concept of bitopological spaces. He defined a bitopological space $(X, \tau_1, \tau_2)$ to be a set $X$ equipped with two topologies $\tau_1$ and $\tau_2$ on $X$ and initiated the systematic study of bitopological space. He extended the notions of separation axioms of single topological space to bitopological space. Also Maki, Sundaram and Balachandran [15] have introduced the concept of $\tau_i-\sigma_k$ continuous, bi-continuous and strongly bi-continuous maps in bitopological spaces.

Here we present some of the definitions, which are used in our study.

In sections 2 of this chapter, we introduce and study the concept of a new class of sets called $(\tau_i, \tau_j)$-minimal open sets, $(\tau_i, \tau_j)$-maximal open sets, $(\tau_i, \tau_j)$-minimal closed sets and $(\tau_i, \tau_j)$-maximal closed sets in bitopological spaces. Also we introduce a new class of bitopological spaces called pairwise-$T_{\text{min}}$ spaces and pairwise-$T_{\text{max}}$ spaces.

In sections 3 of this chapter, we introduce and investigate a new class of maps called $(\sigma_k, \sigma_l)$-$M_iO(Y)-\tau_i$-continuous, $(\sigma_k, \sigma_l)$-$M_iO(Y)-\tau_j$-continuous, $(\sigma_k, \sigma_l)$-$M_iO(Y)-(\tau_i, \tau_j)$-irresolute and $(\sigma_k, \sigma_l)$-$M_iO(Y)-(\tau_j, \tau_j)$-irresolute maps in bitopological spaces.

In sections 4 of this chapter, we introduce and study the concept of a new class of maps called $(\tau_i, \tau_j)$-$M_iO(X)-\sigma_k$-open, $(\tau_i, \tau_j)$-$M_iO(X)-\sigma_k$-open, $(\tau_i, \tau_j)$-$M_iO(X)-(\sigma_k, \sigma_l)$-strongly open and $(\tau_i, \tau_j)$-$M_iO(X)-(\sigma_k, \sigma_l)$-strongly open maps in bitopological spaces. Also we introduce $(\tau_i, \tau_j)$-$M_iC(X)-\sigma_k$-closed, $(\tau_i, \tau_j)$-$M_iC(X)-\sigma_k$-closed, $(\tau_i, \tau_j)$-$M_iC(X)-(\sigma_k, \sigma_l)$-strongly closed and
(τ₁, τ₂)-MₐC(X)-(σₖ, σ₁)-strongly closed maps in bitopological spaces and discuss some of their properties.

In sections 5 of this chapter, we introduce the concept of a new class of homeomorphisms called (σₖ, σ₁)-MᵣO(Y)-τᵣ-homeomorphisms and (σₖ, σ₁)-MₒO(Y)-τᵢ-homeomorphisms in bitopological spaces and studied their properties.

Throughout this chapter (X, τ₁, τ₂), (Y, σ₁, σ₂) and (Z, η₁, η₂) denote nonempty bitopological spaces on which no separation axioms are assumed, unless otherwise mentioned and the fixed integers i, j, k, l, m, nє {1, 2}.

4.1.1 Definition: [11] Let X be a set and τ₁ and τ₂ be two different topologies on X. Then (X, τ₁, τ₂) is called a bitopological space.

4.1.2 Definition: A map f: (X, τ₁, τ₂) → (Y, σ₁, σ₂) is called

i) τᵢ-σₖ-continuous [15] if f⁻¹(V)є τᵢ for every Vє σₖ,

ii) bi-continuous [15] if f is τ₁-σ₁-continuous and τ₂-σ₂-continuous,

iii) strongly-bi-continuous [15] (briefly, s-bi-continuous) if f is bi-continuous, τ₁-σ₂-continuous and τ₂-σ₁-continuous,

4.2 Pairwise minimal open sets and pairwise maximal open sets

4.2.1 Definition: Let i, jє {1, 2} be the fixed integers and (X, τᵢ, τⱼ) be a bitopological space.

i) A proper nonempty τᵢ-open subset M in X is said to be a (τᵢ, τⱼ)-minimal open (briefly (τᵢ, τⱼ)-min open) set if any τⱼ-open set which is contained in M is either ∅ or M itself.

ii) A proper nonempty τᵢ-open subset M in X is said to be a (τᵢ, τⱼ)-maximal open (briefly (τᵢ, τⱼ)-maximal open) set if any τⱼ-open set which contains M is either X or M itself.
iii) A proper nonempty \( \tau_i \)-closed subset \( F \) in \( X \) is said to be a \( (\tau_i, \tau_j) \)-minimal closed (briefly \( (\tau_i, \tau_j) \)-min closed) set if any \( \tau_j \)-closed set which is contained in \( F \) is either \( \emptyset \) or \( F \) itself.

iv) A proper nonempty \( \tau_i \)-closed subset \( F \) in \( X \) is said to be a \( (\tau_i, \tau_j) \)-maximal closed (briefly \( (\tau_i, \tau_j) \)-maximal closed) set if any \( \tau_j \)-closed set which contains \( F \) is either \( X \) or \( F \) itself.

The family of all \( (\tau_i, \tau_j) \)-minimal open (resp. \( (\tau_i, \tau_j) \)-minimal closed) sets in a bitopological space \( (X, \tau_i, \tau_2) \) is denoted by \( (\tau_i, \tau_j)\text{-Min}O(X) \) (resp. \( (\tau_i, \tau_j)\text{-Min}C(X) \)). The family of all \( (\tau_i, \tau_j) \)-maximal open (resp. \( (\tau_i, \tau_j) \)-maximal closed) sets in a bitopological space \( (X, \tau_i, \tau_2) \) is denoted by \( (\tau_i, \tau_j)\text{-Max}O(X) \) (resp. \( (\tau_i, \tau_j)\text{-Max}C(X) \)).

4.2.2 Remark: If \( \tau_i = \tau_2 = \tau \) in the Definition 4.2.1, \( (\tau_i, \tau_j) \)-minimal open sets, \( (\tau_i, \tau_j) \)-maximal open sets, \( (\tau_i, \tau_j) \)-minimal closed sets and \( (\tau_i, \tau_j) \)-maximal closed sets reduce to minimal open sets, maximal open sets, minimal closed sets and maximal closed sets in the sense of F.Nakaoka and N.Oda ([22] and [23]).

4.2.3 Remark:

i) Every \( (\tau_i, \tau_j) \)-minimal open set is an \( \tau_i \)-open set but not conversely.

ii) Every \( (\tau_i, \tau_j) \)-maximal open set is an \( \tau_i \)-open set but not conversely.

4.2.4 Example: Let \( X = \{a, b, c, d\} \) be with topologies \( \tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\} \) and \( \tau_2 = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, X\} \). Then

\[
(\tau_1, \tau_2)\text{-Min}O(X) = \{\{a\}, \{b\}\}; \quad (\tau_2, \tau_1)\text{-Min}O(X) = \{\{a\}, \{c\}\}
\]

\[
(\tau_1, \tau_2)\text{-Max}O(X) = \{\{a, b, c\}\}; \quad (\tau_2, \tau_1)\text{-Max}O(X) = \{\{a, b, c\}\}.
\]

i) The set \( \{a, b\} \) is an \( \tau_1 \)-open set but it is not a \( (\tau_1, \tau_2) \)-minimal open set.

ii) The set \( \{a, b\} \) is an \( \tau_1 \)-open set but it is not a \( (\tau_1, \tau_2) \)-maximal open set.
4.2.5 Remark:

i) \((\tau_i, \tau_j)\)-minimal open sets and \(\tau_j\)-open sets are independent of each other.

ii) \((\tau_i, \tau_j)\)-maximal open sets and \(\tau_j\)-open sets are independent of each other.

iii) \((\tau_i, \tau_j)\)-minimal open sets and \(\tau_i\)-minimal open sets are independent of each other.

iv) \((\tau_i, \tau_j)\)-maximal open sets and \(\tau_i\)-maximal open sets are independent of each other.

v) \((\tau_i, \tau_j)\)-minimal open sets and \((\tau_i, x^-)\)-maximal open sets are independent of each other.

4.2.6 Example: Let \(X=\{a, b, c, d\}\) be with topologies \(\tau_1=\{\phi, \{a\}, \{a, b\}, X\}\) and \(\tau_2=\{\phi, \{a, b\}, \{a, b, c\}, X\}\). Then

\(\tau_1\)-minimal open sets = \{a\}; \(\tau_2\)-minimal open sets = \{a, b\}

\(\tau_1\)-maximal open sets = \{a, b\}; \(\tau_2\)-maximal open sets = \{a, b, c\}

\((\tau_1, \tau_2)\)-\(M\_O(X) = \{\{a\}, \{a, b\}\}; \quad (\tau_2, \tau_1)\)-\(M\_O(X) = - NIL -

\((\tau_1, \tau_2)\)-\(M\_O(X) = - NIL - \quad (\tau_2, \tau_1)\)-\(M\_O(X) = \{\{a, b\}, \{a, b, c\}\}.

i) The set \{a\} is a \((\tau_1, \tau_2)\)-minimal open set but it is not a \(\tau_2\)-open set and the set \{a, b, c\} is an \(\tau_2\)-open set but it is not a \((\tau_1, \tau_2)\)-minimal open set.

ii) The set \{a, b, c\} is a \((\tau_2, \tau_1)\)-maximal open set but it is not a \(\tau_1\)-open set and the set \{a\} is a \(\tau_1\)-open set but it is not a \((\tau_2, \tau_1)\)-maximal open set.

iii) The set \{a, b\} is a \((\tau_1, \tau_2)\)-minimal open set but it is not a \(\tau_1\)-minimal open set and the set \{a, b\} is a \(\tau_2\)-minimal open set but it is not a \((\tau_2, \tau_1)\)-minimal open set.
iv) The set \{a, b\} is a \((\tau_2, \tau_1)\)-maximal open set but it is not a \(\tau_2\)-maximal open set and the set \{a, b\} is a \(\tau_1\)-maximal open set but it is not a \((\tau_1, \tau_2)\)-maximal open set.

v) The set \{a\} is a \((\tau_1, \tau_2)\)-minimal open set but it is not a \((\tau_1, \tau_2)\)-maximal open set and the set \{a, b, c\} is a \((\tau_2, \tau_1)\)-maximal open set but it is not a \((\tau_2, \tau_1)\)-minimal open set.

4.2.7 Remark: From the above discussions and known results we have the following implications.

4.2.8 Remark:

i) Every \((\tau_1, \tau_j)\)-minimal closed set is a \(\tau_j\)-closed set but not conversely.

ii) Every \((\tau_j, \tau_j)\)-maximal closed set is a \(\tau_j\)-closed set but not conversely.

4.2.9 Example: In Example 4.2.4,

\((\tau_1, \tau_2)\)-M\(_j\)C(X) = \{d\}; \quad (\tau_2, \tau_1)\)-M\(_j\)C(X) = \{d\};

\((\tau_1, \tau_2)\)-M\(_a\)C(X) = \{b, c, d\}, \{a, c, d\}\

\((\tau_2, \tau_1)\)-M\(_a\)C(X) = \{b, c, d\}, \{a, b, d\}.

i) The set \{c, d\} is a \(\tau_1\)-closed set but it is not a \((\tau_1, \tau_2)\)-minimal closed set.

ii) The set \{c, d\} is a \(\tau_1\)-closed set but it is not a \((\tau_1, \tau_2)\)-maximal closed set.
4.2.10 Remark:
i) $(\tau_1, \tau_2)$-minimal closed sets and $\tau_1$-closed sets are independent of each other.

ii) $(\tau_1, \tau_2)$-maximal closed sets and $\tau_1$-closed sets are independent of each other.

iii) $(\tau_1, \tau_2)$-minimal closed sets and $\tau_1$-minimal closed sets are independent of each other.

iv) $(\tau_1, \tau_2)$-maximal closed sets and $\tau_1$-maximal closed sets are independent of each other.

v) $(\tau_1, \tau_2)$-minimal closed sets and $(\tau_1, \tau_2)$-maximal closed sets are independent of each other.

4.2.11 Example: In Example 4.2.6,

$\tau_1$-minimal closed sets = \{c, d\}; $\tau_2$-minimal closed sets = \{d\};

$\tau_1$-maximal closed sets = \{b, c, d\}; $\tau_2$-maximal closed sets = \{c, d\};

$(\tau_1, \tau_2)$-$M_{\tau_1}\subset X$ = -NIL- $(\tau_2, \tau_1)$-$M_{\tau_1}\subset X$ = \{d\}, \{c, d\};

$(\tau_1, \tau_2)$-$M_{\tau_1}\subset X$ = \{c, d\}, \{b, c, d\}; $(\tau_2, \tau_1)$-$M_{\tau_1}\subset X$ = -NIL-

i) The set \{d\} is a $(\tau_2, \tau_1)$-minimal closed set but it is not a $\tau_1$-closed set and

the set \{b, c, d\} is a $\tau_1$-closed set but it is not a $(\tau_2, \tau_1)$-minimal closed set.

ii) The set \{b, c, d\} is a $(\tau_1, \tau_2)$-maximal closed set but it is not a $\tau_2$-closed set and the set \{d\} is a $\tau_2$-closed set but it is not a $(\tau_1, \tau_2)$-maximal closed set.

iii) The set \{c, d\} is a $(\tau_2, \tau_1)$-minimal closed set but it is not a $\tau_2$-minimal closed set and the set \{c, d\} is a $\tau_1$-minimal closed set but it is not a $(\tau_1, \tau_2)$-minimal closed set.
iv) The set \( \{c, d\} \) is a \((\tau_1, \tau_2)\)-maximal closed set but it is not a \(\tau_1\)-maximal closed set and the set \( \{c, d\} \) is a \(\tau_2\)-maximal closed set but it is not a \((\tau_2, \tau_1)\)-maximal closed set.

v) The set \( \{d\} \) is a \((\tau_2, \tau_1)\)-minimal closed set but it is not a \((\tau_2, \tau_1)\)-maximal closed set and the set \( \{b, c, d\} \) is a \((\tau_1, \tau_2)\)-maximal closed set but it is not a \((\tau_1, \tau_2)\)-minimal closed set.

