Chapter IV

Essential edges in a graph with respect to radius for contraction of edges....
ESSENTIAL EDGES IN A GRAPH WITH RESPECT TO THE RADIUS FOR CONTRACTION OF EDGES

1. INTRODUCTION:
As in Chapter 1, we consider the problem of finding essential edges in a graph with respect to the elementary contraction of an edge in $G$. In Chapter 1 we had considered the property of diameter -- invariance for elementary edge contractions. In contrast, in this chapter we consider the property being radius -- invariance for elementary edge contractions. All definitions defined in a general set up stand the same as in Chapter 1. Here we redefine some definitions with respect to radius -- invariance.

DEFINITION 1.1: An edge $e$ in a graph $G$ is said to be radius -- essential or simply essential with respect to the contraction, if $\text{rad} (G/e) < \text{rad} (G)$, otherwise non-essential.

The definition is well-defined as $\text{rad} (G/e) \leq \text{rad} (G)$, for every edge $e$ of $G$.

The edge essential number of a graph $G$ can also be redefined as follows: $\sigma_r (G) = |\{e \in E(G) | \text{rad} (G/e) < \text{rad} (G)\}|$, i.e. $\sigma_r (G)$ is the number of essential edges in a graph $G$ with respect to radius of $G$ for elementary edge contractions.

Clearly, from the definition of $\sigma_r (G)$, one can see that $0 \leq \sigma_r (G) \leq q$, where $q$ is the number of edges in $G$.

In this chapter, we show the existence of a graph $G$ with given value of $\sigma_r (G)$ and also we give bounds on $\sigma_r (G)$, in terms of radius, order and size of $G$. 
Further, we study some properties of graphs for the extreme values of $\sigma_r(G)$, viz. 0 and $q$.

2. EXISTING RESULTS:

In this section we list some existing results without proof, which we make use to prove results of this chapter.

**PROPOSITION 2.1 [6]:** The maximum number of edges in a graph on $p$ nodes with radius $r$ is

- $p(p-1)$, \( r = 1; \)
- $\frac{p(p-2)}{2}$, \( r = 2; \)
- $\frac{(p^2-4pr +5p +4r^2 - 6r )}{2}$, \( r \geq 3. \)

**PROPOSITION 2.2 [4]:** A unique eccentric node graph $G$ is self-centered if and only if each vertex of $G$ is eccentric.

3. RESULTS:

In this section we prove some results regarding $\sigma_r(G)$. First proposition ensures the existence of a graph with given $\sigma_r(G)$ except for few values.

**PROPOSITION 3.1:** For any positive integer $n$ satisfying $0 \leq n \leq q$, there exists a graph $G$ with $\sigma_r(G) = n$, except for the values $q = 4$, $n = 1, 2, 3$; and $q=3$, $n=1, 2$ and $q=2$, $n=1, 2$. 
**PROOF:** To prove the proposition, we consider the following cases:

**Case 1 :** Let $q \geq 5$, $n \leq q-3$ and $n \geq 4$ is even. Consider the following graph $G$ as labeled in the Figure 4.1

From the Figure 4.1, it is not difficult to see that the contraction of the edges $u_1u_2$, $u_1v_1$ and $u_2v_1$ does not alter the radius of $G$ and $\text{rad}(G/e) < \text{rad}(G)$ for the edges of the form $e = v_iv_{i+1}$, $1 \leq i \leq n$, since $\text{rad}(G/e) = n-1 < n = \text{rad}(G)$.

**Case 2 :** Let $q \geq 5$ and $n$ is odd.

Consider the graph $G$ as labeled in the Figure 4.2

For the graph $G$ of the Figure 4.2, $\text{rad}(G) = n$, and the contraction of the edges $e_i = u_iu_{i+1}$, for $1 \leq i \leq n$ in $G$, the radius of $G/e$ reduces by one and no other edge reduces the radius by contraction. If $q-n=0$, then a path of odd length serves the purpose, that is, $\sigma_{f}(G) = n = q.$
Case 3: Let $q \leq 4$ and $n \leq q$.

By Table A1, p.p. 214 - 224, Harary [3], for $q=4$, there are only three connected graphs viz: $P_5, C_4$. Thus, for $q = 4$, $n = 4$, the graph $C_4$ serves the purpose and for $P_5$ and $\sigma_T(G) = 0$. Hence, there does not exist a graph $G$ with $q = 4$, and $n = 3, 2$ and 1.

