3. Theoretical Model

3.1. Concept of Magnetohydrodynamic Wave

The motion of incompressible conductive nonuniformly and isotropically
magnetizable fluid in a magnetic field studied extensively. Velocity of the motion of
the fluid is assumed to be nonrelativistic and processes in its quazi-stationary. Under
the above condition the velocity of the motion of the fluid is assumed to be
nonrelativistic and process is quazi-stationary [1]. The corresponding system of
equations has in the form

(i) Continuity : \( \frac{d \rho}{dt} + \rho \, \text{div} \vec{V} = 0 \) (3.1)

(ii) Moment equation:
\[
\rho \frac{d \vec{V}}{dt} = - \nabla (p + \psi) + \frac{1}{4\pi} (\text{rot} \vec{H} \times \vec{B}) + M \cdot \nabla \vec{H} + \text{div} \tau
\]

(iii) Induction :
\[
\frac{d \vec{B}}{dt} = \text{rot} (\vec{V} \times \vec{B}) - \frac{1}{4\pi\sigma} \text{rot} \text{rot} \vec{H}, \quad \text{div} \vec{B} = 0
\]

(iv) Energy :
\[
\rho T \frac{ds'}{dt} = \text{div} (\lambda^0 \nabla T) + \tau_{ij} \cdot V_{ij} + v_m (\text{rot} \vec{H})^2
\]

where, \( \rho \) is the density, \( \vec{V} \) is the velocity of the medium, \( \lambda^0 \) and \( \sigma \) are the thermo and
electric conductivity, \( v_m \) is the magnetic viscosity, \( c \) is the thermal capacity of liquid,
\{\( \tau_{ij} \)\}is viscous stress tensor whose relation to strain velocity tensor \( \{V_{ij}\} \) is defined by
Newton’s constitutive law

(v) \( \tau_{ij} = \eta_1 V_{ij} + \left( \frac{2}{3} \eta_2 \right) \delta_{ij} \text{div} \vec{V} \) (3.2)

Here, \( \eta_1 \) and \( \eta_2 \) are coefficient of viscosity. In the energy equation, \( s' \) is the full
entropy of the unit medium bulk

\[
s' = s + s^e, \quad s' = s + \frac{1}{4\pi\rho} \int_0^H \left( \frac{d \mu}{dT} \right)_{\rho,H} H \, dH
\]

where, \( s \) is the hydrodynamic entropy, \( s^e \) is the part of the full entropy due to magneto
caloric effects.
In the place of the equation determining the change of the full entropy, it is possible to use the equation of conservation of energy in the form

\[
\frac{d}{dt}\left(\rho U + \frac{1}{2} \rho V^2 - \psi^T - \frac{H^2}{8\pi} + \frac{\mu H^2}{4\pi}\right) = -\text{div}\left(\nabla\left(\rho U + \frac{1}{2} \rho V^2 + p + \psi^\rho - \psi^T\right) + \frac{c}{4\pi} (\mathbf{E} \times \mathbf{H}) - \chi^0 V T - \nabla \varphi\right)\quad (3.3)
\]

Where, \(U\) is the internal energy of the medium in the absence of electromagnetic field,

\[
\psi^\rho = \frac{1}{4\pi} \int_0^H (\mu - 1 - \rho \mu_\varphi) \mathbf{H} \, d\mathbf{H}, \quad \psi^T = \frac{1}{4\pi} \int_0^H (\mu - 1 - T \mu_\varphi) \mathbf{H} \, d\mathbf{H}
\]

Electric field from magnetohydrodynamic approximation is found from the equation

\[
\overline{E} = \frac{\nu_m}{c} \text{rot}\overline{H} - \frac{1}{c}\left(\overline{V} \times \overline{B}\right)
\]

The system of equations (3.1) – (3.3) is completed by the equations of state which can be written for a compressible medium in the form

\[
P = P(\rho, s), \quad T = T(\rho, s)
\]

(3.5)

For an incompressible liquid, \(\rho = \text{constant}\).