4.2.12 Remark: From the above discussions and known results we have the following implications.

[Diagram 4.2]

4.2.13 Theorem: A proper nonempty subset \( U \) of a bitopological space \( X \) is \((\tau_\iota, \tau_j)\)-minimal open set if and only if \( X-U \) is a \((\tau_\iota, \tau_j)\)-maximal closed set

Proof: Let \( U \) be a \((\tau_\iota, \tau_j)\)-minimal open set in \( X \). Suppose \( X-U \) is not a \((\tau_\iota, \tau_j)\)-maximal closed set in \( X \). Then there exists a \(\tau_\iota\)-closed set \( F \) in \( X \) such that \( X-U\subseteq F \). That is \( X-F\subseteq U \) and \( X-F \) is an \(\tau_\iota\)-open set in \( X \). This is contradiction to \( U \) is \((\tau_\iota, \tau_j)\)-minimal open set in \( X \). Therefore \( X-U \) is a \((\tau_\iota, \tau_j)\)-maximal closed set in \( X \).

Conversely, let \( X-U \) be a \((\tau_\iota, \tau_j)\)-maximal closed set in \( X \). Suppose \( U \) is not a \((\tau_\iota, \tau_j)\)-minimal open set in \( X \). Then there exists an \(\tau_\iota\)-open set \( V \) in \( X \).
such that $\phi \neq V \subseteq U$. That is $X-U \subseteq X-V$ and $X-V$ is a $\tau_j$-closed set in $X$. This is contradiction to $X-U$ is $(\tau_i, \tau_j)$-maximal open set in $X$. Therefore $U$ is a $(\tau_i, \tau_j)$-minimal open set in $X$.

4.2.14 Theorem: A proper nonempty subset $U$ of a bitopological space $X$ is $(\tau_i, \tau_j)$-maximal open set if and only if $X-U$ is a $(\tau_i, \tau_j)$-minimal closed set.

Proof: Let $U$ be a $(\tau_i, \tau_j)$-maximal open set in $X$. Suppose $X-U$ is not a $(\tau_i, \tau_j)$-minimal closed set in $X$. Then there exists a $\tau_j$-closed set $F$ in $X$ such that $\phi \neq F \subseteq X-U$. That is $U \subseteq X-F$ and $X-F$ is an $\tau_j$-open set in $X$. This is contradiction to $U$ is $(\tau_i, \tau_j)$-maximal open set in $X$. Therefore $X-U$ is a $(\tau_i, \tau_j)$-minimal closed set in $X$.

Conversely, Let $X-U$ be a $(\tau_i, \tau_j)$-minimal closed set in $X$. Suppose $U$ is not a $(\tau_i, \tau_j)$-maximal open set in $X$. Then there exists an $\tau_j$-open set $V$ in $X$ such that $U \subseteq V \neq X$. That is $X-V \subseteq X-U$ and $X-V$ is a $\tau_j$-closed set in $X$. This is contradiction to $X-U$ is $(\tau_i, \tau_j)$-minimal open set in $X$. Therefore $U$ is a $(\tau_i, \tau_j)$-maximal open set in $X$.

4.2.15 Remark: i) Union of $(\tau_i, \tau_j)$-minimal open (resp. $(\tau_i, \tau_j)$-minimal closed) sets need not be a $(\tau_i, \tau_j)$-minimal open (resp. $(\tau_i, \tau_j)$-minimal closed) set.

ii) Union of $(\tau_i, \tau_j)$-maximal open (resp. $(\tau_i, \tau_j)$-maximal closed) sets need not be a $(\tau_i, \tau_j)$-maximal open (resp. $(\tau_i, \tau_j)$-maximal closed) set.

iii) Intersection of $(\tau_i, \tau_j)$-minimal open (resp. $(\tau_i, \tau_j)$-minimal closed) sets need not be a $(\tau_i, \tau_j)$-minimal open (resp. $(\tau_i, \tau_j)$-minimal closed) set.

iv) Intersection of $(\tau_i, \tau_j)$-maximal open (resp. $(\tau_i, \tau_j)$-maximal closed) sets need not be a $(\tau_i, \tau_j)$-maximal open (resp. $(\tau_i, \tau_j)$-maximal closed) set.
4.2.16 Example: Let $X=\{a, b, c, d\}$ be with topologies $\tau_1=\{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, d\}, X\}$ and $\tau_2=\{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, X\}$ Then

$(\tau_1, \tau_2)-M_1O(X) = \{\{a\}, \{b\}\}$;
$(\tau_2, \tau_1)-M_2O(X) = \{\{a\}, \{b\}, \{c\}\}$;
$(\tau_1, \tau_2)-M_3O(X) = \{\{a, b, c\}, \{a, b, d\}\}$;
$(\tau_2, \tau_1)-M_4O(X) = \{\{a, b, c\}, \{a, b, d\}\}$.

$(\tau_1, \tau_2)-M_1C(X) = \{\{c\}, \{d\}\}$;
$(\tau_2, \tau_1)-M_2C(X) = \{\{c\}, \{d\}\}$;
$(\tau_1, \tau_2)-M_3C(X) = \{\{a, c, d\}, \{b, c, d\}\}$;
$(\tau_2, \tau_1)-M_4C(X) = \{\{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$.

i) The sets $\{a\}, \{b\} \in (\tau_1, \tau_2)-M_1O(X)$, then $\{a\} \cup \{b\} = \{a, b\} \notin (\tau_1, \tau_2)-M_1O(X)$ and $\{c\}, \{d\} \in (\tau_1, \tau_2)-M_2C(X)$, then $\{c\} \cup \{d\} = \{c, d\} \notin (\tau_1, \tau_2)-M_2C(X)$.

ii) The sets $\{a, b, c\}, \{a, b, d\} \in (\tau_1, \tau_2)-M_3O(X)$, then $\{a, b, c\} \cup \{a, b, d\} = X \notin (\tau_1, \tau_2)-M_3O(X)$ and $\{a, c, d\}, \{b, c, d\} \in (\tau_1, \tau_2)-M_4C(X)$, then $\{a, c, d\} \cup \{b, c, d\} = X \notin (\tau_1, \tau_2)-M_4C(X)$.

iii) The sets $\{a\}, \{b\} \in (\tau_1, \tau_2)-M_1O(X)$, then $\{a\} \cap \{b\} = \phi \notin (\tau_1, \tau_2)-M_1O(X)$ and $\{c\}, \{d\} \in (\tau_1, \tau_2)-M_2C(X)$, then $\{c\} \cap \{d\} = \phi \notin (\tau_1, \tau_2)-M_2C(X)$.

ii) The sets $\{a, b, c\}, \{a, b, d\} \in (\tau_1, \tau_2)-M_3O(X)$, then $\{a, b, c\} \cap \{a, b, d\} = \{a, b\} \notin (\tau_1, \tau_2)-M_3O(X)$ and $\{a, c, d\}, \{b, c, d\} \in (\tau_1, \tau_2)-M_4C(X)$, then $\{a, c, d\} \cap \{b, c, d\} = \{c, d\} = \phi \notin (\tau_1, \tau_2)-M_4C(X)$.

4.2.17 Remark:

i) The family of $(\tau_1, \tau_2)-M_1O(X)$ is generally not equal to the family of $(\tau_1, \tau_2)-M_1O(X)$.
ii) The family of \((\tau_i, \tau_j)-M_aO(X)\) is generally not equal to the family of 
\((\tau_i, \tau_j)-M_aO(X)\).

iii) The family of \((\tau_i, \tau_j)-M_aC(X)\) is generally not equal to the family of 
\((\tau_i, \tau_j)-M_aC(X)\).

iv) The family of \((\tau_i, \tau_j)-M_aC(X)\) is generally not equal to the family of 
\((\tau_i, \tau_j)-M_aC(X)\).

4.2.18 Example: Let \(X=\{a, b, c, d\}\) be with topologies \(\tau_1=\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X\) and \(\tau_2=\emptyset, \{a\}, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, c, d\}, X\) Then

\[(\tau_1, \tau_2)-M_aO(X) = \{\{a\}, \{b\}\}; \quad (\tau_1, \tau_2)-M_aC(X) = \{\{c\}, \{d\}\};\]
\[(\tau_2, \tau_1)-M_aO(X) = \{\{a\}, \{c\}\}; \quad (\tau_2, \tau_1)-M_aC(X) = \{\{b\}, \{d\}\};\]
\[(\tau_1, \tau_2)-M_aO(X) = \{\{a, b, c\}, \{a, b, d\}\};
\[(\tau_1, \tau_2)-M_aC(X) = \{\{a, b, c\}, \{a, c, d\}\}.
\[(\tau_2, \tau_1)-M_aC(X) = \{\{a, b, d\}, \{b, c, d\}\}.

i) \((\tau_1, \tau_2)-M_aO(X) \neq (\tau_2, \tau_1)-M_aO(X)\)

ii) \((\tau_1, \tau_2)-M_aC(X) \neq (\tau_2, \tau_1)-M_aC(X)\)

iii) \((\tau_1, \tau_2)-M_aO(X) \neq (\tau_2, \tau_1)-M_aO(X)\)

ii) \((\tau_1, \tau_2)-M_aC(X) \neq (\tau_2, \tau_1)-M_aC(X)\)

4.2.19 Definition: Let \(i, j \in \{1, 2\}\) be the fixed integers. A bitopological 
space \((X, \tau_i, \tau_j)\) is said to be pairwise-\(T_{\min}\) space if every nonempty proper 
\(\tau_i\)-open set is \((\tau_i, \tau_j)\)-minimal open set.

4.2.20 Definition: Let \(i, j \in \{1, 2\}\) be the fixed integers. A bitopological 
space \((X, \tau_i, \tau_j)\) is said to be pairwise-\(T_{\max}\) space if every nonempty proper 
\(\tau_i\)-open set is \((\tau_i, \tau_j)\)-maximal open set.
4.2.21 **Theorem:** A bitopological space \((X, \tau_1, \tau_2)\) is pairwise-\(T_{\min}\) space if and only if every nonempty proper \(\tau_i\)-closed set is \((\tau_i, \tau_j)\)-maximal closed set.

**Proof:** The proof follows from the definition and fact that the complement of \((\tau_i, \tau_j)\)-minimal open set is \((\tau_i, \tau_j)\)-maximal closed set.

4.2.22 **Theorem:** A bitopological space \((X, \tau_1, \tau_2)\) is pairwise-\(T_{\max}\) space if and only if every nonempty proper \(\tau_i\)-closed set is \((\tau_i, \tau_j)\)-minimal closed set.

**Proof:** The proof follows from the definition and fact that the complement of \((\tau_i, \tau_j)\)-maximal open set is \((\tau_i, \tau_j)\)-minimal closed set.

4.3 **Pairwise minimal and pairwise maximal continuous maps.**

4.3.1 **Definition:** Let \(i, j, k, l \in \{1, 2\}\) be fixed integers. Let \((X, \tau_1, \tau_2)\) and \((Y, \sigma_1, \sigma_2)\) be two bitopological spaces. A map \(f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)\) is called

i) \((\sigma_k, \sigma_i)\)-M\(_i\)O(Y)-\(\tau_i\)-continuous if \(f^{-1}(M)\in \tau_i\)-open set in \(X\) for every \(M\in (\sigma_k, \sigma_i)\)-M\(_i\)O(Y).

ii) \((\sigma_k, \sigma_i)\)-M\(_s\)O(Y)-\(\tau_i\)-continuous if \(f^{-1}(M)\in \tau_i\)-open set in \(X\) for every \(M\in (\sigma_k, \sigma_i)\)-M\(_s\)O(Y).

iii) \((\sigma_k, \sigma_i)\)-M\(_i\)O(Y)-(\(\tau_i, \tau_j\))-irresolute if \(f^{-1}(M)\in (\tau_i, \tau_j)\)-M\(_i\)O(X) for every \(M\in (\sigma_k, \sigma_i)\)-M\(_i\)O(Y).

iv) \((\sigma_k, \sigma_i)\)-M\(_s\)O(Y)-(\(\tau_i, \tau_j\))-irresolute if \(f^{-1}(M)\in (\tau_i, \tau_j)\)-M\(_s\)O(X) for every \(M\in (\sigma_k, \sigma_i)\)-M\(_s\)O(Y).

