4.1 INTRODUCTION

This chapter would deal with two problems. Firstly motivated by the results of Richard Beal [10], Masser and Oesterlie, [9], [11], the equation \( A^x + B^y + C^z = D^w \) with \( x, y, z, w \) pairwise relatively prime is studied for its integer solutions. Motivation to undertake this investigation is due to the conjectures by Richard Beal and Masser and Oesterlie. Secondly, a computational method developed by Boyd, is considered and applied in obtaining primitive solutions to the equation \( x^2 + y^5 = z^6 \), [1].

4.2 THE EQUATION \( A^x + B^y + C^z = D^w \)

Solutions to diophantine equations of the form \( x^1 + y^m = z^n \) are those solutions for which \( xyz \neq 0 \) and \( x, y, z \) are pairwise relatively prime. Existence of such solutions depends upon a well known Conjecture known as abc-Conjecture. Masser and Oesterlie, in the year 1980, together for the first time formulated related to a Diophantine inequality and came to be known as abc- Conjecture. This conjecture is still open and may be stated as follows,

\[ \text{abc – Conjecture (Masser, Oesterlie)} : \text{For a given each } \varepsilon > 0 \text{, there is a constant } \mu \text{, } \mu > 1 \text{ such that if } a \text{ and } b \text{ are relatively prime and } c = a + b \text{ then} \]

\[ \max \left( |a|, |b|, |c| \right) \leq \mu N(a, b, c)^{1+\varepsilon} \]

\( N(a, b, c) \) here denotes a square free part of the product abc. In other words, it is a product in which no prime divisors repeat.
Also, if X is any given positive integer then by Sieve theory one can extricate squares from X and thus obtain all square free integers N located between $1 < N < X$. A survey of this conjecture can be found in the works of Lang [8] and Goldfeld[6]. Let us look for its application.

1. Fermat’s equation: Consider the Fermat’s equation $x^n + y^n = z^n$, $xyz \neq 0$ and are pair wise relatively prime.

Assume that the abc - Conjecture holds. Then in that case the exponent n is bounded. It is a well known fact that the assertion $x^n + y^n = z^n$, $n \geq 3$ implies $xyz = 0$.

1. Let the abc - Conjecture hold. Then there are infinitely many primes p such that

$$2^{p-1} \equiv 1 \pmod{p^2}.$$ 

In 1909, Wieferich proved that if p is a prime satisfying $2^{p-1} \equiv 1 \pmod{p^2}$ then the equation $x^p + y^p = z^p$ has no non - trivial integral solutions satisfying $p \mid xyz$.

It is still not known that without assuming abc - Conjecture whether it is possible for us to ascertain $2^{p-1} \equiv 1 \pmod{p^2}$ for infinitely many primes.

Recently, Beal came up with a conjecture which states that:

If $A, B, C, x, y$ and $z$ are positive integers with $x, y, z > 2$. If $A^x + B^y = C^z$, then $A, B, C$ have a common factor.

In other words,
The equation $A^x + B^y = C^z$ has no solution in positive integers $A$, $B$, $C$, $x$, $y$, and $z$ with $x$, $y$, and $z$ exponent values at least 3 and $A, B, C$ being co-prime.\[^{10}\]

In fact, very similar conjectures were made in the past. V. Brun in the year 1914 stated several similar problems.\[^{2}\] However, this particular conjecture gains significance as it resembles Fermat's Last theorem; in fact for $x = y = z$ indeed it is.

In the year 1995, Darmon and Andrew Granville,\[^{4}\] have showed that if the positive integers $x$, $y$, and $z$ are such that $1/x + 1/y + 1/z < 1$, then there are only finitely many triples of co-prime integers $A$, $B$, $C$ satisfying Beal's conjecture.

In case of $x = y = z = 3$, there can be only finitely many solutions to diophantine equation $A^x + B^y = C^z$. It seems, Euler and probably Fermat were knowing that there are no solutions to it, by considering other possibilities, they studied the conjecture. Following is an important result.

Fermat – Catalan Conjecture: There are only finitely many triples of co-prime integer powers $x^p, y^q, z^r$ for which

$x^p + y^q = z^r$ with $1/p + 1/q + 1/r < 1$

so far, as mentioned\[^{10}\], ten solutions have been found for this equation. The first five solutions, which are called as small solutions, are given by

$1 + 2^3 = 3^2$, \quad $2^5 + 7^2 = 3^4$, \quad $7^3 + 12^2 = 2^9$, \quad $2^7 + 17^2 = 71^2$, \quad $3^5 + 11^4 = 122^2$
Proposition 4.2.1: For each $\varepsilon > 0$ given and some constant $\mu$, $\mu > 1$ and for integers $a,b,c$ and $d$ relatively prime such that $d = a + b + c$ and $\max(\lvert a \rvert, \lvert b \rvert, \lvert c \rvert, \lvert d \rvert) < p N(a, b, c, d)^{1+\varepsilon}$ then the equation $A^x + B^y + C^z = D^w$, where $A,B,C$ and $D$ are relatively prime do not have solutions in positive integers for higher exponents.