Next, we consider $q = 3$, $n \leq q$. The only connected graphs with three edges are $C_3$, $K_{1,3}$ and $P_4$. Hence for $q = 3$ and $n = 3$, $P_4$ and $K_{1,3}$ are the graphs for which $\sigma_T(G) = 3$. For $q = 3$ and $n = 0$, $C_3$ is the required graph. Therefore, there does not exist a graph $G$ with $q = 3$ and $n = 1, 2$.

For $q = 2$, $P_3$ is the only connected graph for which $\sigma_T(G) = 0$. Hence there does not exist a graph $G$ with $q = 2$, $n = 1, 2$. Also for $q = 1$, $K_2$ is the graph with $\sigma_T(G) = 1$. □

**PROPOSITION 3.2:** For any $(p,q)$-graph $G$ with $rad(G) = r$, the following inequalities hold:

1. $\sigma_T(G) \leq p - 1$, for $r = 1$;
2. $\sigma_T(G) \leq \frac{(p-1)(p+3)}{4} + \frac{p(p-2)}{2}$, for $r = 2$;
3. $\sigma_T(G) \leq 2r(p-1) - 2(p-1)^2 + 2r(p^2 - 4pr + 5p + 4r^2 - 6r)$, for $r \geq 3$.

**PROOF:** Consider a $(p,q)$-graph $G$ with $rad(G) = r$. For any central vertex $u$ in $G$, set $A_0 = \{u\}$ and define the sets $A_i = \{v \in V(G)/d(u,v) = i\}$, for $i = 1, 2, \ldots, r$. Let us denote $|A_i| = n_i$, $n_i \geq 1$. For all edges $e \in <A_i>$, it is clear that $rad(G/e) = rad(G)$. Let $B = \bigcup_{i+1} \{e/e \in <A_i>\}$. 

Then \( q - \sigma_r(G) = \left| \{ e \in E(G) / \text{rad}(G/e) = \text{rad}(G) \} \right| \)

\[
\geq \left| B \right| = \sum_{i=1}^{r} \left( \frac{n_i}{2} \right)^2 = \sum_{i=1}^{r} \frac{n_i^2 - n_i}{2}
\]

\[
= \sum_{i=1}^{r} n_i^2 - \sum_{i=1}^{r} n_i
\]

Therefore,

\[
q - \sigma_r(G) + \frac{\sqrt{2}}{2} (p - 1) \geq \frac{1}{2} \left( \sum_{i=1}^{r} n_i^2 \right) - \frac{1}{2} \left( \sum_{i=1}^{r} n_i \right)
\]

\[
= \frac{(p - 1)^2}{2r}
\]

This implies that,

\[
\sigma_r(G) \leq \frac{1}{2} (p - 1) + q \cdot \frac{1}{2} (p - 1)^2
\]

\[
= \frac{r (p - 1) - (p - 1)^2 + 2rq}{2r}
\]

Using Proposition 2.1[6], we have the maximum value for \( q \) with given radius. So the value of \( \sigma_r(G) \) turns out to the following depending upon the value of \( r \). If \( r = 1 \), then \( q \leq \frac{p(p-1)}{2} \) and hence \( \sigma_r(G) \leq 1(p-1) - (p-1)^2 + p(p-1) = p-1 \). If \( r = 2 \), then \( q \leq \frac{p(p-2)}{2} \) the value of \( \sigma_r(G) \) turns out to,
\[ \sigma_r(G) \leq \frac{p(p-2)}{2} + \frac{(p-1)(-p+3)}{4} \]

For \( r \geq 3 \), \( q \leq \frac{p^2 - 4pr + 5p + 4r^2 - 6r}{2} \), hence,

\[ \sigma_r(G) \leq \frac{2r(p-1) - 2(p-1)^2 + 2r(p^2 - 4pr + 5p + 4r^2 - 6r)}{4r} \]

\[ \Box \]

**PROPOSITION 3.3** : Every graph \( G \) can be embedded in a graph \( H \) such that \( \sigma_r(H) = 0 \), with given radius.