4.3.2 **Remark:** If \(\tau_1=\tau_2=\tau\) and \(\sigma_1=\sigma_2=\sigma\) in the Definition 4.3.1, then the \((\sigma_k, \sigma_i)\)-M\(_i\)O(Y)-\(\tau_i\)-continuous, \((\sigma_k, \sigma_i)\)-M\(_s\)O(Y)-\(\tau_i\)-continuous, \((\sigma_k, \sigma_i)\)-M\(_i\)O(Y)-(\(\tau_i, \tau_j\))-irresolute and \((\sigma_k, \sigma_i)\)-M\(_s\)O(Y)-(\(\tau_i, \tau_j\))-irresolute maps coincide with minimal continuous, maximal continuous, minimal irresolute and maximal irresolute maps respectively.
4.3.3 Theorem: Every $\tau_r$-$\sigma_k$ continuous map is $(\sigma_k, \sigma_i)$-M$_i$O(Y)-$\tau_r$-continuous but not conversely.
Proof: Let $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a $\tau_r$-$\sigma_k$ continuous map. To prove that $f$ is $(\sigma_k, \sigma_i)$-M$_i$O(Y)-$\tau_r$-continuous. Let $N$ be any $(\sigma_k, \sigma_i)$-minimal open set in $Y$. Since every $(\sigma_k, \sigma_i)$-minimal open set is an $\sigma_k$-open set, $N$ is an $\sigma_k$-open set in $Y$. Since $f$ is $\tau_r$-$\sigma_k$ continuous, $f^{-1}(N)$ is an $\tau_r$-open set in $X$. Hence $f$ is a $(\sigma_k, \sigma_i)$-M$_i$O(Y)-$\tau_r$-continuous.

4.3.4 Example: Let $X=Y=\{a, b, c, d\}$ be with topologies $\tau_1=\{\emptyset, \{a\}, \{a, b\}, X\}$, $\tau_2=\{\emptyset, \{a\}, \{a, c\}, X\}$, $\sigma_1=\{\emptyset, \{a\}, \{a, b, c\}, Y\}$ and $\sigma_2=\{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}, Y\}$. Let $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be an identity map. Then $f$ is a $(\sigma_1, \sigma_2)$-M$_i$O(Y)-$\tau_1$-continuous but it is not a $\tau_1$-$\sigma_1$-continuous map, since for the $\sigma_1$-open set $\{a, b, c\}$ in $Y$, $f^{-1}(\{a, b, c\})=\{a, b, c\}$ which is not an $\tau_1$-open set in $X$.

4.3.5 Theorem: Let $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a $(\sigma_k, \sigma_i)$-M$_i$O(Y)-$\tau_r$-continuous, onto map and $(Y, \sigma_1, \sigma_2)$ be a pairwise-T$_{\min}$ space. Then $f$ is a $\tau_r$-$\sigma_k$ continuous.
Proof: Let $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a $(\sigma_k, \sigma_i)$-M$_i$O(Y)-$\tau_r$-continuous, onto map. Note that the inverse image of $\emptyset$ and $Y$ are always $\tau_r$-open sets in a bitopological space $X$. Let $N$ be any nonempty proper $\sigma_k$-open set in $Y$. By hypothesis, $(Y, \sigma_1, \sigma_2)$ is pairwise-T$_{\min}$ space, it follows that $N$ is a $(\sigma_k, \sigma_i)$-minimal open set in $Y$. Since $f$ is $(\sigma_k, \sigma_i)$-M$_i$O(Y)-$\tau_r$-continuous, $f^{-1}(N)$ is an $\tau_r$-open set in $X$. Therefore $f$ is a $\tau_r$-$\sigma_k$ continuous.

4.3.6 Theorem: Every $\tau_r$-$\sigma_k$ continuous map is $(\sigma_k, \sigma_i)$-M$_2$O(Y)-$\tau_r$-continuous but not conversely.
Proof: Similar to that of Theorem 4.3.3.
4.3.7 Example: Let $X=Y=\{a, b, c, d, e\}$ be with topologies $\tau_1=\{\emptyset, \{a\}, \{a, b, c, d\}, X\}$, $\tau_2=\{\空集, \{b\}, \{a, b, c, d\}, X\}$, $\sigma_1=\{\emptyset, \{a, b\}, \{a, b, c, d\}, Y\}$ and $\sigma_2=\{\emptyset, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}, Y\}$. Let $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be an identity map. Then $f$ is a $(\sigma_1, \sigma_2)$-MaO(Y)-$\tau_1$-continuous but it is not a $\tau_1$-$\sigma_1$ continuous map, since for the $\sigma_1$-open set $\{a, b\}$ in $Y$, $f^{-1}(\{a, b\}) =\{a, b\}$ which is not an $\tau_1$-open set in $X$.

4.3.8 Theorem: Let $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a $(\sigma_k, \sigma_i)$-MaO(Y)-$\tau_r$-continuous, onto map and let $(Y, \sigma_1, \sigma_2)$ be a pairwise-$T_{\text{max}}$ space. Then $f$ is a $\tau_r$-$\sigma_k$ continuous.

**Proof:** Similar to that of Theorem 4.3.5.

4.3.9 Remark: $(\sigma_k, \sigma_i)$-MaO(Y)-$\tau_r$-continuous and $(\sigma_k, \sigma_i)$-MaO(Y)-$\tau_r$-continuous maps are independent of each other.

4.3.10 Example: In Example 4.3.4, $f$ is a $(\sigma_1, \sigma_2)$-MaO(Y)-$\tau_1$-continuous but it is not a $(\sigma_1, \sigma_2)$-MaO(Y)-$\tau_1$-continuous. In Example 4.3.7, $f$ is a $(\sigma_1, \sigma_2)$-MaO(Y)-$\tau_1$-continuous but it is not a $(\sigma_1, \sigma_2)$-MaO(Y)-$\tau_1$-continuous.

4.3.11 Theorem: Let $(X, \tau_1, \tau_2)$ and $(Y, \sigma_1, \sigma_2)$ be two bitopological spaces. A map $f: X \rightarrow Y$ is a $(\sigma_k, \sigma_i)$-MaO(Y)-$\tau_r$-continuous if and only if the inverse image of each $(\sigma_k, \sigma_i)$-maximal closed set in $Y$ is a $\tau_r$-closed set in $X$.

**Proof:** The proof follows from the definition and fact that the complement of $(\sigma_k, \sigma_i)$-minimal open set is $(\sigma_k, \sigma_i)$-maximal closed set.

4.3.12 Theorem: Let $(X, \tau_1, \tau_2)$ and $(Y, \sigma_1, \sigma_2)$ be two bitopological spaces. A map $f: X \rightarrow Y$ is a $(\sigma_k, \sigma_i)$-MaO(Y)-$\tau_r$-continuous if and only if the inverse image of each $(\sigma_k, \sigma_i)$-minimal closed set in $Y$ is a $\tau_r$-closed set in $X$.

**Proof:** The proof follows from the definition and fact that the complement of $(\sigma_k, \sigma_i)$-maximal open set is $(\sigma_k, \sigma_i)$-minimal closed set.
4.3.13 Theorem: Let \((X, \tau_1, \tau_2)\) and \((Y, \sigma_1, \sigma_2)\) be two bitopological spaces and \(A\) be a nonempty subset of \(X\). If \(f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)\) is a \((\sigma_k, \sigma_l)\)-\(M_iO(Y)\)-\(\tau_i\)-continuous then the restriction map \(f_A: (A, \tau_{1A}, \tau_{2A}) \to (Y, \sigma_1, \sigma_2)\) is a \((\sigma_k, \sigma_l)\)-\(M_iO(Y)\)-\(\tau_{iA}\)-continuous map.

Proof: Let \(f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)\) is a \((\sigma_k, \sigma_l)\)-\(M_iO(Y)\)-\(\tau_i\)-continuous. To prove \(f_A: (A, \tau_{1A}, \tau_{2A}) \to (Y, \sigma_1, \sigma_2)\) is a \((\sigma_k, \sigma_l)\)-\(M_iO(Y)\)-\(\tau_{iA}\)-continuous map. Let \(N\) be any \((\sigma_k, \sigma_l)\)-minimal open set in \(Y\). Since \(f\) is \((\sigma_k, \sigma_l)\)-\(M_iO(Y)\)-\(\tau_i\)-continuous, \(f^{-1}(N)\) is an \(\tau_i\)-open set in \(X\). By definition of relative topology, \(f_A^{-1}(N)=A \cap f^{-1}(N)\). Therefore \(A \cap f^{-1}(N)\) is an \(\tau_{iA}\)-open set in \(A\). Therefore \(f_A\) is a \((\sigma_k, \sigma_l)\)-\(M_iO(Y)\)-\(\tau_{iA}\)-continuous map.

4.3.14 Theorem: Let \((X, \tau_1, \tau_2)\) and \((Y, \sigma_1, \sigma_2)\) be two bitopological spaces and let \(A\) be a nonempty subset of \(X\). If \(f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)\) is a \((\sigma_k, \sigma_l)\)-\(M_iO(Y)\)-\(\tau_i\)-continuous then the restriction map \(f_A: (A, \tau_{1A}, \tau_{2A}) \to (Y, \sigma_1, \sigma_2)\) is a \((\sigma_k, \sigma_l)\)-\(M_iO(Y)\)-\(\tau_{iA}\)-continuous map.

Proof: Similar to that of Theorem 4.3.13.

4.3.15 Remark: The composition of \((\sigma_k, \sigma_l)\)-\(M_iO(Y)\)-\(\tau_i\)-continuous maps need not be a \((\sigma_k, \sigma_l)\)-\(M_iO(Y)\)-\(\tau_i\)-continuous map.

4.3.16 Example: Let \(X=Y=Z=\{a, b, c, d\}\) be with topologies \(\tau_1=\{\phi, \{a\}, \{a, c\}, X\}, \tau_2=\{\phi, \{a\}, \{a, d\}, X\}\), \(\sigma_1=\{\phi, \{a\}, \{a, b\}, \{a, b, c\}, Y\}\), \(\sigma_2=\{\phi, \{a, b\}, \{a, c\}, \{a, b, c\}, Y\}\), \(\eta_1=\{\phi, \{a, b\}, \{a, b, c\}, Z\}\) and \(\eta_2=\{\phi, \{a, b\}, \{a, c, d\}, Z\}\). Let \(f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)\) and \(g: (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2)\) be the identity maps. Then clearly \(f\) is a \((\sigma_1, \sigma_2)\)-\(M_iO(Y)\)-\(\tau_i\)-continuous and \(g\) is a \((\eta_1, \eta_2)\)-\(M_iO(Y)\)-\(\sigma_i\)-continuous but \(gof: (X, \tau_1, \tau_2) \to (Z, \eta_1, \eta_2)\) is not a \((\eta_1, \eta_2)\)-\(M_iO(Y)\)-\(\tau_i\)-continuous map, since for the
(η₁, η₂)-minimal open set {a, b} in Z, \((gof)^{-1}({a, b})=\{a, b\}\) which is not a \(τ_1\)-open set in X.

**4.3.17 Theorem:** Let \((X, τ_1, τ_2)\), \((Y, σ_1, σ_2)\) and \((Z, η_1, η_2)\) be three bitopological spaces. If \(f: X→Y\) is \(τ_1-σ_k\)-continuous and \(g: Y→Z\) is \((η_m, η_n)-M_0(Y)-σ_k\)-continuous maps, then \(gof: X→Z\) is a \((η_m, η_n)-M_0(Y)-τ_τ\)-continuous.

**Proof:** Let \(N\) be any \((η_m, η_n)\)-minimal open set in Z. Since \(g\) is \((η_m, η_n)-M_0(Y)-σ_k\)-continuous, \(g^{-1}(N)\) is an \(σ_k\)-open set in Y. Again since \(f\) is \(τ_1-σ_k\) continuous, \(f^{-1}(g^{-1}(N))=(gof)^{-1}(N)\) is an \(τ_1\)-open set in X. Hence \(gof\) is a \((η_m, η_n)-M_0(Y)-τ_τ\)-continuous.

**4.3.18 Remark:** The composition of \((σ_k, σ_i)-M_0(Y)-τ_i\)-continuous maps need not be a \((σ_k, σ_i)-M_0(Y)-τ_i\)-continuous map.

**4.3.19 Example:** Let \(X=Y=Z^{\{a, b, c, d\}}\) be with topologies \(τ_1=\{\emptyset, \{a\}, \{a, b, c, d\}\}, \tau_2=\{\emptyset, \{b\}, \{a, b, c, d\}\}, \sigma_1=\{\emptyset, \{a\}, \{a, b, c, d\}\}, \sigma_2=\{\emptyset, \{a\}, \{a, b\}, \{a, b, c, d\}\}, \eta_1=\{\emptyset, \{b\}, \{a, b, c, d\}\}, \eta_2=\{\emptyset, \{a\}, \{a, b\}\}\) and \(η_2=\{\emptyset, \{a\}, \{a, b\}, Z\}\) and \(η_2=\{\emptyset, \{a\}, \{a, b\}, Z\}\).

Let \(f: (X, τ_1, τ_2)→(Y, σ_1, σ_2)\) and \(g: (Y, σ_1, σ_2)→(Z, η_1, η_2)\) be the identity maps. Then clearly \(f\) is a \((σ_1, σ_2)-M_0(Y)-τ_τ\)-continuous and \(g\) is a \((η_1, η_2)-M_0(Y)-σ_1\)-continuous but \(gof: (X, τ_1, τ_2)→(Z, η_1, η_2)\) is not a \((η_1, η_2)-M_0(Y)-τ_τ\)-continuous map, since for the \((η_1, η_2)\)-maximal open set \(\{a, b\}\) in Z, \((gof)^{-1}({a, b})=\{a, b\}\) which is not a \(τ_1\)-open set in X.