Proof: Let $k = (\log \mu / \log 2) + 4(1+\varepsilon)$ and let $\min(x,y,z,w) > k$.

Assume $A, B, C, D$ are relatively prime integers such that

$$A^x + B^y + C^z = D^w \quad (4.2.1)$$

In (4.2.1), set $a = A^x$, $b = B^y$ and $c = C^z$ then we see that $d = a + b + c$, where $a, b, c, d$ are co-primes. Further,

$$N(A^x, B^y, C^z, D^w) \leq \mu (ABCD)^{1+\varepsilon} \quad (4.2.2)$$

Now, if $\max(A^x, B^y, C^z, D^w) = A$. Then (4.2.2) would imply

$$A^x \leq \mu A^{4+4\varepsilon}$$

Which is the same as

$$x \leq (\log x / \log a) + (4 + 4\varepsilon) \leq k, \text{ since } \min(x,y,z,w) > k.$$ 

This is absurd.

Similar arguments hold in the case of $y, z$ and $w$ also.

Therefore, our observation in the beginning assuming that (4.2.1) holds for integral values $A, B, C, D$ and exponents $x, y, z, w$ large enough. Proposition follows.
Remark 4.2.1: It seems that any integer $a \not\equiv 4$ or $5 \mod 9$ can be expressible as a sum of three cubes. However to tackle such a question appears to be out of reach with any of the present available techniques. [10]

Here is an example of the above remark. The integer $36$ is not congruent to $4$ or $5$ modulo $9$. But $36$ can be expressible as

$$36 = 1^3 + 2^3 + 3^3$$

Incidentally $36$ is again a square.

2. Conjecture: There are only finitely many quadruples of co-prime integer powers $x^p$, $y^q$, $z^r$, $w^s$ with $1/p + 1/q + 1/r + 1/s < 1$ for which $x^p + y^q + z^r = w^s$ (This conjecture is still open)

4.3 THE EQUATION $x^2 + y^5 = z^6$

Consider the equation $x^2 + y^m = z^n$ where $m$, $n$ are relatively prime for its solution in integers. For $m = 2$, $n = 1$, it reduces to Fermat's equation $x^2 + y^2 = z^2$ and whose solution in integers or rationals are well known to us. Thus we assume that $(m, n) \neq (2, 1)$ and study the equation $x^2 + y^5 = z^6$ for its integer solutions.

The equation $x^2 + y^3 = z^4$ has been investigated by Mathieu [5] (see Dickson). His solutions are suggested by the well known identity on the sum of $n$ cubes given by

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left( \frac{n(n + 1)}{2} \right)^2$$
Thus if one takes $x = n(n-1)/2$, $y = n$, and $z = n(n+1)/2$, then $x^2 + y^3 = z^4$.

In order for $z$ to be an integer, one must solve $2z^2 = n(n+1)$ which is essentially a Pell's equation and thus has infinitely many solutions. E.Kiss [7] found all solutions in which $x$, $y$, $z$ are relatively prime. In one of the cases he considered factorization in the field $\mathbb{Q}(\sqrt{5})$. Although he does not obtain all solutions of $x^2 + y^3 = z^4$, his result that there are infinitely many solutions with $x$, $y$ and $z$ relatively prime is stronger than the statement that there are infinitely many primitive solutions. Hence, it appears that more difficult methods are needed in solving the more general equation $x^2 + y^m = z^{2n}$ and whether these equations will have infinitely many solutions is also not known in which $x$, $y$ and $z$ are relatively prime is also not known. However, David W. Boyd [1] considered the equation $x^2 + y^m = z^{2n}$ with $(m,n) = 1$ and gave a method to show that the equation admitted an infinitude of primitive solutions in integers.

His technique which required a little more than unique factorization property enabled him to obtain integer solutions to the equations $x^2 + y^3 = z^4$ and $x^2 + y^4 = z^6$. Employing the technique developed by Boyd the equation $x^2 + y^5 = z^6$ is solved for its integer solutions.
In this section we briefly sketch the procedure of Boyd in case of a general equation \( x^2 + y^m = z^{2n} \), for its integer solutions. By a primitive solution to a given equation we mean a solution in \( x, y, z \) with \( xyz \neq 0 \) and \( x > 0, z > 0 \). Further, such a solution is not of the form \( (k^{mn} x, k^{2n} y, k^m z) \) for any integer \( k > 0 \). In case \( x_0, y_0, z_0 \) forms a primitive solution to the equation then each solution is thus of the form

\[
( k^{mn} x_0, k^{2n} y_0, k^m z_0 ) , k \geq 1 .
\]

We see that the two solutions to the equations lie in the same class if they correspond to the same primitive solution. Thus, \((28, 8, 6)\) constitute a primitive solution of the equation \( x^2 + y^3 = z^4 \). The solutions determined by Kiss are not all primitive solutions of this equation. Based on a very simple observation, Boyd gave a method which we now describe. Given an integer \( m > 1 \), any integer \( N > 1 \) can be written uniquely in the form

\[
N = s_1 s_2^2 \cdots s_{m-1}^{m-1} U^m
\]

Where \( s_1, s_2, \cdots, s_{m-1} \) are positive square-free and pair wise relatively prime.