**PROOF**: Let \( G \) be any graph of order \( p \) with vertices \( v_1, v_2, ..., v_p \). We construct a graph \( H \) containing \( G \) as an induced subgraph with given radius \( \text{rad}(H) \) of \( H \).

**Case 1**: \( \text{rad}(H) = 1 \).

Consider a graph \( H = K_1 + G \). Clearly \( \text{rad}(H) = 1 \), and it is not difficult to see that the contraction any edge does not alter the radius of \( H \) and hence \( \sigma_r(H) = 0 \).

**Case 2**: Let \( \text{rad}(H) \geq 2 \).

Consider a graph \( H \) as the sequential join of \( K_1 \)'s and \( G \) as in the following the Figure 4.3.

![Figure 4.3](image-url)
For any edge $e$ either of the form $u_ju_{j+1}$, $u_iv_j$, $v_jw_1$, $w_iw_{i+1}$, $\text{rad } (H/e) = \text{rad } (H)$ hold. □

**PROPOSITION 3.4:** For a tree $T$, $\sigma_r(T) = 0$, if and only if $T$ has exactly one central vertex.

**PROOF:** Let $T$ be a tree with $\sigma_r(T) = 0$. We prove that $T$ has exactly one central vertex. If possible assume that $T$ has two central vertices, say $u$ and $v$. Then, for an edge $e = uv$, $\text{rad } (T/e) < \text{rad } (T)$, a contradiction to the hypothesis that $\sigma_r(T) = 0$. Hence, $T$ has exactly one central vertex.

Conversely, suppose that a tree $T$ has exactly one central vertex $u$ (say). Let $\text{rad } (T) = r = e(u)$. But, in a tree $T$, only end vertices are eccentric vertices, so there are at least two peripheral vertices in $T$ say $v_1$ and $v_2$ with $d(v_1, v_2) = \text{diam } (T)$. But $\text{diam } (T) = 2 \text{ rad } (T)$, since $T$ has exactly one central vertex. Therefore, there exists a diametrical path $v_1-v_2$ containing $u$. So $d(u, v_1) = d(u, v_2) = r$. Let $e$ be any edge in a tree $T$. Clearly the contraction of an edge $e$ whether it is on the geodesic of the paths $u-v_1$ or $u-v_2$ or not, does not alter the radius of $T$. Hence, $\sigma_r(T) = 0$. □

**PROPOSITION 3.5:** For a bicentral tree $T$, $\sigma_r(T) = q$, if only if $T$ is a path.

**PROOF:** Let $T$ be a bicentral tree and $\sigma_r(T) = q$. We prove that $T$ is a path. Let us label the central vertices as $u$ and $v$. We claim that each of $u$ and $v$ have exactly one eccentric vertex. On the contrary assume that either $u$ or $v$ has at least two eccentric vertices. Without loss of generality assume that $u$ has two eccentric vertices say $u_1$ and $u_2$. Contraction of any edge on $u-u_1$, geodesic does not alter the eccentricity, $e(u)$, of $u$, as there is a vertex $u_2$ in $T$ such that $d(u, u_2) = \text{rad } (T)$; a contradiction to $\sigma_r(T) = q$; hence the claim. Let $u'$ and $v'$ be the eccentric vertices of $u$ and $v$ respectively. Hence, $u'-v'$ is the diametrical path in $T$. Now, we claim that $T$ is this...
u'-v' path only. If not, then there exists at least one vertex w on u'-v' path such that \( \deg w \geq 3 \). But the contraction of an edge incident with w, not on u'-v' path does not alter the radius of T. We arrive at a similar contradiction as above. Hence, T is a path.

Conversely, suppose T is a path containing two central vertices. Then, it is obvious that T=\( P_{2n} \), for some n. Clearly rad (T/e) < rad (T), for any edge e. Hence \( \sigma_f (T) = q \) holds. □

**PROPOSITION 3.6:** For any unicyclic graph G, \( \sigma_f(G) = q \), if and only if G is obtained by the identification of a vertex of the cycle with an end vertex of the path \( P_{n+2} \).