**4.3.20 Theorem:** Let \((X, τ_1, τ_2)\), \((Y, σ_1, σ_2)\) and \((Z, η_1, η_2)\) be three bitopological spaces. If \(f: X→Y\) is \(τ_1-σ_k\)-continuous and \(g: Y→Z\) is \((η_m, η_n)-M_0(Y)-σ_k\)-continuous maps, then \(gof: X→Z\) is a \((η_m, η_n)-M_0(Y)-τ_τ\)-continuous.

**Proof:** Similar to that of Theorem 4.3.17.
4.3.21 **Theorem:** Every \((\sigma_k, \sigma_i)-M_iO(Y)-(\tau_i, \tau_j)\)-irresolute map is \((\sigma_k, \sigma_i)-M_iO(Y)-\tau_i\)-continuous but not conversely.

**Proof:** Let \(f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) be a \((\sigma_k, \sigma_i)-M_iO(Y)-(\tau_i, \tau_j)\)-irresolute map. To prove \((\sigma_k, \sigma_i)-M_iO(Y)-\tau_i\)-continuous. Let \(N\) be any \((\sigma_k, \sigma_i)\)-minimal open set in \(Y\). Since \(f\) is \((\sigma_k, \sigma_i)-M_iO(Y)-(\tau_i, \tau_j)\)-irresolute, \(f^{-1}(N)\) is a \((\tau_i, \tau_j)\)-minimal open set in \(X\). Since every \((\tau_i, \tau_j)\)-minimal open set is an \(\tau_i\)-open set, \(f^{-1}(N)\) is an \(\tau_i\)-open set in \(X\). Hence \(f\) is a \((\sigma_k, \sigma_i)-M_iO(Y)-\tau_i\)-continuous.

4.3.22 **Example:** Let \(X=Y=\{a, b, c, d\}\) be with topologies \(\tau_1=\{\phi, \{a, b, c\}\}, \tau_2=\{\phi, \{a, b\}\}, \sigma_1=\{\phi, \{a, b, c\}\}, \sigma_2=\{\phi, \{a, b\}\}\) and \(\sigma_1=\{\phi, \{a, b\}\}, \sigma_2=\{\phi, \{a, b\}\}\). Let \(f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) be an identity map. Then \(f\) is a \((\sigma_1, \sigma_2)-M_iO(Y)-(\tau_i, \tau_j)\)-irresolute but it is not a \((\sigma_1, \sigma_2)-M_iO(Y)-(\tau_i, \tau_j)\)-irresolute map, since for the \((\sigma_1, \sigma_2)\)-minimal open set \(\{a, b\}\) in \(Y\), \(f^{-1}(\{a, b\})=\{a, b\}\) which is not a \((\tau_i, \tau_j)\)-minimal open set in \(X\).

4.3.23 **Theorem:** Let \(f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) be a \((\sigma_k, \sigma_i)-M_iO(Y)-(\tau_i, \tau_j)\)-irresolute, onto map and let \((Y, \sigma_1, \sigma_2)\) be a pairwise-\(T_{\text{min}}\) space. Then \(f\) is a \(\tau_i-\sigma_k\)-continuous.

**Proof:** Proof follows from the Theorems 4.3.21 and 4.3.3.

4.3.24 **Theorem:** Every \((\sigma_k, \sigma_i)-M_iO(Y)-(\tau_i, \tau_j)\)-irresolute map is \((\sigma_k, \sigma_i)-M_iO(Y)-\tau_i\)-continuous but not conversely.

**Proof:** Similar to that of Theorem 4.3.21.

4.3.25 **Example:** Let \(X=Y=\{a, b, c, d\}\) be with topologies \(\tau_1=\{\phi, \{a, b, c\}\}, \tau_2=\{\phi, \{a, b\}\}, \sigma_1=\{\phi, \{a, b, c\}\}, \sigma_2=\{\phi, \{a, b\}\}\). Let \(f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) be an identity map. Then \(f\) is a \((\sigma_1, \sigma_2)-M_iO(Y)-\tau_i\)-continuous but it is not a \((\sigma_1, \sigma_2)-M_iO(Y)-(\tau_i, \tau_j)\)-irresolute.
irresolute map, since for the \((\sigma_1, \sigma_2)\)-maximal open set \(\{a, b\}\) in \(Y\), 
\(f^{-1}(\{a, b\}) = \{a, b\}\) which is not a \((\tau_1, \tau_2)\)-maximal open set in \(X\).

4.3.26 Theorem: Let \(f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) be a \((\sigma_k, \sigma_l)\)-\(M\)\(O(Y)-(\tau_i, \tau_j)\)-irresolute, onto map and let \((Y, \sigma_1, \sigma_2)\) be a pairwise-\(T_{\min}\) space. Then \(f\) is a \(\tau_i\)-\(\sigma_k\)-continuous.

Proof: Proof follows from the Theorems 4.3.24 and 4.3.8.

4.3.27 Remark: \((\sigma_k, \sigma_l)\)-\(M\)\(O(Y)-(\tau_i, \tau_j)\)-irresolute and \(\tau_i\)-\(\sigma_k\) continuous maps are independent of each other.

4.2.28 Example: Let \(X=Y=\{a, b, c, d\}\) be with topologies \(\tau_1=\{\phi, \{a, b\}, \{a, c\}, \{a, b, c\}, X\}\), \(\tau_2=\{\phi, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}, X\}\), \(\sigma_1=\{\phi, \{a, b\}, Y\}\) and \(\sigma_2=\{\phi, \{a, c\}, Y\}\). Let \(f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) be an identity map. Then \(f\) is a \(\tau_1\)-\(\sigma_1\) continuous but it is not a \((\sigma_1, \sigma_2)\)-\(M\)\(O(Y)-(\tau_1, \tau_2)\)-irresolute map, since for the \((\sigma_1, \sigma_2)\)-minimal open set \(\{a, b\}\) in \(Y\), 
\(f^{-1}(\{a, b\}) = \{a, b\}\) which is not a \((\tau_1, \tau_2)\)-minimal open set in \(X\). In Example 4.3.4, \(f\) is a \((\sigma_1, \sigma_2)\)-\(M\)\(O(Y)-(\tau_1, \tau_2)\)-irresolute but it is not a \(\tau_1\)-\(\sigma_1\) continuous map, since for the \(\sigma_1\)-open set \(\{a, b, c\}\) in \(Y\), 
\(f^{-1}(\{a, b, c\}) = \{a, b, c\}\) which is not a \(\tau_1\)-open set in \(X\).

4.3.29 Remark: \((\sigma_k, \sigma_l)\)-\(M\)\(O(Y)-(\tau_i, \tau_j)\)-irresolute and \(\tau_i\)-\(\sigma_k\) continuous maps are independent of each other.

4.3.30 Example: In Example 4.3.28, \(f\) is a \(\tau_1\)-\(\sigma_1\) continuous but it is not a \((\sigma_1, \sigma_2)\)-\(M\)\(O(Y)-(\tau_1, \tau_2)\)-irresolute map, since for the \((\sigma_1, \sigma_2)\)-maximal open set \(\{a, b\}\) in \(Y\), 
\(f^{-1}(\{a, b\}) = \{a, b\}\) which is not a \((\tau_1, \tau_2)\)-maximal open set in \(X\). In Example 4.3.7, \(f\) is a \((\sigma_1, \sigma_2)\)-\(M\)\(O(Y)-(\tau_1, \tau_2)\)-irresolute but it is not a \(\tau_1\)-\(\sigma_1\) continuous map, since for the \(\sigma_1\)-open set \(\{a, b\}\) in \(Y\), 
\(f^{-1}(\{a, b\}) = \{a, b\}\) which is not a \(\tau_1\)-open set in \(X\).
4.2.31 **Remark:** \((\sigma_k, \sigma_i)-M_iO(Y)-(\tau_i, \tau_j)-irresolute\) and \((\sigma_k, \sigma_i)-M_O(Y)-(\tau_i, \tau_j)-irresolute\) maps are independent of each other.

4.2.32 **Example:** In Example 4.3.4, \(f\) is a \((\sigma_1, \sigma_2)-M_iO(Y)-(\tau_1, \tau_2)-irresolute\) but it is not a \((\sigma_1, \sigma_2)-M_O(Y)-(\tau_1, \tau_2)-irresolute\) map. In Example 4.3.7, \(f\) is an \((\sigma_1, \sigma_2)-M_O(Y)-(\tau_1, \tau_2)-irresolute\) but it is not a \((\sigma_1, \sigma_2)-M_O(Y)-(\tau_1, \tau_2)-irresolute\) map.

4.3.33 **Theorem:** Let \((X, \tau_1, \tau_2)\) and \((Y, \sigma_1, \sigma_2)\) be two bitopological spaces. A map \(f: X \rightarrow Y\) is a \((\sigma_k, \sigma_i)-M_iO(Y)-(\tau_i, \tau_j)-irresolute\) if and only if the inverse image of each \((\sigma_k, \sigma_i)-maximal\ closed set in Y\) is a \((\tau_i, \tau_j)-maximal\ closed set in X\).

**Proof:** The proof follows from the definition and fact that the complement of \((\sigma_k, \sigma_i)-minimal\ open set\) is \((\sigma_k, \sigma_i)-maximal\ closed set\).

4.3.34 **Theorem:** Let \((X, \tau_1, \tau_2)\) and \((Y, \sigma_1, \sigma_2)\) be two bitopological spaces. A map \(f: X \rightarrow Y\) is a \((\sigma_k, \sigma_i)-M_iO(Y)-(\tau_i, \tau_j)-irresolute\) if and only if the inverse image of each \((\sigma_k, \sigma_i)-minimal\ closed set in Y\) is a \((\tau_i, \tau_j)-minimal\ closed set in X\).

**Proof:** The proof follows from the definition and fact that the complement of \((\sigma_k, \sigma_i)-maximal\ open set\) is \((\sigma_k, \sigma_i)-minimal\ closed set\).

4.3.35 **Theorem:** Let \((X, \tau_1, \tau_2)\), \((Y, \sigma_1, \sigma_2)\) and \((Z, \eta_1, \eta_2)\) be three bitopological spaces. If \(f: X \rightarrow Y\) is \((\sigma_k, \sigma_i)-M_iO(Y)-(\tau_i, \tau_j)-irresolute\) and \(g: Y \rightarrow Z\) is \((\eta_m, \eta_i)-M_iO(Z)-(\sigma_k, \sigma_i)-irresolute\) maps, then \(gof: X \rightarrow Z\) is a \((\eta_m, \eta_i)-M_O(Z)-(\tau_i, \tau_j)-irresolute\).

**Proof:** Let \(N\) be any \((\eta_m, \eta_i)-minimal\ open set in Z\). Since \(g\) is \((\eta_m, \eta_i)-M_iO(Z)-(\sigma_k, \sigma_i)-irresolute\), \(g^{-1}(N)\) is a \((\sigma_k, \sigma_i)-minimal\ open set in Y\). Again since \(f\) is \((\sigma_k, \sigma_i)-M_iO(Y)-(\tau_i, \tau_j)-irresolute\), \(f^{-1}(g^{-1}(N))=(gof)^{-1}(N)\) is a
(τ₁, τ₂)-minimal open set in X. Therefore gof is a (ηₘ, ηₙ)-MₐO(Z)-(τ₁, τ₂)-irresolute.

4.3.36 Theorem: Let (X, τ₁, τ₂), (Y, σ₁, σ₂) and (Z, η₁, η₂) be three bitopological spaces. If f: X→Y is (σₖ, σ₀)-MₐO(Y)-(τ₁, τ₂)-irresolute and g: Y→Z is (ηₘ, ηₙ)-MₐO(Z)-(σₖ, σ₁)-irresolute maps, then gof: X→Z is a (ηₘ, ηₙ)-MₐO(Z)-(τ₁, τ₂)-irresolute.

Proof: Similar to that of Theorem 4.3.35.

4.3.37 Definition: Let i, j, k, l∈ {1, 2} be fixed integers. Let (X, τ₁, τ₂) and (Y, σ₁, σ₂) be two bitopological spaces. A map f: (X, τ₁, τ₂) → (Y, σ₁, σ₂) is called

i) minimal bi-continuous if (σ₁, σ₂)-MₐO(Y)-τ₁-continuous and (σ₂, σ₁)-MₐO(Y)-τ₂-continuous.

ii) maximal bi-continuous if (σ₁, σ₂)-MₐO(Y)-τ₁-continuous and (σ₂, σ₁)-MₐO(Y)-τ₂-continuous.

iii) minimal bi-irresolute if (σ₁, σ₂)-MₐO(Y)-(τ₁, τ₂)-irresolute and (σ₂, σ₁)-MₐO(Y)-(τ₂, τₙ)-irresolute.

iv) maximal bi-irresolute if (σ₁, σ₂)-MₐO(Y)-(τ₁, τ₂)-irresolute and (σ₂, σ₁)-MₐO(Y)-(τ₂, τ₁)-irresolute.

4.3.38 Theorem: Let f: (X, τ₁, τ₂) → (Y, σ₁, σ₂) be a map.

i) If f is bi-continuous then f is minimal bi-continuous.

ii) If f is bi-continuous then f is maximal bi-continuous.

iii) If f is minimal bi-irresolute then f is minimal bi-continuous.

iv) If f is maximal bi-irresolute then f is maximal bi-continuous.