(Here we consider 1 to be square-free and allow \( s_1 = 1 \).)

To this end,

let \( N = \Pi p^{\nu(p)} \) be the natural decomposition of \( N \) into prime powers and define

\[
s_k = \Pi \{ p : a(p) \equiv k \pmod{m} \} , \text{ for } 1 \leq k \leq m-1 , \text{ where } s_k = 1 \text{ if the product is empty.}
\]

Clearly, \( N / (s_1 s_2^2 \cdots s_{m-1}^{m-1}) \) is an \( m^{th} \) power.

Suppose that \((x, y, z)\) is a solution of

\[
x^2 + y^m = z^{2n}
\]

(4.4.1)
with \(xyz \neq 0\) and \(x > 0, z > 0\).

Then (4.4.1) can be written as a product decomposed into
\[
(z^n + x)(z^n - x) = y^m
\] (4.4.2)

By writing \(z^n + x\) as
\[
z^n + x = s_1 s_2^{m-1} u^m
\] (4.4.3)
\[
z^n - x = s_1 s_2^{m-1} v^m
\] (4.4.4)

with \((s_1, s_2, \ldots, s_{m-1})\) as a set of parameters and \(u > 0\).

Consequently,
\[
y = s_1 s_2^{m-1} u^m v
\] (4.4.5)

for some integer \(v\), with signature of \(v\) is same as that of \(y\).

Changing \((x, y, z)\) to \((k^m x, k^{2n} y, k^m z)\) then \((z^n \pm x)\) gets multiplied by the factor \(k^m = (k^n)^m\). As a result \(s_1, s_2, \ldots, s_{m-1}\) are determined by the class of the solution. Further, it would also determine the ratio \(u / v = a / b\) where \(a\), \(b\) are relatively prime and \(a > 0\).

If \(s_1 s_2^{m-1} a^m > s_1^{m-1} s_2^{m-2} \ldots s_{m-1} |b|^m\) (4.4.6)

Which is true in fact from the assumption that \(x, z\) are positive.

Then (4.4.1) will have a unique solution satisfying (4.4.3), (4.4.4) and (4.4.5), with
\[
u / v = a / b.
\]

From (4.4.3) and (4.4.4) and writing \(u = a w\) and \(v = b w\) where \(w > 0\) which is to be determined, we see that
\[
2x = (s_1 s_2^{m-1} a^m - s_1^{m-1} s_2^{m-2} \ldots s_{m-1} b^m) w^m = A w^m
\] (4.4.7)
2z^n = (s_1s_2^{-1} \cdots s_{m-1}^{m-1}a^m + s_1^{-1}s_2^{-1} \cdots s_{m-1}^{-1}b^m)w^m = Bw^m \quad (4.4.8)

where A, B are constants determined by s_1, s_2, \ldots, s_{m-1} and a, b. The inequality (4.4.6) ensures that A, B are positive. Since, they have the same parity, if the equation (4.4.8) has a solution (z, w) then the equation (4.4.1) has a solution (x, y, z) where x is given by the equation (4.4.7) and y by the equation (4.4.5).

4.5 THE EQUATION \( x^2 + y^5 = z^6 \)

From this sub-title equation we have \( m = 5 \) and \( n = 3 \). So, \( m, n \) are relatively prime. Adopting the procedure given in the earlier section, we can at once write expressions for the constants A and B, thus given by

\[
A = s_1 s_2^{-2} s_3^{-3} s_4^{-4} a^4 - s_1^{-4} s_2^{-3} s_3^{-2} s_4 b^4
\]

and

\[
B = s_1 s_2^{-2} s_3^{-3} s_4^{-4} a^4 + s_1^{-4} s_2^{-3} s_3^{-2} s_4 b^4
\]

where each \( s_i \) are square free and satisfy the condition if (4.6) and a, b are chosen accordingly.