**PROOF:** Let G be a unicyclic graph. Now we, claim that G contains atmost one pendent vertex. If possible assume that there are at least two pendent vertices, say u and v in G. Clearly, these two pendent vertices are peripheral vertices in G. Contraction of any edge incident with either u or v does not alter the radius of G. Hence, the claim. If G does not contain any pendent vertex, then G must be cycle. This cycle must be of even length, since, the contraction of an edge in an odd cycle does not alter the radius. Let G contain exactly one pendent vertex. We claim that the vertex on cycle which has its degree equal to three is a central vertex of G. If not then, the contraction of the pendent edge does not alter the radius of G, a contradiction, hence, the claim. Now, we claim that the cycle is of even length. If not the vertex of degree three will have two eccentric vertices and contraction of an edge on any path does not alter the radius of G. Hence G, is as follows:

\[ G : \]

FIGURE 4.4
**PROPOSITION 3.7:** For any self-centered graph $G$, $\sigma_r(G) = q$ if and only if $G$ is unique eccentric node graph.

**PROOF:** Let $G$ be a self-centered graph of radius $r$. Suppose $\sigma_r(G) = q$. Then, we prove that $G$ is unique eccentric node graph. On the contrary assume that there is a vertex $u$ in $G$ having two (or more) eccentric vertices. Without loss of generality, let $u_1$ and $u_2$ be the eccentric vertices of $u$. The contraction of any edge incident with either $u_1$ or $u_2$ does not alter the radius of $G$; a contradiction to $\sigma_r(G) = q$.

Conversely suppose $G$ is a self-centered unique eccentric node graph. By the result in [4], a self-centered graph is unique eccentric node graph, if and only if each vertex of $G$ is eccentric. So, any vertex $u$ in $G$ is an eccentric vertex of exactly one vertex say $u'$. Hence $d(u,u') = e(u') = \text{rad}(G) = \text{diam}(G)$. The contraction of an edge on $u-u'$ geodesic gives $e_{G/e}(u) < e_G(u) = \text{rad}(G)$. Since each edge of $G$ is on some shortest path joining two eccentric vertices. Thus, the contraction of each edge reduces the radius in $G/e$ and hence $\sigma_r(G) = q$. □

**COROLLARY 3.8:** Let $G$ be a self-centered graph of radius $r$. Then $G/e$ is a self-centered graph of radius $r-1$, if and only if $G$ is unique eccentric node graph.

**PROOF:** Let $G$ be a self-centered graph of radius $r$. The vertex set of $G/e$ can be partitioned into two sets $V_1$ and $V_2$ as follows $V_1 = \{u \in V(G/e) / e_{G/e}(u) = r\}$ and $V_2 = \{v \in V(G/e) / e_{G/e}(v) = r-1\}$. It is clear that if $\sigma_r(G) = q$, then $V_1 = \emptyset$ and if $\sigma_r(G) = 0$, then $V_2 = \emptyset$.

In this proposition first we have to prove that $G$ is unique eccentric node graph if $G/e$ is self-centered of radius $r-1$. If $G/e$ is self-centered with radius $r-1$, then $\sigma_r(G) = q$, by the definition of $\sigma_r(G)$. Hence $V_1 = \emptyset$. And the proof follows from the Proposition 3.7 above. □

$\{G \in \mathcal{G} \mid r \}$

\[ B \]
**Proposition 3.9**: For any graph $G$, $\sigma_r(G) = 0$, if and only if every central vertex of $G$ has at least two eccentric vertices joined by disjoint paths.

**Proof**: Let $G$ be a graph in which each central vertex has at least two eccentric vertices joined by disjoint paths. Let $u_1, u_2$ be any two eccentric vertices of a central vertex $u$. Paths $u-u_1$ and $u-u_2$ are disjoint (edge). Contraction of an edge on either $u-u_1$ path or $u-u_2$ path does not alter the eccentricity of $u$ and hence radius of $G$, as contraction of any other edge not on these paths does not alter the eccentricity of $u$. Hence $\sigma_r(G) = 0$.