Proof: i) Let f: (X, τ₁, τ₂) → (Y, σ₁, σ₂) be a bi-continuous map. Therefore by definition, f is τ₁-σ₁ continuous and τ₂-σ₂ continuous and so by Theorem
4.3.3, \( f \) is \((\sigma_1, \sigma_2)\)-\(M_1O(Y)\)-\(\tau_1\)-continuous and \((\sigma_2, \sigma_1)\)-\(M_1O(Y)\)-\(\tau_2\)-continuous. Thus, \( f \) is a minimal bi-continuous.

ii) Similar to (i), using Theorem 4.3.6.

iii) Let \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be a minimal bi-irresolute map. Therefore by definition, \( f \) is \((\sigma_1, \sigma_2)\)-\(M_1O(Y)\)-(\(\tau_1\), \(\tau_2\))-irresolute and \((\sigma_2, \sigma_1)\)-\(M_1O(Y)\)-(\(\tau_2\), \(\tau_1\))-irresolute and so by Theorem 4.3.21, \( f \) is \((\sigma_1, \sigma_2)\)-\(M_1O(Y)\)-\(\tau_1\)-continuous and \((\sigma_2, \sigma_1)\)-\(M_1O(Y)\)-\(\tau_2\)-continuous. Thus, \( f \) is a minimal bi-continuous.

iv) Similar to (iii), using Theorem 4.3.24.

4.3.39 Definition: Let \( i, j, k, l \in \{1, 2\} \) be fixed integers. Let \((X, \tau_1, \tau_2)\) and \((Y, \sigma_1, \sigma_2)\) be two bitopological spaces. A map \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is called

i) minimal-s-bi-continuous if minimal bi-continuous, \((\sigma_1, \sigma_2)\)-\(M_1O(Y)\)-\(\tau_2\)-continuous and \((\sigma_2, \sigma_1)\)-\(M_1O(Y)\)-\(\tau_1\)-continuous.

ii) maximal-s-bi-continuous if maximal bi-continuous, \((\sigma_1, \sigma_2)\)-\(M_1O(Y)\)-\(\tau_2\)-continuous and \((\sigma_2, \sigma_1)\)-\(M_1O(Y)\)-\(\tau_1\)-continuous.

iii) minimal-s-bi-irresolute if minimal bi-irresolute, \((\sigma_1, \sigma_2)\)-\(M_1O(Y)\)-(\(\tau_2\), \(\tau_1\))-irresolute and \((\sigma_2, \sigma_1)\)-\(M_1O(Y)\)-(\(\tau_1\), \(\tau_2\))-irresolute.

iv) maximal-s-bi-irresolute if maximal bi-irresolute, \((\sigma_1, \sigma_2)\)-\(M_1O(Y)\)-(\(\tau_2\), \(\tau_1\))-irresolute and \((\sigma_2, \sigma_1)\)-\(M_1O(Y)\)-(\(\tau_1\), \(\tau_2\))-irresolute.

From the definitions we have the following results.

4.3.40 Theorem: Let \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be a map.

i) If \( f \) is minimal-s-bi-continuous then \( f \) is minimal bi-continuous.

ii) If \( f \) is maximal-s-bi-continuous then \( f \) is maximal bi-continuous.

iii) If \( f \) is minimal-s-bi-irresolute then \( f \) is minimal bi-irresolute.
iv) If \( f \) is maximal-s-bi-irresolute then \( f \) is maximal bi-irresolute

4.3.41 Remark: Let \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be a map. Then we get the following diagram.

\[
\begin{align*}
\text{(\( \sigma_k \), \( \sigma_l \))-\text{M}_i \text{O}(Y)-\tau_i \text{-continuous}} & \quad \leftrightarrow \quad \text{(\( \sigma_k \), \( \sigma_l \))-\text{M}_a \text{O}(Y)-\tau_i \text{-continuous}} \\
\text{\( \tau_i \)-\sigma_k \text{-continuous}} & \quad \leftrightarrow \quad \text{\( \tau_i \)-\sigma_k \text{-continuous}} \\
\text{(\( \sigma_k \), \( \sigma_l \))-\text{M}_i \text{O}(Y)-(\tau_i, \tau_j \)-irresolute} & \quad \leftrightarrow \quad \text{(\( \sigma_k \), \( \sigma_l \))-\text{M}_a \text{O}(Y)-(\tau_i, \tau_j \)-s-irresolute}
\end{align*}
\]

Diagram 4.3

4.4 Pairwise minimal open and pairwise maximal open maps.

5.4.1 Definition: Let \( i, j, k, l \in \{1, 2\} \) be fixed integers. Let \((X, \tau_i, \tau_j)\) and \((Y, \sigma_1, \sigma_2)\) be two bitopological spaces. A map \( f: (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_1, \sigma_2) \) is called

i) \( \tau_i \)-\( \sigma_k \) open if \( f(M) \) is \( \sigma_k \)-open set in \( Y \) for every \( \tau_i \)-open set \( M \) in \( X \).

ii) \( (\tau_i, \tau_j) \)-\text{M}_i \text{O}(X)-\sigma_k \)-open if \( f(M) \) is \( \sigma_k \)-open set in \( Y \) for every \( M \in (\tau_i, \tau_j) \)-\text{M}_i \text{O}(X) \).

iii) \( (\tau_i, \tau_j) \)-\text{M}_a \text{O}(X)-\sigma_k \)-open if \( f(M) \) is \( \sigma_k \)-open set in \( Y \) for every \( M \in (\tau_i, \tau_j) \)-\text{M}_a \text{O}(X) \).

iv) \( (\tau_i, \tau_j) \)-\text{M}_i \text{O}(X)-(\sigma_k, \sigma_l \)-\text{-strongly open if \( f(M) \in (\sigma_k, \sigma_l) \)-\text{M}_i \text{O}(Y) \) for every \( M \in (\tau_i, \tau_j) \)-\text{M}_i \text{O}(X) \).

v) \( (\tau_i, \tau_j) \)-\text{M}_a \text{O}(X)-(\sigma_k, \sigma_l \)-\text{-strongly open if \( f(M) \in (\sigma_k, \sigma_l) \)-\text{M}_a \text{O}(Y) \) for every \( M \in (\tau_i, \tau_j) \)-\text{M}_a \text{O}(X) \).

4.4.2 Remark: If \( \tau_1=\tau_2=\tau \) and \( \sigma_1=\sigma_2=\sigma \) in the Definition 4.4.1, then the \( \tau_i \)-\( \sigma_k \)-open, \( (\tau_i, \tau_j) \)-\text{M}_i \text{O}(X)-\sigma_k \)-open, \( (\tau_i, \tau_j) \)-\text{M}_a \text{O}(X)-\sigma_k \)-open, \( (\tau_i, \tau_j) \)-\text{M}_i \text{O}(X)-(\sigma_k, \sigma_l \)-\text{-strongly open and \( (\tau_i, \tau_j) \)-\text{M}_a \text{O}(X)-(\sigma_k, \sigma_l \)-\text{-strongly open

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maps coincide with open, minimal open, maximal open, strongly minimal open and strongly maximal open maps respectively.

4.4.3 Theorem: Every $\tau_i$-$\sigma_k$ open map is $\left(\tau_i, \tau_j\right)$-M$_4$O(X)-$\sigma_k$-open map but not conversely.

Proof: Let $f: (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_1, \sigma_2)$ be an $\tau_i$-$\sigma_k$-open map and $N$ be any $(\tau_i, \tau_j)$-minimal open set in $X$. Since every $(\tau_i, \tau_j)$-minimal open set is an $\tau_i$-open set, $N$ is an $\tau_i$-open set in $X$. Since $f$ is $\tau_i$-$\sigma_k$ open map, $f(N)$ is an $\sigma_k$-open set in $Y$. Hence $f$ is a $(\tau_i, \tau_j)$-M$_4$O(X)-$\sigma_k$-open map.

4.4.4 Example: Let $X=Y=\{a, b, c, d\}$ be with topologies $\tau_1=\{\phi, \{a\}, \{a, b\}, X\}$, $\tau_2=\{\phi, \{a\}, \{a, c\}, X\}$, $\sigma_1=\{\phi, \{a\}, \{a, b, c\}, Y\}$ and $\sigma_2=\{\phi, \{a\}, \{a, b, d\}, Y\}$. Let $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be an identity map. Then $f$ is a $(\tau_1, \tau_2)$-M$_4$O(X)-$\sigma_1$-open but it is not a $\tau_i$-$\sigma_1$-open map, since for the $\tau_i$-open set $\{a, b\}$ in $X$, $f(\{a, b\})=\{a, b\}$ which is not an $\sigma_1$-open set in $Y$.

4.4.5 Theorem: Every $\tau_i$-$\sigma_k$ open map is $\left(\tau_i, \tau_j\right)$-M$_4$O(X)-$\sigma_k$-open map but not conversely.

Proof: Similar to that of Theorem 4.4.3.

4.4.6 Example: Let $X=Y=\{a, b, c, d\}$ be with topologies $\tau_1=\{\phi, \{a\}, \{a, b, c\} X\}$, $\tau_2=\{\phi, \{b\}, \{a, b, c\}, X\}$, $\sigma_1=\{\phi, \{a, b\}, \{a, b, c\}, Y\}$ and $\sigma_2=\{\phi, \{a, c\}, \{a, b, c\}, Y\}$. Let $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be an identity map. Then $f$ is a $(\tau_1, \tau_2)$-M$_4$O(X)-$\sigma_1$-open but it is not a $\tau_i$-$\sigma_1$ open map, since for the $\tau_i$-open set $\{a\}$ in $X$, $f(\{a\})=\{a\}$ which is not an $\sigma_1$-open set in $Y$.

4.4.7 Remark: $(\tau_i, \tau_j)$-M$_4$O(X)-$\sigma_k$-open and $(\tau_i, \tau_j)$-M$_4$O(X)-$\sigma_k$-open maps are independent of each other.
4.4.8 Example: In Example 4.4.4, \( f \) is a \((\tau_1, \tau_2)\)-\(M_1O(X)\)-\(\sigma_1\)-open but it is not a \((\tau_1, \tau_2)\)-\(M_2O(X)\)-\(\sigma_1\)-open map. In Example 4.4.6, \( f \) is a \((\tau_1, \tau_2)\)-\(M_3O(X)\)-\(\sigma_1\)-open but it is not a \((\tau_1, \tau_2)\)-\(M_4O(X)\)-\(\sigma_1\)-open map.

4.4.9 Theorem: A map \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is a \((\tau_1, \tau_2)\)-\(M_1O(X)\)-\(\sigma_k\)-open if and only if for any subset \( S \) in \( Y \) and each \((\tau_1, \tau_2)\)-maximal closed set \( M \) in \( X \) containing \( f^{-1}(S) \), there is a \( \sigma_k \)-closed set \( W \) in \( Y \) such that \( S \subseteq W \) and \( f^{-1}(W) \subseteq M \).

Proof: Suppose \( f \) is a \((\tau_1, \tau_2)\)-\(M_1O(X)\)-\(\sigma_k\)-open map. Let \( S \) be any subset in \( Y \) and \( M \) is a \((\tau_1, \tau_2)\)-maximal closed set in \( X \) such that \( f^{-1}(S) \subseteq M \). Then \( W = Y - f(X - M) \) is a \( \sigma_k \)-closed set in \( Y \) containing \( S \) such that \( f^{-1}(W) \subseteq M \).

Conversely, suppose that \( N \) is a \((\tau_1, \tau_2)\)-minimal open set in \( X \). Then \( f^{-1}(Y - f(N)) \subseteq X - N \) and \( X - N \) is a \((\tau_1, \tau_2)\)-maximal closed set in \( X \). By hypothesis, there is a \( \sigma_k \)-closed set \( W \) in \( Y \) such that \( Y - f(N) \subseteq W \) and \( f^{-1}(W) \subseteq X - N \). Therefore \( N \subseteq X - f^{-1}(W) \). Hence \( Y - W \subseteq f(N) \subseteq f(X - f^{-1}(W)) \subseteq Y - W \) which implies \( f(N) = Y - W \). Since \( Y - W \) is an \( \sigma_k \)-open set in \( Y \). Therefore \( f(N) \) is an \( \sigma_k \)-open set in \( Y \). Hence \( f \) is a \((\tau_1, \tau_2)\)-\(M_1O(X)\)-\(\sigma_k\)-open map.

4.4.10 Theorem: A map \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is a \((\tau_1, \tau_2)\)-\(M_4O(X)\)-\(\sigma_k\)-open if and only if for any subset \( S \) in \( Y \) and each \((\tau_1, \tau_2)\)-minimal closed set \( N \) in \( X \) containing \( f^{-1}(S) \), there is a \( \sigma_k \)-closed set \( W \) in \( Y \) such that \( S \subseteq W \) and \( f^{-1}(W) \subseteq N \).

Proof: Similar to that of Theorem 4.4.9.

