CHAPTER III

THE EQUATION $x^3 = Ny^2 \pm 8$

3.1 INTRODUCTION

In this chapter equation of the form $x^3 = Ny^2 \pm 8$ are studied. J.H.E. Cohn, [1], [2] and R.J. Stroeker, [3] have studied the diophantine equation $x^3 = Ny^2 \pm 1$ for their integer solutions. Cohn generalized the results of Stroeker in case of $N = 5$ and proved the following theorems.

**Theorem 3.1.1:** Let $N$ denote a positive integer with no prime factor $p \equiv 1 \pmod{3}$. Then the equation $x^3 = Ny^2 + 1$ has no integer solution in positive integers.

**Theorem 3.1.2:** Let $N$ be as in the earlier theorem. Then the equation $x^3 = Ny^2 - 1$ has no solution in positive integers; the following cases excepted:

<table>
<thead>
<tr>
<th>$N$</th>
<th>1</th>
<th>2</th>
<th>2</th>
<th>8</th>
<th>9</th>
<th>18</th>
<th>72</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>2</td>
<td>1</td>
<td>23</td>
<td>23</td>
<td>2</td>
<td>23</td>
<td>23</td>
</tr>
<tr>
<td>$y$</td>
<td>3</td>
<td>1</td>
<td>78</td>
<td>39</td>
<td>1</td>
<td>26</td>
<td>13</td>
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In the course of his proof Cohn formulated the following Lemmas.

**Lemma 3.1.1:** The equation $x^4 - 3y^2 = 1$ has no solution in positive integers. **Lemma 3.1.2**

The equation $4x^4 - py^2 = 1$ where $p$ denotes a prime has no solution in positive integers except for $p = 3$, $x = 1$, $y = 1$ and for $p = 7$, $x = 2$, and $y = 3$.

For proofs we refer to [1].
In this chapter the equations \( x^3 = N y^2 \pm 8 \) for their integral solutions are studied, on the lines similar to the one developed by Cohn and Stroeker. In the course of their proofs, we come across certain equations which are formulated in the form of Lemmas; these Lemmas run parallel to the Lemmas obtained by them in case of the equations \( x^3 = N y^2 \pm 1 \).

Consequently, these Lemmas lead to the theorems in case of the equations for their integer solutions.

### 3.2. THE EQUATIONS \( x^3 = N y^2 \pm 8 \).

Firstly we formulate the following Lemmas.

**Lemma 3.2.1:** The equation \( 4x^4 - 3y^2 = 8 \) has no solution in positive integers.

**Proof:** The given equation may be written as,

\[
4x^4 - y^2 = 2(y^2 + 4) \tag{3.2.1}
\]

which is the same as

\[
(2x^2 + y)(2x^2 - y) = 2(y^2 + 4) \tag{3.2.2}
\]

Since a square is either congruent to 0 or 1 (mod 4) \((3.2.2)\) does not hold for any integer values of \(x\) and \(y\).

Therefore, LEMMA follows.

**Lemma 3.2.2:** The equation \( 6x^4 - 5y^2 = 2 \)

**Proof:** \( 6x^4 - 5y^2 = 2 \) can be written as

\[
6x^4 - y^2 = 2(2y^2 + 1) \tag{3.2.3}
\]
by some argument (3.2.3) is not satisfied by any integral values of \( x \) and \( y \). We now formulate a theorem and make use of the above Lemmas to ascertain it.

**Theorem 3.2.1**: Let \( N \) denote a positive integer with no prime factor \( p \equiv 1 \pmod{3} \). Then the equation \( x^3 = N y^2 + 8 \) has no solution in positive integers.

**Proof**: Without loss of generality assume that \( N \) is square free. Then by rewriting the equation we note that

\[
N y^2 = x^3 - 8 = (x - 2)(x^2 + 2x + 4)
\]

We show that \((x^2 + 2x + 4, N) = 1\) or \(3\)

If \( p \mid N \) then \( p \mid (x - 2)(x^2 + 2x + 4) \)

Since \( p \) is prime this implies either

\[
p \mid (x - 2) \text{ or } p \mid (x^2 + 2x + 4)
\]

and moreover \( p > 3 \).

Hence,

\[
(x - 2, x^2 + 2x + 4) = 1 \text{ or } 3
\]

\(N\) has no prime factor \( = (x + 1)^2 + 3\) and \((-3/p) = -1\). So \( x^2 + 2x + 4 \) has no prime factor \( p \equiv 2 \pmod{3} \).

Thus we have either

\[
x^2 + 2x + 4 = a^2 \quad x - 2 = Nb^2 \quad \text{(i)}
\]

or

\[
x^2 + 2x + 4 = 3a^2 \quad x - 2 = 3Nb^2 \quad \text{(ii)}
\]
(3.2.4) would imply

\[(x + 1)^2 + 3 = a^2\]  (3.2.5)

and (3.2.5) is possible only for \(x = 0\) which is not a solution to the original equation.

Eliminating \(x\) from (3.2.4) (ii) we notice that

\[a^2 = 1 + 3(2Nb^2 + N^2b^4 + 1)\]  (3.2.6)

here \(a\) is odd hence \(Nb^2 = -1\) or \(0\) (mod 8).

We discuss the above congruence by turn.

Case I: Let \(Nb^2 = -1\) (mod 8).

From (3.2.6) we notice that

\[4a^2 = (4 + 3Nb^2)^2 + 3N^2b^4\]  (3.2.7)

which is the same as

\[3N^2b^4 = (2a + 4 + 3Nb^2)(2a - 4 - 3Nb^2)\]  (3.2.8)

On the right hand side of (3.2.8) observe that no two factors have factors common to them.

\[2a \pm (4 + 3Nb^2) = P^2c^4, 2a \pm (4 + 3Nb^2) = 3Q^2d^4\]  (3.2.9)

Where

\[N = PQ \text{ and } b = cd\]

Hence

\[\pm 2, (4 + 3Nb^2) = P^2c^4 - 3Q^2d^4\]  (3.2.10)

The lower sign must be rejected modulo 3. Thus,
Here we must have \( P = 1 \). For, \( N = PQ \) and \( N \) is odd, \( P \) must be odd.

If a prime \( p \) divides \( P \), then by hypothesis \( p \equiv 2 \pmod{3} \) which would make (3.2.11) impossible. Thus (3.2.11) reduces to

\[
8 = 4c^4 - 3e^2
\]

(3.2.12)

Where

\[
e = c^2 + Nd^2
\]

(3.2.13)

By Lemma 3.2.1 this case cannot arise.

Case II: \( Nb^2 \equiv 0 \pmod{8} \). Then since \( N \) is square free \( b \) must be even.

Suppose \( b = 2\beta \), for some \( \beta \) a non-negative integer. Then

\[
4a^2 = (4 + 12N\beta^2)^2 + 48N^2\beta^4
\]

which on simplification gives

\[
a^2 = (2 + 6N\beta^2)^2 + 12N^2\beta^4
\]

(3.2.14)

(3.2.14), in turn can be written as

\[
12Nb^2\beta^4 = (a+2+6N\beta^2)(a-2-6N\beta^2)
\]

(3.2.15)

From (3.2.15) we observe that,

\[
a \pm (2 + 6N\beta^2) = 6P^2c^4, a\beta (2 + 6N\beta^2) = 2Q^2d^4
\]

(3.2.16)

where \( N = PQ \), \( b = cd \).

Hence

\[
\pm Q (2 + 6N\beta^2) = 3P^2c^4 - Q^2d^4
\]

(3.2.17)
As before, the sign modulo 3 is again discarded. Thus,

\[ 2 = 3P^2 c^4 - 6PQc^2d^2 - Q^2 d^4 \]  

(3.2.18)

Once again we notice that \( p \) cannot have any other odd factor than 1. As a result (3.2.18) becomes

\[ 2 = 6 c^4 - 5 e^2 \]  

(3.2.19)

where \( e = (c^2 + N d^2) \)  

(3.2.20)

By Lemma 3.2.2 this case also does not arise.

In case \( P = 2 \) even then (3.2.18) would imply

\[ 2 = 24 c^4 - 3 e^2 \]  

(3.2.21)

where \( e = (2c^2 + Qd^2) \)  

(3.2.22)

which is impossible modulo 8. This concludes the proof of the theorem.

REMARK 3.2.1:

1. The decomposition

\[ a + (2 + 6 N \beta^2) = 4 P^2 c^4 \]

\[ a - (2 + 6 N \beta^2) = 3Q^2 d^4 \]

For their product would lead to similar conclusions resulting in some diophantine equations not admitting integer solutions.

2. Same is the case with the other combination of factors of 12.

3. The equation \( x^3 = Ny^2 - 8 \)

Can be similarly dealt with N as in theorem 3.2.1
Theorem 3.2.2: Let $N$ denote a positive integer with no prime factor $p \equiv 1 \pmod{3}$. Then the equation $x^3 = Ny^2 - 8$ has no solution in positive integers.

Proof: From the given equation we note that

$$x^3 + 8 = (x + 2) (x^2 - 2x + 4)$$

We show that

$$(x^2 - 2x + 4, N) = 1 \text{ or } 3$$

Again as noted earlier,

If $p | (x^2 - 2x + 4, N)$.

Then $p | (x^2 - 2x + 4)$, or $p | N$ and moreover $p > 3$

Hence $(x + 2, x^2 - 2x + 4) = 1 \text{ or } 3$.

$N$ has no prime factor $\equiv 1 \pmod{3}$

Since

$$x^2 - 2x + 4 = (x - 1)^2 + 3$$

Since

$$(-3/p) = -1 \text{ so } x^2 - 2x + 4 \text{ has no prime factors } p \equiv 2 \pmod{3}$$

Thus we have either of the following possibilities

$$x^2 - 2x + 4 = a^2 \quad x + 2 = Nb^2 \quad (3.2.23)$$

$$x^2 - 2x + 4 = 3a^2 \quad x + 2 = 3Nb^2 \quad (3.2.24)$$

$(3.2.23)$ would imply

$$(x - 1)^2 + 3 = a^2$$

This is possible if $x = 0$ and $x = 2$
Since $N \equiv (\text{mod } 3)$

\[ 16 = N y^2 \iff N = 1 \]

\[ x = 3N b^2 - 2 \]

Substituting this value in (3.2.24) we get,

\[ 3a^2 = (3N b^2 - 2)^2 - 2 (3N b^2 - 2) + 4 \]

Which on simplification reduces to

\[ a^2 = 3N b^4 - 6Nb^2 + 4 \]

\[ = 1 + 3(N^2 b^4 - 2Nb^2 + 1) \quad (3.2.25) \]

Here $a$ is odd hence $N b^2 \equiv -1$ or $0 \text{ (mod } 8)$

Case 1: Let $N b^2 \equiv -1 \text{ (mod } 8)$

From (3.2.25) we notice that

\[ 4a^2 = (4 - 3Nb^2)^2 + 3N^2 b^4 \]

which is same as

\[ 3N^2 b^4 = (2a + (4 - 3Nb^2)) (2a - (4 - 3Nb^2)) \]

The factors in the above expression are relatively prime. Therefore, we must have

\[ 2a \pm (4 - 3Nb^2) = L^2 r^4 \]

\[ 2a + (4 - 3Nb^2) = 3T^2 q^4 \]

where $N = L T$ and $b = rq$

Hence $\pm 2 (4 - 3Nb^2) = L^2 r^4 - 3T^2 q^4$

Again the lower sign is discarded modulo 3. Thus

\[ 8 = L^2 r^4 + 6Nb^2 - 3T^2 q^4 \]
\[ 8 = L^2 r^4 + 6 LTb^2 - 3T^2 q^4 \]  \hspace{1cm} (3.2.26)

Here we must have \( L = 1 \), for \( N = LT \) and \( N \) is odd, \( L \) must be odd. If a prime \( p \mid L \), then by hypothesis \( p \equiv 2 \pmod{3} \) which would make (3.2.26) impossible.

REFERENCES:

1. J.H.E. Cohn  
   The diophantine equations \( x^3 = Ny^2 \pm 1 \)  

2. J.H.E. Cohn  
   On the diophantine equations \( x^4 - Dy^2 = 1 \)  

3. R.J. Stroeker  
   On the diophantine equations \( x^3 - Dy^2 = 1 \)  
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