For the converse if $\sigma_r(G) = 0$, then we have to prove that each central vertex has at least two eccentric vertices joined by disjoint paths. Suppose a central vertex of $G$, say $u$, has only one eccentric vertex say $u'$, contraction of an edge on $u-u'$ geodesic reduces the eccentricity of $u$ by one, a contradiction $\sigma_r(G) = 0$. Let all central vertices have at least two eccentric vertices. Suppose these two eccentric vertices are not joined by edge disjoint paths, then contraction of any common edge joining the eccentric vertices with any central vertex reduces the eccentricity of the central vertex and thereby the radius of $G$, a contradiction to $\sigma_r(G) = 0$. Hence each central vertex has at least two eccentric vertices joined by edge disjoint paths. Hence the result. □

**Proposition 3.10**: For any graph $G$, if $\sigma_r(G) = 0$, then $C(G) \subseteq C(G/e)$.

**Proof**: Let $G$ be a graph with $\sigma_r(G) = 0$. We prove that $C(G) \subseteq C(G/e)$. Suppose that $C(G) \nsubseteq C(G/e)$ then there exists at least one vertex, say $x$, such that $x \in C(G) \setminus C(G/e)$. Therefore $e_G(x) = \text{rad}(G)$ and $e_{G/e}(x) \neq \text{rad}(G/e)$, i.e.
\(e_{G/e}(x) > \text{rad}(G/e)\). So \(\text{rad}(G/e) < e_{G/e}(x) = e_G(x) = \text{rad}(G)\). This implies that \(\text{rad}(G/e) < \text{rad}(G)\), a contradiction to the fact that \(\sigma_r(G) = 0\). Hence the result. □

**Proposition 3.11**: For a graph \(G\), if \(\sigma_r(G) = 0\), then \(P(G) \subseteq EC(G)\).

**Proof**: Let \(G\) be a graph with \(\sigma_r(G) = 0\). By Proposition 3.9 above, each central vertex of \(G\) has at least two eccentric vertices joined by edge disjoint paths.

We have to prove that \(P(G) \subseteq EC(G)\). On the contrary assume that \(P(G) \not\subseteq EC(G)\) then there exists at least one vertex, say \(x\), such that \(x \in P(G) \setminus EC(G)\). Hence \(e(x) = \text{diam}(G)\) and \(\forall u \in C(G), d(u, x) \neq \text{rad}(G)\). Let \(u_1\) and \(u_2\) be any two eccentric vertices of \(u\) joined by edge disjoint paths. Hence \(d(u, u_1) = d(u, u_2) = \text{rad}(G)\). Hence \(d(u, x) \leq \text{rad}(G) - 1\). Let \(y\) be an eccentric vertex of \(x\). So \(d(x, y) = \text{diam}(G)\). There are two possibilities for \(y\), either \(y\) is an eccentric vertex of \(u\) or not. If \(y\) is an eccentric vertex of \(u\) then \(d(u, y) = \text{rad}(G)\), if not \(d(u, y) < \text{rad}(G)\). Hence \(d(x, y) \leq \text{rad}(G) - 1 + \text{rad}(G) = 2\text{rad}(G) - 1\). But \(d(u_1, u_2) = 2\text{rad}(G)\), through \(u\) as \(u-u_1, u-u_2\) are edge disjoint paths. Since \(x, y \in P(G)\), \(d(x, y) \geq d(u_1, u_2)\) this implies \(2\text{rad}(G) - 1 \geq 2\text{rad}(G)\), a contradiction. Hence \(P(G) \subseteq EC(G)\). □

**Proposition 3.12**: For a self-centered graph with radius two and \(\sigma_r(G) = 0\), the following inequality holds:

\[2p - 5 \leq q \leq 2p^3\]

where \(p\) is the order and \(q\) is the size of \(G\).

**Proof**: Let \(G\) be a self-centered graph with radius two, and \(\sigma_r(G) = 0\). The left hand inequality follows from [1]. For the right hand inequality we have the
following argument. Since \( \sigma_1(G) = 0 \), for each edge \( e \in E(G) \), \( \text{rad}(G/e) = \text{rad}(G) = 2 \). Hence for every edge \( e \), the edge degree, \( \text{deg}_{G/e} e \leq p-3 \), otherwise, \( \text{rad}(G/e) = 1 \).

\[
\sum \text{deg} e_i \leq q (p-3).
\]

Hence \( \sum \text{deg} e_i \leq q (p-3) \).