4.4.11 Remark: The composition of \((\tau_1, \tau_2)\)-\(M_4O(X)\)-\(\sigma_k\)-open maps is need not be a \((\tau_1, \tau_2)\)-\(M_4O(X)\)-\(\sigma_k\)-open map.
4.4.12 Example: Let \( X=Y=Z=\{a, b, c, d\} \) be with topologies \( \tau_1=\{\phi, \{a, c\}, \{a, b, c\}, X\} \), \( \tau_2=\{\phi, \{a, c\}, \{a, c, d\}, X\} \), \( \sigma_1=\{\phi, \{a\}, \{a, c\}, \{a, b, c\}, Y\} \), \( \sigma_2=\{\phi, \{a\}, \{a, c\}, \{a, c, d\}, Y\} \), \( \eta_1=\{\phi, \{a\}, \{a, b\}, Z\} \) and \( \eta_2=\{\phi, \{a\}, \{a, b\}, Z\} \). Let \( f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) and \( g: (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2) \) be the identity maps. Then clearly \( f \) is a \( (\tau_1, \tau_2)\)-\( M_1O(X)\)-\( \sigma_1 \)-open and \( g \) is a \( (\sigma_1, \sigma_2)\)-\( M_1O(Y)\)-\( \eta_1 \)-open maps but \( gof: (X, \tau_1, \tau_2) \to (Z, \eta_1, \eta_2) \) is not a \( (\tau_1, \tau_2)\)-\( M_1O(X)\)-\( \eta_1 \)-open map, since for the \( (\tau_1, \tau_2)\)-minimal open set \( \{a, c\} \) in \( X \), \( (gof)(\{a, c\})=\{a, c\} \) which is not an \( \eta_1 \)-open set in \( Z \).

4.4.13 Theorem: Let \( f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) be a \( (\tau_1, \tau_2)\)-\( M_1O(X)\)-\( \sigma_k \)-open and \( g: (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2) \) be an \( \sigma_k\)-\( \eta_m \)-open maps. Then \( gof: (X, \tau_1, \tau_2) \to (Z, \eta_1, \eta_2) \) is a \( (\tau_1, \tau_2)\)-\( M_1O(X)\)-\( \eta_m \)-open map.

Proof: Let \( N \) be any \( (\tau_1, \tau_2)\)-minimal open set in \( X \). Since \( f \) is \( (\tau_1, \tau_2)\)-\( M_1O(X)\)-\( \sigma_k \)-open, \( f(N) \) is an \( \sigma_k \)-open set in \( Y \). Again since \( g \) is an \( \sigma_k\)-\( \eta_m \)-open, \( g(f(N))=(gof)(N) \) is an \( \eta_m \)-open set in \( Z \). Hence \( gof \) is a \( (\tau_1, \tau_2)\)-\( M_1O(X)\)-\( \eta_m \)-open map.

4.4.14 Remark: The composition of \( (\tau_1, \tau_2)\)-\( M_1O(X)\)-\( \sigma_k \)-open maps is need not be a \( (\tau_1, \tau_2)\)-\( M_1O(X)\)-\( \sigma_k \)-open map.

4.4.15 Example: Let \( X=Y=Z=\{a, b, c, d\} \) be with topologies \( \tau_1=\{\phi, \{a\}, \{a, b\}, X\} \), \( \tau_2=\{\phi, \{a\}, \{a, b\}, X\} \), \( \sigma_1=\{\phi, \{a\}, \{a, b\}, \{a, c\}, Y\} \), \( \sigma_2=\{\phi, \{a\}, \{a, b\}, \{a, c\}, Y\} \), \( \eta_1=\{\phi, \{a\}, \{a, b, c\}, Z\} \) and \( \eta_2=\{\phi, \{a\}, \{a, b, c\}, Z\} \). Let \( f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) and \( g: (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2) \) be the identity maps. Then clearly \( f \) is a \( (\tau_1, \tau_2)\)-\( M_1O(X)\)-\( \sigma_1 \)-open and \( g \) is a \( (\sigma_1, \sigma_2)\)-\( M_1O(Y)\)-\( \eta_1 \)-open maps but \( gof: (X, \tau_1, \tau_2) \to (Z, \eta_1, \eta_2) \) is not a \( (\tau_1, \tau_2)\)-\( M_1O(X)\)-\( \eta_1 \)-open map, since for the \( (\tau_1, \tau_2)\)-maximal open set \( \{a, b\} \) in \( X \), \( (gof)(\{a, b\})=\{a, b\} \) which is not an \( \eta_1 \)-open set in \( Z \).
4.4.16 **Theorem:** Let \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be a \((\tau_1, \tau_2)\)-Mi\(O\)(X)-\(\sigma_k\)-open and \( g: (Y, \sigma_1, \sigma_2) \rightarrow (Z, \eta_1, \eta_2) \) be an \(\sigma_k\)-\(\eta_m\)-open maps. Then \( gof: (X, \tau_1, \tau_2) \rightarrow (Z, \eta_1, \eta_2) \) is a \((\tau_1, \tau_2)\)-Mi\(O\)(X)-\(\eta_m\)-open map.

**Proof:** Similar to that of Theorem 4.4.13.

4.4.17 **Theorem:** Let \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) and \( g: (Y, \sigma_1, \sigma_2) \rightarrow (Z, \eta_1, \eta_2) \) be the maps and let \( gof: (X, \tau_1, \tau_2) \rightarrow (Z, \eta_1, \eta_2) \) be a \((\tau_1, \tau_2)\)-Mi\(O\)(X)-\(\eta_m\)-open map. Then

i) \( g \) is \((\sigma_k, \sigma_1)\)-Mi\(O\)(X)-\(\eta_m\)-open if \( f \) is \((\sigma_k, \sigma_1)\)-Mi\(O\)(Y)-\(\tau_1\)-irresolute and surjective.

ii) \( f \) is \((\tau_1, \tau_2)\)-Mi\(O\)(X)-\(\sigma_k\)-open if \( g \) is \(\sigma_k\)-\(\eta_m\)-continuous and injective.

**Proof:** i) Let \( N \) be any \((\sigma_k, \sigma_1)\)-minimal open set in \( Y \). Since \( f \) is \((\sigma_k, \sigma_1)\)-Mi\(O\)(Y)-\(\tau_1\)-irresolute and surjective, \( f^{-1}(N) \) is a \((\tau_1, \tau_2)\)-minimal open set in \( X \). Again since \( gof \) is \((\tau_1, \tau_2)\)-Mi\(O\)(X)-\(\eta_m\)-open, \( (gof)(f^{-1}(N)) \) is an \(\eta_m\)-open set in \( Z \). But \( (gof)(f^{-1}(N))=g(f^{-1}(N))=g(N) \) is an \(\eta_m\)-open set in \( Z \). Hence \( g \) is a \((\sigma_k, \sigma_1)\)-Mi\(O\)(X)-\(\eta_m\)-open map.

ii) Let \( N \) be any \((\tau_1, \tau_2)\)-minimal open set in \( X \). Since \( gof \) is \((\tau_1, \tau_2)\)-Mi\(O\)(X)-\(\eta_m\)-open, \( (gof)(N) \) is an \(\eta_m\)-open set in \( Z \). Again since \( g \) is \(\sigma_k\)-\(\eta_m\)-continuous, \( g^{-1}((gof)(N)) \) is an \(\sigma_k\)-open set in \( Y \). But \( g^{-1}((gof)(N))=g^{-1}(g(f(N))=f(N) \) is an \(\sigma_k\)-open set in \( Y \). Hence \( f \) is a \((\tau_1, \tau_2)\)-Mi\(O\)(X)-\(\sigma_k\)-open map.

4.4.18 **Theorem:** Let \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) and \( g: (Y, \sigma_1, \sigma_2) \rightarrow (Z, \eta_1, \eta_2) \) be the maps and let \( gof: (X, \tau_1, \tau_2) \rightarrow (Z, \eta_1, \eta_2) \) be a \((\tau_1, \tau_2)\)-Mi\(O\)(X)-\(\eta_m\)-open map. Then

i) \( g \) is \((\sigma_k, \sigma_1)\)-Mi\(O\)(X)-\(\eta_m\)-open if \( f \) is \((\sigma_k, \sigma_1)\)-Mi\(O\)(Y)-\(\tau_1\)-irresolute and surjective.

ii) \( f \) is \((\tau_1, \tau_2)\)-Mi\(O\)(X)-\(\sigma_k\)-open if \( g \) is \(\sigma_k\)-\(\eta_m\)-continuous and injective.
Proof: Similar to that of Theorem 4.4.17.

4.4.19 Theorem: Every \((\tau_1, \tau_2)-M_0(X)-(\sigma_1, \sigma_2)\)-strongly open map is \((\tau_1, \tau_2)-M_0(X)-(\sigma_1, \sigma_2)\)-open map but not conversely.

Proof: Let \(f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)\) be a \((\tau_1, \tau_2)-M_0(X)-(\sigma_1, \sigma_2)\)-strongly open map and let \(N\) be any \((\tau_1, \tau_2)\)-minimal open set in \(X\). Since \(f\) is \((\tau_1, \tau_2)-M_0(X)-(\sigma_1, \sigma_2)\)-strongly open, \(f(N)\) is a \((\sigma_1, \sigma_2)\)-minimal open set in \(Y\). Since every \((\sigma_1, \sigma_2)\)-minimal open is an \(\sigma_k\)-open set, \(f(N)\) is an \(\sigma_k\)-open set in \(Y\). Hence \(f\) is a \((\tau_1, \tau_2)-M_0(X)-(\sigma_1, \sigma_2)\)-open map.

4.4.20 Example: Let \(X=Y=\{a, b, c, d\}\) be with topologies \(\tau_1=\{\phi, \{a, b\}, \{a, b, c\}, X\}\), \(\tau_2=\{\phi, \{b\}, \{a, b, d\}, X\}\), \(\sigma_1=\{\phi, \{a\}, \{a, b\}, Y\}\) and \(\sigma_2=\{\phi, \{b\}, \{a, b\}, Y\}\). Let \(f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)\) be an identity map. Then \(f\) is a \((\tau_1, \tau_2)-M_0(X)-(\sigma_1, \sigma_2)\)-open map but it is not a \((\tau_1, \tau_2)-M_0(X)-(\sigma_1, \sigma_2)\)-strongly open map, since for the \((\tau_1, \tau_2)\)-minimal open set \(\{a, b\}\) in \(X\), \(f(\{a, b\})=\{a, b\}\) which is not a \((\sigma_1, \sigma_2)\)-minimal open set in \(Y\).

4.4.21 Theorem: Every \((\tau_1, \tau_2)-M_0(X)-(\sigma_1, \sigma_2)\)-strongly open map is \((\tau_1, \tau_2)-M_0(X)-(\sigma_1, \sigma_2)\)-open map but not conversely.

Proof: Similar to that of Theorem 4.4.19.

4.4.22 Example: Let \(X=Y=\{a, b, c, d\}\) be with topologies \(\tau_1=\{\phi, \{a\}, \{a, b\}\) \(X\}\), \(\tau_2=\{\phi, \{b\}, \{a, b, d\}, X\}\), \(\sigma_1=\{\phi, \{a, b\}, \{a, b, c\}, Y\}\) and \(\sigma_2=\{\phi, \{a, b\}, \{a, b, d\}, Y\}\). Let \(f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)\) be an identity map. Then \(f\) is a \((\tau_1, \tau_2)-M_0(X)-(\sigma_1, \sigma_2)\)-open map but it is not a \((\tau_1, \tau_2)-M_0(X)-(\sigma_1, \sigma_2)\)-strongly open map, since for the \((\tau_1, \tau_2)\)-maximal open set \(\{a, b\}\) in \(X\), \(f(\{a, b\})=\{a, b\}\) which is not a \((\sigma_1, \sigma_2)\)-maximal open set in \(Y\).

4.4.23 Remark: \((\tau_1, \tau_2)-M_0(X)-(\sigma_1, \sigma_2)\)-strongly open and \(\tau_1-\sigma_k\) open maps are independent of each other.
4.4.24 Example: Let $X=Y=\{a, b, c, d\}$ be with topologies $\tau_1=\{\emptyset, \{a, b\}, X\}$, $\tau_2=\{\emptyset, \{a, c\}, X\}$, $\sigma_1=\{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, Y\}$ and $\sigma_2=\{\emptyset, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}, Y\}$. Let $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be an identity map. Then $f$ is a $\tau_1$-$\sigma_1$ open but it is not a $(\tau_1, \tau_2)$-$M_0(X)$-$(\sigma_1, \sigma_2)$-strongly open map, since for the $(\tau_1, \tau_2)$-minimal open set $\{a, b\}$ in $X$, $f(\{a, b\}) = \{a, b\}$ which is not a $(\sigma_1, \sigma_2)$-minimal open set in $Y$. In Example 4.4.4, $f$ is a $(\tau_1, \tau_2)$-$M_0(X)$-$(\sigma_1, \sigma_2)$-strongly open but it is not a $\tau_1$-$\sigma_1$ open map, since for the $\tau_1$-open set $\{a, b\}$ in $X$, $f(\{a, b\})=\{a, b\}$ which is not an $\sigma_1$-open set in $Y$.

4.4.25 Remark: $(\tau_1, \tau_2)$-$M_0(X)$-$(\sigma_k, \sigma_i)$-strongly open and $\tau_1$-$\sigma_k$ open maps are independent of each other.