The pressure is found from the equation of motion and \(T = T(s)\). If the thermo capacity \(c\) of the liquid is constant, than the entropy and the internal energy of the liquid are as follows

\[
s = \tilde{c} \ln T + \text{const}, \quad U = \tilde{c} T + \text{const}
\]

(3.6)

In the case of isotropically magnetization magnetic induction \(\overline{B}\) strength \(\overline{H}\) and the magnetization density \(\overline{M}\) of the medium are related by

\[
\overline{B} = \overline{H} + 4\pi M(\rho, T, H) \quad \frac{\overline{H}}{H} = \mu(\rho, T, H). \overline{H}
\]

(3.7)

### 3.2. Monodispersed System

The magnetic fluid is formed by the dispersion of ferromagnetic particles in carrier liquid. The theoretical model obtained on the basis of a magnetic fluid model consider as a nonhomogeneously and isotropically magnetized medium. It is known that nonhomogeneities due to aggregation and restructuring of dispersed particles in magnetic fluid which depend on particle concentration, magnetic field strength, temperature and other factors. These formation influence anisotropic properties of
magnetic fluid in magnetic field and change the equation of magnetic state and thermodynamical properties of magnetic fluid. While taking the thermodynamic state equation in the most general form \( p = p(\rho, s) \) and \( T = T(\rho, s) \). Considering the propagation of small perturbation in such a medium, it has been shown that the unperturbed state of the medium be characterized by constant values of its parameters \( \rho_0, V_0, S_0, P_0, H_0 \) [2] & [3]

\[
\begin{align*}
P_0 &= \left( \frac{dp}{dp} \right)_0, P_s = \left( \frac{dp}{ds} \right)_0, T_p = \left( \frac{d\mu}{d\rho} \right)_0, \mu_m = \left( \frac{d\mu}{dH} \right)_0, \mu_r = \left( \frac{d\mu}{dT} \right)_0
\end{align*}
\] (3.8)

The propagation of small amplitude waves in a medium can be described by the following system of seven equations:

\[
\begin{align*}
\frac{du_i}{dt} + x_{ik} \frac{du_k}{dx} &= d_{ik} \frac{d^2u_k}{dx^2} \quad (i, k = 1, 2, 3, 4, \ldots 7)
\end{align*}
\] (3.9)

\[
\begin{align*}
u_1 &= \rho', \ u_2 = S', \ u_3 = v_x', \ u_4 = B_y', \ u_5 = B_z', \ u_6 = E_y', \ u_7 = E_z'
\end{align*}
\]

Here, \( u_i \) denote perturbation of the variables and the matrices \( ||x_{ik}|| \) and \( ||d_{ik}|| \). The expressions for these coefficients written in terms of the matrix elements \( x_{ik} \) are sufficiently intricate and are not given here but details are given in reference [2].

We shall consider small perturbations without taking account of dissipation. We shall assume \( d_{ik} = 0 \). Seeking a solution of equation (3.9) in the form

\[
\begin{align*}
u_i &= \nu_i^* \cdot \exp(\text{i}kx - \text{i} \omega t)
\end{align*}
\] (3.10)

Obtain the phase velocity of a wave \( \lambda = \omega / k \), a characteristic equation of the form

\[
\begin{align*}
\lambda [\lambda^6 - \lambda^4 (A_1 + C'^2 A_2) + \lambda^2 C'^2 (A_3 + C'^2 A_4) + C'^4 A_5] &= 0
\end{align*}
\] (3.11)

Here, the \( A_i \) are independent of C. Obtain the roots of characteristic equation in the form of magnetohydrodynamic wave with phase velocity [3]

\[
\begin{align*}
\lambda^2 = \left[ L_o - \left\{ L_1 \left( \frac{B_x^2 + B_z^2}{B^2} \right) \left( 1 + L_2 \frac{B_y^2 + B_z^2}{B^2} \right)^{-1} \right\} \right]
\end{align*}
\] (3.12)

Thus, in a simple magnetohydrodynamic wave only the electric field does not vary and similarity this wave is also plane polarized (i.e. \( B_z = 0 \)). From equation (3.12), in magnetizable medium the phase velocity may become purely imaginary at certain values of the parameters. If \( \theta = \arcsin B_y B^{-1} \) is found such that phase velocity \( \lambda \) becomes imaginary. This may occur in following cases.
\( L_0 > 0; \quad 1 + L_2 > 0; \quad (1+L_2)L_0 - L_1 > 0 \)

For perturbation velocity \( \lambda \) obtain from the equation (3.12)

\[
\lambda^2 = L_0 - \frac{L_1 \sin^2 \theta}{1 + L_2 \sin^2 \theta}
\]  

(3.13)

In this equation, \( \theta \) is the angle between the magnetic field direction and the direction of ultrasonic wave propagation. The parameters \( L_0, L_1 \) and \( L_2 \) are functions of magnetization, field and temperature. In this equation, in the case of \( \theta = 90^\circ \) and \( 0 \leq \xi < 1 \) has two minima, one will get one maxima at \( \xi = \xi_3 \) and two minima at \( \xi = 0 \) and \( \xi = \xi_2 \). For \( \xi >> 1 \), ultrasonic velocity saturated with applied magnetic field as shown in Figure 3(a).