But \( \text{deg} e = \text{deg} u + \text{deg} v - 2 \), where \( e = uv \) is an edge of a graph. Hence

\[
\sum \text{deg} e_i \leq \sum (\text{deg} u_j + \text{deg} u_k - 2)
\]

\[
= \text{deg} u_1 + \sum_{k=1}^{p} \text{deg} u_k - 2p^2 + \text{deg} u_2 + \sum_{k=1}^{p} \text{deg} u_k - 2p^2 + \ldots
\]

\[
= \text{deg} u_p + \sum_{k=1}^{p} \text{deg} u_k - 2p^2
\]

\[
= \sum_{j=1}^{p} \text{deg} u_j + p \sum_{k=1}^{p} \text{deg} u_k - p(2p^2)
\]

\[
= 2q + 2pq - 2p^3
\]

Hence \( 2p + 2pq - 2p^3 \leq q (p-3) \)

\[
\Rightarrow pq + 2q - 2p^3 + 3q \leq 0
\]

\[
\Rightarrow q(p+5) \leq 2p^3
\]

\[
\Rightarrow q \leq \frac{2p^3}{p+5}
\]

Hence the result. □
**Proposition 3.13**: Let $G$ be a graph with $\text{rad}(G) = 3$. Then $\sigma_r(G) = q$ if and only if either $G = P_6$ or $G = K_{1} + \bar{K}_{m} + K_1$ where $m \geq 2$ and $\bar{K}_m$. $\bar{K}_m$ is a partite graph with partite sets $V_1$ and $V_2$ and every vertex of $V_1$ is adjacent to at least one vertex in $V_2$ and vice-versa.

**Proof**: Let $G$ be a graph with $\text{rad}(G) = 3$. Suppose $\sigma_r(G) = q$. If $G$ is a tree then by Proposition 3.5 above $G$ is a path of even order of radius three. Hence $G = P_6$. If $G$ is not a tree, consider a central vertex $u$ in $G$. Then define a partition by setting $A_0 = \{u\}$ and $A_i = \{w/\delta(u, w) = i, i = 1, 2, 3\}$. Then there exists a $u - u_1 - v_1 - v$ path such that $u_1 \in A_1$, $v_1 \in A_2$, $v \in A_3$.

Consider the partition of the set $V(G) - \{u, v\}$ as defined below.

$A_u = \{u_i/u_{ui} \text{ is an edge in } G\}$

$A_v = \{v_i/v_{vi} \text{ is an edge in } G\}$

$A_w = \{w_i/w_{wi} \text{ is neither adjacent to } u \text{ nor } v\}$

Clearly, $A_u$ and $A_v$ are not empty because at least one $u_1 \in A_1$, $v_1 \in A_2$. To prove the assertion we prove the following claims.

**Claim 1**: $A_w$ is empty.

For, if $A_w \neq \emptyset$, then there exists a vertex $w_i \in A_w$ such that $w_i$ is not adjacent to both $u$ and $v$. Since $G$ is connected, there are $u-w_i$ paths and $v-w_i$, paths in $G$ (each of length two). Let $e$ be an edge incident with $w_i$. Then $\text{rad}(G/e) = \text{rad}(G) = 3$ holds. This is a contradiction to the fact that $\sigma_r(G) = q$. 


Claim 2: The sets $A_u$ and $A_v$ are independent.

If $A_u$ is not independent, then there exists at least one edge $e \in \langle A_u \rangle$ whose contraction does not alter the radius. This again leads to the same contradiction as in Claim 1. The similar argument holds for $A_v$ too.

Claim 3: Every vertex in $A_u$ is adjacent to at least one vertex in $A_v$ and vice-versa.

If not, let $u_i$ be a vertex in $A_u$ not adjacent to any vertex $v_i$ of $A_v$. Then $e_G(u_i) \geq 4$. Contraction of an edge of the form $e = uu_i$, gives $e_{G}(uu_i) = 3$ as there exists a path $uu_i - u_1 - v_1 - v$ of length three in $G/e$ which gives $\text{rad}(G/e) = 3$. This is a contradiction to $\sigma_r(G) = q$.

From the facts proved in above claims, $G = K_1 + \overline{K}_m$. $\overline{K}_m + K_1$ as defined in the hypothesis.