4.4.26 Example: In Example 4.4.25, $f$ is a $\tau_1$-$\sigma_1$ open but it is not a $(\tau_1, \tau_2)$-$M_0(X)$-$(\sigma_1, \sigma_2)$-strongly open map, since for the $(\tau_1, \tau_2)$-maximal open set $\{a, b\}$ in $X$, $f(\{a, b\})=\{a, b\}$ which is not a $(\sigma_1, \sigma_2)$-maximal open set in $Y$. In Example 4.4.6, $f$ is a $(\tau_1, \tau_2)$-$M_0(X)$-$(\sigma_1, \sigma_2)$-strongly open but it is not a $\tau_1$-$\sigma_1$ open map, since for the $\tau_1$-open set $\{a\}$ in $X$, $f(\{a\})=\{a\}$ which is not an $\sigma_1$-open set in $Y$.

4.4.27 Remark: $(\tau_1, \tau_2)$-$M_0(X)$-$(\sigma_k, \sigma_i)$-strongly open and $(\tau_1, \tau_2)$-$M_0(X)$-$(\sigma_k, \sigma_i)$-strongly open maps are independent of each other.

4.4.28 Example: In Example 4.4.4, $f$ is a $(\tau_1, \tau_2)$-$M_0(X)$-$(\sigma_1, \sigma_2)$-strongly open but it is not a $(\tau_1, \tau_2)$-$M_0(X)$-$(\sigma_1, \sigma_2)$-strongly open map. In Example 4.4.6, $f$ is a $(\tau_1, \tau_2)$-$M_0(X)$-$(\sigma_1, \sigma_2)$-strongly open but it is not a $(\tau_1, \tau_2)$-$M_0(X)$-$(\sigma_1, \sigma_2)$-strongly open map.

4.4.29 Theorem: A map $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a $(\tau_1, \tau_2)$-$M_0(X)$-$(\sigma_k, \sigma_i)$-strongly open if and only if for any subset $S$ in $Y$ and each $(\tau_1, \tau_2)$-
maximal closed set \( M \) in \( X \) containing \( f^{-1}(S) \), there is a \((\sigma_k, \sigma)\)-maximal closed set \( W \) in \( Y \) containing \( S \) such that \( f^{-1}(W) \subseteq M \).

**Proof:** Suppose \( f \) is a \((\tau_i, \tau_j)\)-\(M_\alpha(X)-(\sigma_k, \sigma)\)-strongly open map. Let \( S \) be a subset in \( Y \) and \( M \) is a \((\tau_i, \tau_j)\)-maximal closed set in \( X \) such that \( f^{-1}(S) \subseteq M \). Then \( W = Y - f(X - M) \) is a \((\sigma_k, \sigma)\)-maximal closed set in \( Y \) containing \( S \) such that \( f^{-1}(W) \subseteq M \).

Conversely, suppose that \( N \) is a \((\tau_i, \tau_j)\)-minimal open set in \( X \). Then \( f^{-1}(Y - f(N)) \subseteq X - N \) and \( X - N \) is a \((\tau_i, \tau_j)\)-maximal closed set in \( X \). By hypothesis, there is a \((\sigma_k, \sigma)\)-maximal closed set \( W \) in \( Y \) such that \( Y - f(N) \subseteq W \) and \( f^{-1}(W) \subseteq X - N \). Therefore \( N \cap X - f^{-1}(W) \subseteq X - W \) which implies \( f(N) = Y - W \). Since \( Y - W \) is a \((\sigma_k, \sigma)\)-minimal open set in \( Y \). Therefore \( f(N) \) is a \((\sigma_k, \sigma)\)-minimal open set in \( Y \). Hence \( f \) is a \((\tau_i, \tau_j)\)-\(M_\alpha(X)-(\sigma_k, \sigma)\)-strongly open map.

**4.4.30 Theorem:** A map \( f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is a \((\tau_i, \tau_j)\)-\(M_\alpha(X)-(\sigma_k, \sigma)\)-strongly open if and only if for any subset \( S \) in \( Y \) and each \((\tau_i, \tau_j)\)-minimal closed set \( N \) in \( X \) containing \( f^{-1}(S) \), there is a \((\sigma_k, \sigma)\)-minimal closed set \( W \) in \( Y \) such that \( S \subseteq W \) and \( f^{-1}(W) \subseteq N \).

**Proof:** Similar to that of Theorem 4.4.29.

**4.4.31 Theorem:** Let \( f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) be a \((\tau_i, \tau_j)\)-\(M_\alpha(X)-(\sigma_k, \sigma)\)-strongly open and \( g: (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2) \) be a \((\sigma_k, \sigma)\)-\(M_\alpha(Y)-(\eta_m, \eta)\)-strongly open maps. Then \( g \circ f: (X, \tau_1, \tau_2) \to (Z, \eta_1, \eta_2) \) is a \((\tau_i, \tau_j)\)-\(M_\alpha(X)-(\eta_m, \eta)\)-strongly open map.

**Proof:** Let \( N \) be any \((\tau_i, \tau_j)\)-minimal open set in \( X \). Since \( f \) is \((\tau_i, \tau_j)\)-\(M_\alpha(X)-(\sigma_k, \sigma)\)-strongly open, \( f(N) \) is a \((\sigma_k, \sigma)\)-minimal open set in \( Y \). Again since \( g \) is \((\sigma_k, \sigma)\)-\(M_\alpha(Y)-(\eta_m, \eta)\)-strongly open, \( g(f(N)) = (g \circ f)(N) \)
is a \((\eta_m, \eta_n)\)-minimal open set in \(Z\). Hence \(gof\) is a \((\tau_i, \tau_j)\)-\(M_iO(X)-(\eta_m, \eta_n)\)-strongly open map.

4.4.32 Theorem: Let \(f: (X, \tau_i, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) be a \((\tau_i, \tau_j)\)-\(M_iO(X)-(\sigma_k, \sigma_l)\)-strongly open and \(g: (Y, \sigma_1, \sigma_2) \rightarrow (Z, \eta_1, \eta_2)\) be a \((\sigma_k, \sigma_l)\)-\(M_iO(Y)-(\eta_m, \eta_n)\)-strongly open maps. Then \(gof: (X, \tau_i, \tau_2) \rightarrow (Z, \eta_1, \eta_2)\) is a \((\tau_i, \tau_j)\)-\(M_iO(X)-(\eta_m, \eta_n)\)-strongly open map.

Proof: Similar to that of Theorem 4.4.31.

4.4.33 Theorem: Let \(f: (X, \tau_i, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) and \(g: (Y, \sigma_1, \sigma_2) \rightarrow (Z, \eta_1, \eta_2)\) be the maps and let \(gof: (X, \tau_i, \tau_2) \rightarrow (Z, \eta_1, \eta_2)\) be a \((\tau_i, \tau_j)\)-\(M_iO(X)-(\eta_m, \eta_n)\)-strongly open map. Then

i) \(g\) is \((\sigma_k, \sigma_l)\)-\(M_iO(Y)-(\eta_m, \eta_n)\)-strongly open if \(f\) is \((\sigma_k, \sigma_l)\)-\(M_iO(Y)-(\tau_i, \tau_j)\)-irresolute and surjective.

ii) \(f\) is \((\tau_i, \tau_j)\)-\(M_iO(X)-(\sigma_k, \sigma_l)\)-strongly open if \(g\) is \((\eta_m, \eta_n)\)-\(M_iO(Z)-(\sigma_k, \sigma_l)\)-irresolute and injective.

Proof: i) Let \(N\) be any \((\sigma_k, \sigma_l)\)-minimal open set in \(Y\). Since \(f\) is \((\sigma_k, \sigma_l)\)-\(M_iO(Y)-(\tau_i, \tau_j)\)-irresolute and surjective, \(f^{-1}(N)\) is a \((\tau_i, \tau_j)\)-minimal open set in \(X\). Again since \(gof\) is \((\tau_i, \tau_j)\)-\(M_iO(X)-(\eta_m, \eta_n)\)-strongly open, \((gof)(f^{-1}(N))\) is a \((\eta_m, \eta_n)\)-minimal open set in \(Z\). But \((gof)(f^{-1}(N))=g(f(f^{-1}(N))=g(N)\) is a \((\eta_m, \eta_n)\)-minimal open set in \(Z\). Hence \(g\) is a \((\sigma_k, \sigma_l)\)-\(M_iO(Y)-(\eta_m, \eta_n)\)-strongly open map.

ii) Let \(N\) be any \((\tau_i, \tau_j)\)-minimal open set in \(X\). Since \(gof\) is \((\tau_i, \tau_j)\)-\(M_iO(X)-(\eta_m, \eta_n)\)-strongly open, \((gof)(N)\) is a \((\eta_m, \eta_n)\)-minimal open set in \(Z\). Again since \(g\) is \((\eta_m, \eta_n)\)-\(M_iO(Z)-(\sigma_k, \sigma_l)\)-irresolute and injective, \(g^{-1}((gof)(N))\) is a \((\sigma_k, \sigma_l)\)-minimal open set in \(Y\). But \(g^{-1}((gof)(N))=g^{-1}(g(f(N))=f(N)\) is a
(\sigma_k, \sigma_l)-minimal open set in Y. Hence \( f \) is a \((\tau_i, \tau_j)-M_iO(X)-(\sigma_k, \sigma_l)\)-strongly open map.

**4.4.34 Theorem:** Let \( f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) and \( g: (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2) \) be the maps and let \( gof: (X, \tau_1, \tau_2) \to (Z, \eta_1, \eta_2) \) be a \((\tau_i, \tau_j)-M_aO(X) -(\eta_m, \eta_n)\)-strongly open map. Then

i) \( g \) is \((\sigma_k, \sigma_l)-M_aO(Y) -(\eta_m, \eta_n)\)-strongly open if \( f \) is \((\sigma_k, \sigma_l)-M_aO(Y) -(\tau_i, \tau_j)\)-irresolute and surjective.

ii) \( f \) is \((\tau_i, \tau_j)-M_aO(X)-(\sigma_k, \sigma_l)\)-strongly open if \( g \) is \((\eta_m, \eta_n)-M_aO(Z)-(\sigma_k, \sigma_l)\)-irresolute and injective.