Now we choose the values for parameters \( s_1, s_2, s_3, s_4, a \) and b as follows:

Let \( s_1 = s_2 = s_3 = a = b = 1 \) and \( s_4 = 2 \). This choice of values is not totally arbitrary, since \( s_k \)'s are guided by the decomposition of N into primes. Moreover \( m, N \) are fixed such choices which are at the least finite. These values of \( s_i \)'s when substituted would give us for A, B the values A = 14 and B = 18.
We solve (4.4.7) and (4.4.8) for $x$ and $z$, by substituting these values. Thus, from (4.4.8),
\[ 2z^3 = 18w^5 \]
which is the same as
\[ z^3 = 9w^5 = 3^2 w^5 \]
Now taking $w = 3^2$ we see that
\[ z^3 = (3^2)^5 = (3^4)^3 \]
and hence
\[ z = 3^4 \]
From (4.4.7)
\[ x = (A w^5) / 2 = 14 / 2 . (3^2)^5 \]
which is the same as
\[ x = 3^{10} . 7 \]
and J's solving (4.4.5) for $y$ gives
\[ y = 2 . 3^4 \]
Consequently
\[ (3^{10} . 7)^2 + (2 . 3^4)^5 \]
\[ = 3^{20} . 7^2 + 2^5 . 3^{20} \]
\[ = 3^{20} (7^2 + 2^5) \]
\[ = 3^{20} . 81 = 3^{24} = (3^4)^6 \]
Thus
\[ (413343, 162, 81) \] constitutes a solution to the equation.
Next, choose $s_3 = 2 = s_4$ and $s_1 = s_2 = a = b = 1$. Then we can give another solution to it.

In this case, the values of $A$ and $B$ are given by $A = 120$, $B = 136$ respectively.

Solving for $z$, $x$ and $y$ as done before, we notice that (4.4.8), (4.4.7) and (4.4.5) respectively would give

$$2z^3 = 136w^5$$

(from (4.4.8)).

which gives

$$z^3 = 2^{2\cdot17}w^5$$

Taking $w = 2^{2\cdot17}$ we observe that

$$z^3 = 2^{2\cdot17}(2^{2\cdot17})^5 = (2^{2\cdot17})^6$$

Thus $z$ is given by

$$z = 2^{4\cdot17^2}.$$ 

From (4.4.7) $x$ can be obtained.

Thus

$$x = (120w^5)/2 = 2^{2\cdot35}(2^{2\cdot17})^5$$

which gives

$$x = 2^{12\cdot3\cdot5\cdot17^5}$$

and finally $y$ is given by

$$y = 2^{6\cdot17^2},$$

(from (4.4.5))

Then,

$$x^2 + y^5 = (2^{12\cdot3\cdot5\cdot17^5})^2 + (2^{6\cdot17^2})^5$$

$$= 2^{24\cdot3\cdot5^2\cdot17^{10}} + 2^{30\cdot17^{10}}$$

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= 2^{24}.17^{10}(3^2.5^2 + 2^6)
= 2^{24}.17^{10}(225 + 64)
= 2^{24}.17^{10}(289)
= 2^{24}.17^{12}
and which is
z^6 = (2^{4}.17^2)^6
So, \((x , y , z)\) in this case is given by \((87236014080 , 18496 , 4624)\) which constitutes yet another solution to the equation.

Observe that the earlier solution that is \((413343 , 162 , 81)\) is smaller compared to the second solution \((87236014080 , 18496 , 4624)\)

Also the parameters \(s , s\) are symmetric in their exponent representation and moreover square-free, the choices are limited and exhaustive.

Further, the solutions are primitive solutions.

Thus for some \(k > 1\),
\((k^{15}.3^{10}.7 , k^6.2.3^4 , k^5.3^4)\)
a family of solutions for the primitive solution \((3^{10}.7 , 2.3^4 , 3^4)\)
while \((k^{15}.2^{12}.3.5.17^5 , k^6.2^6.17^2 , k^5.2^4.17^2)\) in respect of the primitive solution \((2^{12}.3.5.17^5 , 2^6.17^2 , 2^4.17^2)\).

REFERENCES:
1.D.W.Boyd The diophantine equation \(x^2+y^m = z^n\), notes

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the American Mathematical Monthly, pp 544-547, vol 95, no. 6 (1988).


7. E. Kiss
(September 1996, 26-31, 34)
Rezolvarea in numere naturale a ecuatiei
diofanteene \(x^2 + y^3 = 2^4\), studia Univ. Babes –
Bolyia ser. I.Math Phys. no. 1 (1960), 15-19
(MR 26# 1279)

8. S. Lang
Old and new conjectured diophantine
inequalities, Bull. Amer. Math. Soc. 23
(1990), 37-75.

9. D. Masser
Open problems, Proc. sympos. Analytic
Number theory (W.W.L. chened.) Imperial
college, London 1985

10. B. Mazur
Questions about powers of Numbers,
Notices, AMS vol. 47 No 2, Feb 2000, 195-202

11. J. Oesterlie
Nouvelles approaches du "Theorem 'de
Fermat, Asterisque 161/162 (1988), 165-186