Conversely, suppose $G = P_6$ or $G = K_1 + \overline{K}_m$. $\overline{K}_m + K_1$, then we show that $\sigma_r(G) = q$. If $G = P_6$, then obviously $\sigma_r(G) = q$ holds. Now let $G = K_1 + \overline{K}_m \cdot H$. $\overline{K}_m + K_1$, and let us label vertices of $G$ as in the proof of above part. If $e = uu_i$, for $i = 1, 2, \ldots, m$, is contracted then $\text{rad}(G/e) = 2$ and if $e = u_i v_j$ for $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$ is contracted then again $\text{rad}(G/e) = 2$, holds. Similarly, for $e = vv_j$, $j = 1, 2, \ldots, n$, $\text{rad}(G/e) = 2$. Hence $\sigma_r(G) = q$ holds. Hence the result. □

PROPOSITION 3.14: For any positive integers $a$ and $b$ such that $b = 2a$, there exists a graph $G$ with $\text{rad}(G) = a$ and $\text{diam}(G) = b$ and $C(G/e) = C(G)$, for all edges $e$ of $G$.

PROOF: Let $a$ and $b$ be any two positive integers such that $b = 2a$. Consider the graph $G = K_m + \overline{K}_n$. Identify an end vertex of a path of order ‘$a$’ with each vertex
of $\overline{K_n}$, to get a graph $G$. The schematic representation of $G$ will be as shown in Figure 4.5.

Label the vertices of $G$ as in Figure 4.5. Clearly, $v_i$'s, $1 \leq i \leq m$, are central vertices of $G$. And rad $(G) = a$, diam $(G) = 2a = b$. Contraction of an edge of the form $e = u_i^j u_{i+1}^j$, $1 \leq i \leq a, 1 \leq j \leq n$, gives $e_{G/e}(u_i^j u_{i+1}^j) = e_G(u_i^j)$, since $u_i^j u_{i+1}^j$ is a vertex in $G/e$. $e_{G/e}(u_{i-1}^j) = e_{G/e}(u_{i-1}^j)$; $e_{G/e}(u_{k}^j) = e_G(u_{k}^j) - 1$, $k \geq i+2$. So $u_1^j u_2^j$ is such last edge, $1 \leq j \leq a$. Hence in $G/e$, $e_{G/e}(u_1^j u_2^j) = e_G(u_1^j) = a+1$, and for all $u_3^j, u_4^j, \ldots, u_a^j$ and $e_{G/e}(u_3^j) = e_G(u_3^j) - 1, 3 \leq i \leq a, 1 \leq j \leq a$. Hence eccentricity of each vertex of the type $u_i^j$ is reduced by one and is almost equal to $a+1$.

If an edge of the form $e = v_i^j u_1^j$, $1 \leq i \leq m, 1 \leq j \leq n$, then $e_{G/e}(v_i^j u_1^j) = e_G(v_i^j)$, as there are still $n-1$ vertices of the type $u_a^j, 1 \leq j \leq n$, at distance $a$ from the vertex obtained by identifying $v_i$ and $u_i^j$. So $e_{G/e}(v_i^j u_1^j) = e_G(v_i^j) = a$. 
If an edge of the form \( v_j v_{j+1} \), \( 1 \leq j \leq m \), is contracted then clearly no change occurs in the eccentricity of any vertex of \( G \).

Hence in all cases discussed above the vertices of the form \( v_i \)'s, \( 1 \leq i \leq m \), have minimum eccentricity, in \( G \) as well as in \( G/e \). In other words \( C(G/e) = C(G) \) for all edges \( e \) of \( G \). □

**Proposition 3.15:** For any positive integers \( a \) and \( b \), \( a \geq 2 \), \( b = 2a \), there exists a graph \( G \) such that \( \text{rad} (G) = a \), \( \text{diam} (G) = 2a = b \) and \( C(G) \subseteq (G/e) \) for all edges, \( e \) of \( G \).

**Proof:** Let \( a \) and \( b \) be any two positive integers, with \( a \geq 2 \) and \( b = 2a \). Consider a path \( P_{2a+1} \) and label the vertices of \( P_{2a+1} \) as in Figure 4.6.