**Proof:** Similar to that of Theorem 4.4.33.

**4.4.35 Remark:** Let \( f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) be a map. Then we get the following diagram.

```
\begin{diagram}
(\tau_i, \tau_j)-M_iO(X)-(\sigma_k, \sigma_l)-open \leftrightarrow (\tau_i, \tau_j)-M_iO(X)-(\sigma_k, \sigma_l)-open

\tau_i-\sigma_k-open

(\tau_i, \tau_j)-M_iO(X)-(\sigma_k, \sigma_l)-s-open \leftrightarrow (\tau_i, \tau_j)-M_iO(X)-(\sigma_k, \sigma_l)-s-open

\end{diagram}
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Diagram 4.4.

Analogous to \((\tau_i, \tau_j)-M_iO(X)-(\sigma_k, \sigma_l)-open\), \((\tau_i, \tau_j)-M_iO(X)-(\sigma_k, \sigma_l)-open\), \((\tau_i, \tau_j)-M_iO(X)-(\sigma_k, \sigma_l)-strongly open\) and \((\tau_i, \tau_j)-M_iO(X)-(\sigma_k, \sigma_l)-strongly open\) open maps, we define \((\tau_i, \tau_j)-M_iC(X)-(\sigma_k, \sigma_l)-closed\), \((\tau_i, \tau_j)-M_iC(X)-(\sigma_k, \sigma_l)-closed\), \((\tau_i, \tau_j)-M_iC(X)-(\sigma_k, \sigma_l)-strongly closed\) and \((\tau_i, \tau_j)-M_iC(X)-(\sigma_k, \sigma_l)-strongly closed\) closed maps as follows.
4.4.36 Definition: Let \( i, j, k, \ell \in \{1, 2\} \) be fixed integers. Let \((X, \tau_i, \tau_2)\) and \((Y, \sigma_1, \sigma_2)\) be two bitopological spaces. A map \( f: (X, \tau_i, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is called

i) \( \tau_i\)-\( \sigma_k \) closed if \( f(M) \) is \( \sigma_k \)-closed set in \( Y \) for every \( \tau_i \)-closed set \( M \) in \( X \).

ii) \( (\tau_i, \tau_j)\)-\( M_{iC}(X)\)-\( \sigma_k \)-closed if \( f(M) \) is \( \sigma_k \)-closed set in \( Y \) for every \( M \in (\tau_i, \tau_j)\)-\( M_{iC}(X) \).

iii) \( (\tau_i, \tau_j)\)-\( M_{iC}(X)\)-\( \sigma_k \)-closed if \( f(M) \) is \( \sigma_k \)-closed set in \( Y \) for every \( M \in (\tau_i, \tau_j)\)-\( M_{iC}(X) \).

iv) \( (\tau_i, \tau_j)\)-\( M_{iC}(X)\)-(\( \sigma_k, \sigma_i \))-strongly closed if \( f(M) \in (\sigma_k, \sigma_i)\)-\( M_{iC}(Y) \) for every \( M \in (\tau_i, \tau_j)\)-\( M_{iC}(X) \).

v) \( (\tau_i, \tau_j)\)-\( M_{iC}(X)\)-(\( \sigma_k, \sigma_i \))-strongly closed if \( f(M) \in (\sigma_k, \sigma_i)\)-\( M_{iC}(Y) \) for every \( M \in (\tau_i, \tau_j)\)-\( M_{iC}(X) \).

4.4.37 Remark: If \( \tau_1=\tau_2=\tau \) and \( \sigma_1=\sigma_2=\sigma \) in the Definition 4.4.1, then the \( \tau_i\)-\( \sigma_k \)-closed, \( (\tau_i, \tau_j)\)-\( M_{iC}(X)\)-\( \sigma_k \)-closed, \( (\tau_i, \tau_j)\)-\( M_{iC}(X)\)-\( \sigma_k \)-closed, \( (\tau_i, \tau_j)\)-\( M_{iC}(X)\)-(\( \sigma_k, \sigma_i \))-strongly closed and \( (\tau_i, \tau_j)\)-\( M_{iC}(X)\)-(\( \sigma_k, \sigma_i \))-strongly closed maps coincide with closed, minimal closed, maximal closed, strongly minimal closed and strongly maximal closed maps respectively.

From the definitions we have the following results.

4.4.38 Theorem:

i) Every \( \tau_i\)-\( \sigma_k \) closed map is \( (\tau_i, \tau_j)\)-\( M_{iC}(X)\)-\( \sigma_k \)-closed map but not conversely.

ii) Every \( \tau_i\)-\( \sigma_k \) closed map is \( (\tau_i, \tau_j)\)-\( M_{iC}(X)\)-\( \sigma_k \)-closed map but not conversely.

iii) Every \( (\tau_i, \tau_j)\)-\( M_{iC}(X)\)-(\( \sigma_k, \sigma_i \))-strongly closed map is \( (\tau_i, \tau_j)\)-\( M_{iC}(X)\)-\( \sigma_k \)-closed map but not conversely.
iv) Every \((\tau_i, \tau_j)\)-MaC(X)-(0)c, ai)-strongly closed map is \((\tau_i, \tau_j)\)-MaC(X)-
\(\sigma_k\)-closed map but not conversely.

**4.4.39 Theorem:** Let \(f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) be a bijection map. Then
the following statements are equivalent.

i) \(f^{-1}: Y \rightarrow X\) is \((\tau_i, \tau_j)\)-M\(_i\)O(X)-\(\sigma_k\)-continuous.

ii) \(f\) is \((\tau_i, \tau_j)\)-M\(_i\)O(X)-\(\sigma_k\)-open map.

iii) \(f\) is \((\tau_i, \tau_j)\)-M\(_i\)O(X)-\(\sigma_k\)-closed map.

**Proof:** i)\(\rightarrow\)ii): Let \(N\) be any \((\tau_i, \tau_j)\)-minimal open set in \(X\). By assumption
\((f^{-1})^{-1}(N)\) is an \(\sigma_k\)-open set in \(Y\). But \((f^{-1})^{-1}(N)=f(N)\) is an \(\sigma_k\)-open set in \(Y\).
Therefore \(f\) is a \((\tau_i, \tau_j)\)-M\(_i\)O(X)-\(\sigma_k\)-open map.

ii)\(\rightarrow\)iii): Let \(F\) be any \((\tau_i, \tau_j)\)-maximal closed set in \(X\), then \(X-F\) is a \((\tau_i, \tau_j)\)-
minimal open set in \(X\). By assumption \(f(X-F)\) is an \(\sigma_k\)-open set in \(Y\). But
\(f(X-F)=Y-f(F)\) is an \(\sigma_k\)-open set in \(Y\). Therefore \(f(F)\) is a \(\sigma_k\)-closed set in
\(Y\). Hence \(f\) is a \((\tau_i, \tau_j)\)-M\(_i\)O(X)-\(\sigma_k\)-closed map.

iii)\(\rightarrow\)i): Let \(N\) be any \((\tau_i, \tau_j)\)-minimal open set in \(X\), then \(X-N\) is a \((\tau_i, \tau_j)\)-
maximal closed set in \(X\). By assumption \(f(X-N)\) is a \(\sigma_k\)-closed set in \(Y\). But
\(f(X-N)=(f^{-1})^{-1}(X-N)=Y-(f^{-1})^{-1}(N)\) is a \(\sigma_k\)-closed set in \(Y\). Therefore
\((f^{-1})^{-1}(N)\) is an \(\sigma_k\)-open set in \(Y\). Hence \(f^{-1}: Y \rightarrow X\) is a \((\tau_i, \tau_j)\)-M\(_i\)O(X)-\(\sigma_k\)-continuous.

**4.4.40 Theorem:** Let \(f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) be a bijection map. Then
the following statements are equivalent.

i) \(f^{-1}: Y \rightarrow X\) is \((\tau_i, \tau_j)\)-M\(_i\)O(X)-\(\sigma_k\)-continuous.

ii) \(f\) is \((\tau_i, \tau_j)\)-M\(_i\)O(X)-\(\sigma_k\)-open map.

iii) \(f\) is \((\tau_i, \tau_j)\)-M\(_i\)O(X)-\(\sigma_k\)-closed map.

**Proof:** Similar to that of Theorem 4.4.39.
4.4.41 Theorem: Let \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be a bijection map. Then the following statements are equivalent.

i) \( f^{-1}: Y \rightarrow X \) is \((\tau_1, \tau_2)-M\sigma_1O(X)-(\sigma_k, \sigma_i)\)-irresolute.

ii) \( f \) is \((\tau_1, \tau_2)-M\sigma_1O(X)-\sigma_k\)-strongly open map.

iii) \( f \) is \((\tau_1, \tau_2)-M\sigma_1O(X)-\sigma_k\)-strongly closed map.

**Proof:** i)\(\rightarrow\)ii): Let \( N \) be any \((\tau_1, \tau_2)\)-minimal open set in \( X \). By assumption \((f^{-1})^{-1}(N)\) is a \((\sigma_k, \sigma_i)\)-minimal open set in \( Y \). But \((f^{-1})^{-1}(N)=f(N)\) is a \((\sigma_k, \sigma_i)\)-minimal open set in \( Y \). Therefore \( f \) is a \((\tau_1, \tau_2)-M\sigma_1O(X)-\sigma_k\)-strongly open map.

ii)\(\rightarrow\)iii): Let \( F \) be any \((\tau_1, \tau_2)\)-maximal closed set in \( X \), then \( X-F \) is a \((\tau_1, \tau_2)\)-minimal open set in \( X \). By assumption \( f(X-F) \) is a \( \sigma_k\)-minimal open set in \( Y \). But \( f(X-F)=Y-f(F) \) is a \((\sigma_k, \sigma_i)\)-minimal open set in \( Y \). Therefore \( f(F) \) is a \((\sigma_k, \sigma_i)\)-maximal closed set in \( Y \). Hence \( f \) is a \((\tau_1, \tau_2)-M\sigma_1O(X)-\sigma_k\)-strongly closed map.

iii)\(\rightarrow\)i): Let \( N \) be any \((\tau_1, \tau_2)\)-minimal open set in \( X \), then \( X-N \) is a \((\tau_1, \tau_2)\)-maximal closed set in \( X \). By assumption \( f(X-N) \) is a \((\sigma_k, \sigma_i)\)-maximal closed set in \( Y \). But \( f(X-N)=(f^{-1})^{-1}(X-N)=Y-(f^{-1})^{-1}(N) \) is a \((\sigma_k, \sigma_i)\)-maximal closed set in \( Y \). Therefore \((f^{-1})^{-1}(N)\) is a \((\sigma_k, \sigma_i)\)-minimal open set in \( Y \). Hence \( f^{-1}: Y \rightarrow X \) is a \((\tau_1, \tau_2)-M\sigma_1O(X)-(\sigma_k, \sigma_i)\)-irresolute.

4.4.42 Theorem: Let \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be a bijection map. Then the following statements are equivalent.

i) \( f^{-1}: Y \rightarrow X \) is \((\tau_1, \tau_2)-M\sigma_1O(X)-(\sigma_k, \sigma_i)\)-irresolute.

ii) \( f \) is \((\tau_1, \tau_2)-M\sigma_1O(X)-\sigma_k\)-strongly open map.

iii) \( f \) is \((\tau_1, \tau_2)-M\sigma_1O(X)-\sigma_k\)-strongly closed map.

**Proof:** Similar to that of Theorem 4.4.40.
4.5 Pairwise minimal and pairwise maximal homeomorphisms.

5.5.1 Definition: Let $i, j, k, l \in \{1, 2\}$ be fixed integers. Let $(X, \tau_i, \tau_j)$ and $(Y, \sigma_i, \sigma_j)$ be two bitopological spaces. A bijection map $f: (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ is called

i) $\tau_i-\sigma_k$-homeomorphism if $f$ and $f^{-1}$ are $\tau_i-\sigma_k$ continuous maps.

ii) $(\sigma_k, \sigma_l)$-$M_iO(Y)$-$\tau_j$-homeomorphism if $f$ and $f^{-1}$ are $(\sigma_k, \sigma_l)$-$M_iO(Y)$-$\tau_j$-continuous maps.

ii) $(\sigma_k, \sigma_l)$-$M_jO(Y)$-$\tau_i$-homeomorphism if $f$ and $f^{-1}$ are $(\sigma_k, \sigma_l)$-$M_jO(Y)$-$\tau_i$-continuous maps.

5.5.2 Remark: If $\tau_1=\tau_2=\tau$ and $\sigma_1=\sigma_2=\sigma$ in the Definition 4.5.1, then the $\tau_i-\sigma_k$ homeomorphism, $(\sigma_k, \sigma_l)$-$M_iO(Y)$-$\tau_j$-homeomorphism and $(\sigma_k, \sigma_l)$-$M_jO(Y)$-$\tau_i$-homeomorphism coincide with homeomorphism, minimal homeomorphism and maximal homeomorphism respectively.

4.5.3 Theorem: Every $\tau_i-\sigma_k$ homeomorphism is $(\sigma_k, \sigma_l)$-$M_iO(Y)$-$\tau_j$-homeomorphism but not conversely.

Proof: Let $f: (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ be a $\tau_i-\sigma_k$ homeomorphism. Now $f$ and $f^{-1}$ are $\tau_i-\sigma_k$ continuous maps. Then $f$ and $f^{-1}$ are $(\sigma_k, \sigma_l)$-$M_iO(Y)$-$\tau_j$-continuous maps, as every $\tau_i-\sigma_k$ continuous map is $(\sigma_k, \sigma_l)$-$M_iO(Y)$-$\tau_j$-continuous map. Hence $f$ is a $(\sigma_k, \sigma_l)$-$M_iO(Y)$-$\tau_j$-homeomorphism.

4.5.4 Example: In Example 4.3.4, $f$ is a $(\sigma_1, \sigma_2)$-$M_iO(Y)$-$\tau_1$-homeomorphism but it is not a $\tau_1-\sigma_1$ homeomorphism, since $f$ is not a $\tau_1-\sigma_1$ continuous map, for the $\sigma_1$-open set $\{a, b, c\}$ in $Y$, $f^{-1}(\{a, b, c\})=\{a, b, c\}$ which is not an $\tau_1$-open set in $X$.

4.5.5 Theorem: Every $\tau_i-\sigma_k$ homeomorphism is $(\sigma_k, \sigma_l)$-$M_jO(Y)$-$\tau_i$-homeomorphism but not conversely.
**Proof:** Similar to that of Theorem 4.5.3.

4.5.6 **Example:** In Example 4.3.7, \( f \) is a \((\sigma_1, \sigma_2)-M_aO(Y)-\tau_1\)-homeomorphism but it is not a \(\tau_1-\sigma_1\) homeomorphism, since \( f \) is not a \(\tau_1-\sigma_1\) continuous map, for the \(\sigma_1\)-open set \(\{a, b\}\) in \(Y\), \(f^{-1}(\{a, b\}) = \{a, b\}\) which is not a \(\tau_1\)-open set in \(X\).

4.5.7 **Remark:** \((\sigma_k, \sigma_1)-M_aO(Y)-\tau_i\)-homeomorphism and \((\sigma_k, \sigma_1)-M_aO(Y)-\tau_i\)-homeomorphism are independent of each other.

4.5.8 **Example:** In Example 4.3.4, \( f \) is a \((\sigma_1, \sigma_2)-M_aO(Y)-\tau_1\)-homeomorphism but it is not a \((\sigma_1, \sigma_2)-M_aO(Y)-\tau_1\)-homeomorphism. In Example 4.3.7, \( f \) is a \(\tau(\sigma_1, \sigma_2)-M_aO(Y)-\tau_1\)-homeomorphism but it is not a \((\sigma_1, \sigma_2)-M_aO(Y)-\tau_1\)-homeomorphism.

4.5.9 **Theorem:** Let \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be a bijective and \((\sigma_k, \sigma_1)-M_aO(Y)-\tau_i\)-continuous map. Then the following statements are equivalent.

i) \( f \) is \((\tau_i, \tau_j)-M_aO(X)-\sigma_k\)-open map

ii) \( f \) is \((\sigma_k, \sigma_i)-M_aO(Y)-\tau_i\)-homeomorphism

iii) \( f \) is \((\tau_i, \tau_j)-M_aC(X)-\sigma_k\)-closed map

**Proof:** Similar to that of Theorem 4.4.39.

4.5.10 **Theorem:** Let \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be a bijective and \((\sigma_k, \sigma_1)-M_aO(Y)-\tau_i\)-continuous map. Then the following statements are equivalent.

i) \( f \) is \((\tau_i, \tau_j)-M_aO(X)-\sigma_k\)-open map

ii) \( f \) is \((\sigma_k, \sigma_i)-M_aO(Y)-\tau_i\)-homeomorphism

iii) \( f \) is \((\tau_i, \tau_j)-M_aC(X)-\sigma_k\)-closed map

**Proof:** Proof follows from Theorem 4.4.40.