Let us now consider some models of the magnetizable media. The state of an ideal gas in weak magnetic field can be described by the following equations.

\[
\frac{L_0}{a_0^2} = a_0 \left[ q \psi (\psi - a_2) + a_0^{-1} \left\{ N \phi a_1 (\beta q \psi a_0 a_2 - \beta_1) \right\} \right]
\]

\[
\frac{L_1}{a_1^2} = a_0^{-1} q \left\{ \psi - \alpha \beta N a_2 \gamma^2 (q^{-1} - \psi a_2 a_0) \right\} \left\{ \psi - a_2 (1 - \beta \phi N a_0 a_1) \right\}
\]

\[
L_2 = a_0 \frac{\alpha}{\gamma^2} \left[ - \phi + \psi \xi + \beta \psi^2 N a_0 a_2 \gamma^2 \right]
\]  

(3.13a)

Here, \( a_0 \) is the perturbation velocity in the absence of field. The particular law of magnetization is determined by the type of magnetization media under consideration. Thus, for an ideal paramagnetic gas Langevin’s formula can be used

\[
M = M_0, \quad L(\xi), \quad M_0 = n \mu
\]

where, \( \mu \) is the magnetic moment of dispersed magnetic particles and \( n \) the number of magnetic particles per unit of volume. Constant parameters are defined in Table 3.1.

Here,

\[
\phi = 1 - \frac{\xi^2}{s h^2 \xi}, \quad N = a_0^{-1} \left[ 1 + \phi \left( \beta + \frac{\alpha}{\gamma^2} \right) \right]^{-1}, \quad \alpha = \frac{4 \pi \mu M_0}{k_B T}, \quad \beta = \frac{H M_0}{\rho \xi T}, \quad \gamma = \frac{T}{\rho T}, \quad q = \frac{4 \pi M_0^2}{\rho a_2^2}, \quad a_0 = \left( 1 + \frac{\alpha \phi}{\xi^2} \right)^{-1},
\]

\[
a_1 = \left( 1 + \frac{\alpha \gamma \psi}{\xi} \right), \quad a_2 = \left( \frac{\phi}{\gamma \xi} \right), \quad a_3 = \left( 1 + \frac{\alpha \psi}{\xi} \right)
\]

(3.13b)
Table 3.1  Some constant parameters

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Terms</th>
<th>Value (order)</th>
<th>Dimensions</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>Temperature</td>
<td>10^2</td>
<td>K</td>
</tr>
<tr>
<td>k_B</td>
<td>Boltzmann Constant</td>
<td>10^{-23}</td>
<td>J / K</td>
</tr>
<tr>
<td>* T_s</td>
<td>Temperature</td>
<td>10^2</td>
<td>K</td>
</tr>
<tr>
<td>* T_ρ</td>
<td>Temperature</td>
<td>10^2</td>
<td>K</td>
</tr>
<tr>
<td>ρ</td>
<td>Density</td>
<td>10^3</td>
<td>kg / m^3</td>
</tr>
<tr>
<td>N</td>
<td>number density of particle</td>
<td>10^{-2}</td>
<td>m^3</td>
</tr>
<tr>
<td>M_d</td>
<td>Domain magnetization</td>
<td>10^9</td>
<td>A/m</td>
</tr>
<tr>
<td>μ</td>
<td>Magnetic Moment</td>
<td>10^{-19}</td>
<td>Am^2</td>
</tr>
</tbody>
</table>

* Unperturbed state of the medium (T_s and T_ρ) values remain constant from equation (3.8).

C_p and C_v are heat capacities at constant pressure and volume. R is the gas constant.

Rewrite this equation,

$$\lambda^2 = \frac{L_0}{a_s^2} - \frac{1}{a_s^2} \left( \frac{L_1 \sin^2 \theta}{1 + L_2 \sin^2 \theta} \right)$$

(3.14)

For ξ < 1, expanding equation (3.13) into a series [4]

$$\lambda^2 = \left( \frac{\lambda - a_s}{a_s} \right)$$

Here,

$$\left( \frac{\lambda - a_s}{a_s} \right) = \left( \frac{V - V_0}{V_0} \right)_{\text{normalized}} = \left( \frac{\Delta V}{V_0} \right)_{\text{normalized}}$$

ΔV/V_0 is fractional change in sound velocity. ΔV is defined by ΔV = (V−V_0). V and V_0 are ultrasonic propagation velocities with and without an external magnetic field, respectively. For θ = 90°, the variation of velocity anisotropy with field for μ = 1.6 ×10^{-19} Am^2 is calculated shown in figure 3(a). The existence of such anisotropic behavior was confirmed experimentally by Chung and Isler [5]. They show the angle dependence ultrasonic velocity anisotropy as a function of applied magnetic field. The qualitative character of the angle dependences of ultrasound velocity in a magnetic fluid is also predicted by the present theory. Figure 3(b) (open circle), agree well with that predicated in Sokolov [6]. Similar results are obtain figure 3(b) (line) by using one another formula for propagation velocity of fast C_f magnetoelastic wave in the case of (β_⊥ = β|| = 10^6 gm/cm^3) [6]. The sound velocity is higher in case of parallel compare to perpendicular direction. The biggest change in velocity is achieved when the magnetic field passes from a parallel to perpendicular configuration. This change is higher than that obtained when passing from a non ordered to an ordered state [7].
For $\theta = 90^\circ$, the variation of velocity anisotropy with field for $\mu = 1.6 \times 10^{-19}$ Am$^2$ is calculated using equation (3.14). The existence of such anisotropic behavior was confirmed experimentally by Chung and Isler [5]. (b) Ultrasonic velocity anisotropy dependence on the angle at 0.025T, theoretically derived from equation (3.14) (open circle). The ultrasound velocity anisotropy due to the magnetoelastic mechanism (line), obtain from ref. [6]. (c) Contribution of each term A and B of Tarapov's theory (equation (3.14)) along with the resultant graph for $\mu = 3 \times 10^{-19}$ Am$^2$. (see text). (d) Anisotropy in velocity at 2 MHz with magnetic field at 308K. The red line is fit to Tarapov's theory (3.14) for monodispersed system.

Figure 3. (a) For $\theta = 90^\circ$, the variation of velocity anisotropy with field for $\mu = 1.6 \times 10^{-19}$ Am$^2$ is calculated using equation (3.14). The existence of such anisotropic behavior was confirmed experimentally by Chung and Isler [5]. (b) Ultrasonic velocity anisotropy dependence on the angle at 0.025T, theoretically derived from equation (3.14) (open circle). The ultrasound velocity anisotropy due to the magnetoelastic mechanism (line), obtain from ref. [6]. (c) Contribution of each term A and B of Tarapov's theory (equation (3.14)) along with the resultant graph for $\mu = 3 \times 10^{-19}$ Am$^2$. (see text). (d) Anisotropy in velocity at 2 MHz with magnetic field at 308K. The red line is fit to Tarapov's theory (3.14) for monodispersed system.

Using above theory for $\theta = 90^\circ$, the variation of velocity anisotropy with field for $\mu = 3 \times 10^{-19}$ Am$^2$ calculated and the contribution of each term A and B (terms A and B are the first and second terms of equation (3.14), respectively) along with the resultant curve explained as shown in figure 3(c) [8]. It is evident from figure 3(c) that by increasing field the first term A saturates at about 0.0175 T and at same time B starts decreasing. Here, figure 3(d) shows that the anisotropy in velocity at 2 MHz with magnetic field at 308 K. The fit using equation (3.14) generated using $\mu = 3.2 \pm 0.02 \times 10^{19}$ Am$^2$ is shown as red line. The fit is not good at intermediate fields.
**Limitations of Equation (3.14):**

(i) This theory is only valid for the Monodispersed system.

(ii) By using Tarapov theory, we cannot explain frequency dependence change in ultrasonic propagation velocity anisotropy.

### 3.3. Polydisperse System

(i) Problem Resolved: *This theory is only valid for the Monodispersed system.*

In equation (3.14), the parameters \( L_0, \ L_1 \) and \( L_2 \) are functions of magnetization, field and temperature. The magnetic particles dispersed in the magnetic fluids are not mono-dispersed, thus one needs to account for the polydispersity in the moment. Therefore, the change in ultrasonic propagation velocity anisotropy in a fluid determined by equation (3.14) is corrected for the distribution function given by equation (2.7) (Refer Appendix A for details). In this report, we show that incorporating the polydispersity in Tarapov’s theory in terms of magnetic moment, it is possible to explain the change in ultrasonic velocity anisotropy for polydisperse system [8, 9].

![Figure 3.1](image)

**Figure 3.1** Influence of moment distribution on velocity anisotropy calculated using equations (2.7) and (3.14) for (a) different values of \( \sigma \) and \( \mu_0 = 3 \times 10^{-19} \text{Am}^2 \), (b) different values of \( \mu_0 \) and \( \sigma = 0.6 \).

First of all we discuss effect of magnetic moment and polydispersity on ultrasonic velocity anisotropy in magnetic fluid. Using the equations (2.7) and (3.14), variation in velocity anisotropy for constant \( \mu_0 \) and different values of \( \sigma \) as well as for constant value of \( \sigma \) and different values of \( \mu_0 \) are generated and these are shown in figures 3.1(a) and 3.1(b), respectively. With the increase in moment distribution, the minimum first shifts towards the lower field and then decreases in magnitude (Figure 3.1(a)). This change in magnitude is higher when the moment is reduced (Figure 3.1(b)). Thus moment distribution does influence the behavior of velocity anisotropy.
From figure 3.2, it shows that the anisotropy in velocity at 2 MHz with magnetic field at 308 K. The fit using equation (3.14) generated using \( \mu = 3.2 \pm 0.02 \times 10^{-19} \text{ Am}^2 \) is shown as red line. The fit is not good at intermediate fields. The dashed line is fit to Tarapov’s theory after incorporating polydispersity in magnetic moment equation (2.7) using equation (3.14). The fit values are: \( \mu_0 = 3.02 \pm 0.02 \times 10^{-19} \text{ Am}^2 \), \( \sigma (D) = 0.33 \) and \( <D> = 12.4 \text{ nm} \). Results indicate that by incorporating polydispersity in magnetic moment, it is possible to explain the whole behavior of ultrasonic velocity anisotropy due to the influence of magnetic field [8].

Certain theoretical models available in the literature for magnetic fluid assume fluid to be monodispersed. Several interesting experimental results related to ultrasonic propagation velocity were reported but these results have not been clarified by any theoretical models. Incorporating the polydispersity in Tarapov’s theory in terms of magnetic moment here it is shown that it is possible to explain the change in ultrasonic velocity anisotropy for a polydisperse system also.

(ii) Problem Resolved: *By using Tarapov theory, we cannot explain frequency dependence change in ultrasonic propagation velocity anisotropy.*

Interesting point in Tarapov theory [4], the whole theory is independent of frequency. It is observed in figure 3.3 (a) red line that the magnitude of normalized \( \Delta V/V_{H=0} \) decreases initially with field, reaches a minimum and thereafter it increases. At a higher field, it exhibits saturation behaviour. This is similar to the observations by Chung et al [1]. This behaviour can be explained only for one particular frequency. Experimental data of ultrasonic velocity anisotropy are taken at two different frequencies 2 MHz and 9 MHz. By incorporating polydispersity in terms of magnetic moment, it is possible to explain change in ultrasonic velocity anisotropy at given frequency as shown in figure 3.3 (b).
When these data were fitted using equations (2.7) and (3.14), it is observed that $\mu_0$ remains constant $\mu_0 = 3.04 \pm 0.02 \times 10^{-19}$ Am$^2$ but $\sigma$ increases with frequency. The $\sigma$ value is 0.32 at 2 MHz and 1.50 at 9 MHz. This corresponds to increase in effective mean particle diameter 12.4 nm for 2 MHz and 17.7 nm for 9 MHz. Result indicates in the present system, the change in behavior of ultrasonic velocity is a function of frequency of ultrasonic wave [9]. It suggests the relation between propagation velocity and characteristic time of Brownian motion of magnetic particles. The change in velocity anisotropy is dominated by grain–grain interaction rather than grain–field interaction.

The renewal of interest in ultrasonic propagation velocity anisotropy in magnetic and magnetorheological fluid make more importance to understand inner structural properties due to an external magnetic field. The next chapter is devoted to results and discussion.
References

6. V. V. Sokolov, Acoust. Phys. 56 (2010) 972-988