![Figure 4.6](image)

Clearly, \( u_{a+1} \) is the only central vertex of \( G \). Contraction of any edge of \( G \) reduces \( P_{2a+1} \) to \( P_{2a} \). Hence eccentricity of each vertex of \( G \) except \( u_{a+1} \) reduces by one. Hence in \( G/e \), \( e_{G/e}(u_{a+1}) = a \), and \( e_{G/e}(u_a) = a \). Hence \( C(G/e) = \{ u_{a+1}, u_a \} \) or \( C(G/e) = \{ u_a, u_{a+1} \} \). Clearly, \( C(G) \subseteq C(G/e) \), for all \( e \in E(G) \). □

**Observation:** For any positive integer \( a \), there exists a graph \( G \) such that \( \text{rad} (G) = a \), with \( C(G/e) \subseteq C(G) \), for all \( e \in E(G) \).

**Proof:** Let \( a \) be any positive integer. Consider a self-centered graph of radius \( a \). Clearly, \( C(G/e) \subseteq C(G) \) as every vertex of \( G \) is in center. □
**PROPOSITION 3.16**: There exists no graph \( G \) with \( C(G) \cap C(G/e) = \emptyset \), for any edge \( e \) of \( G \).

**PROOF**: On the contrary suppose there exists a graph \( G \) with \( C(G) \cap C(G/e) = \emptyset \), for some edge \( e \) of \( G \). Then the following two conditions hold: (i) \( C(G/e) \subsetneq C(G) \) and (ii) \( C(G) \subsetneq C(G/e) \).

By condition (i) \( \forall x \in C(G/e) \Rightarrow x \notin C(G) \) i.e. \( e_{G/e}(x) = \text{rad}(G/e) \) and \( e_G(x) \geq \text{rad}(G) + 1 \Rightarrow e_{G/e}(x) \geq \text{rad}(G) \).

i.e. \( \text{rad}(G) \leq e_{G/e}(x) = \text{rad}(G/e) \)

\( \Rightarrow \text{rad}(G) = \text{rad}(G/e) \) as \( \text{rad}(G/e) \leq \text{rad}(G) \), holds for all edges \( e \) of \( G \).

By condition (ii) \( \forall y \in C(G) \Rightarrow y \notin C(G/e) \) i.e. \( e_{G/e}(y) = \text{rad}(G/e) \) and \( e_G(y) \geq \text{rad}(G/e) + 1 \) i.e. \( e_{G/e}(y) \leq \text{rad}(G) \).

So \( \text{rad}(G/e) + 1 \leq e_{G/e}(y) \leq \text{rad}(G) \). From above condition (i), \( \text{rad}(G) = \text{rad}(G/e) \), \( \forall e \in E(G) \). Hence \( \text{rad}(G) + 1 \leq e_G(y) \leq \text{rad}(G) \), a contradiction.

Hence the result. \( \square \)

**PROPOSITION 3.17**: For any two positive integers \( a, b \) such that \( a+2 \leq b \leq 2a-2 \), there exists a graph \( G \) so that \( \text{rad}(G) = a \), \( \text{diam}(G) = b \), and \( P(G) \cap EC(G) = \emptyset \) with \( \sigma_t(G) = 0 \).

**PROOF**: Consider the following graph \( G \) with \( P(G) \cap EC(G) = \emptyset \), \( \text{rad}(G) = a \), \( \text{diam}(G) = b \).
We have labeled some vertices of $G$ for clarity. Clearly $C(G) = \{c\}$, $P(G) = \{w,z\}$, $EC(G) = \{x,y\}$. Contraction of any edge on $z-z'$ path or $w-w'$ path does not alter the eccentricity of $c$. So also contraction of an edge on $c-y$ path ($c-x$ path) does not alter the eccentricity of $c$ as there exists $x$ (or $y$) at distance $r$ from $c$. Similarly contraction of any other edge of $G$ does not alter the eccentricity of $c$ and hence the radius of $G$. So $\sigma_r(G) = 0$. □

**Remark 2.18:** There exists a graph $G$ with $P(G) = EC(G)$ and $\sigma_r(G) = q$.

By [5] if $G$ is self-centered then $P(G) = EC(G)$. By Proposition 3.7, above for a self-centered graph $G$, $\sigma_r(G) = q$ if and only if $G$ is unique eccentric node graph. Hence any self-centered unique eccentric node graph serves the purpose. The following graph is an example.
$G$: 

![Graph Image]

FIGURE 4.8
REFERENCES:


