CHAPTER II

THE EQUATION $X^4 + 5X^2Y^2 + Y^4 = Z^2$ AND OTHER RELATED EQUATIONS

2.1 : INTRODUCTION

Despite being remained as an unsolved conjecture for nearly three hundred and odd years, the Fermat's Last Theorem has been a landmark result in the whole of Number theory and in particular Mathematics in general. Its role in ascertaining certain results in number theory and diophantine equations is indeed noteworthy. There are many such results for whose validity depended upon this conjecture. In this chapter we consider one such result due to Lebesgue [3]

Proposition (Lebesgue) : For any natural number n, if $X^n + Y^n = Z^n$ has no non-trivial solutions in integers then so also the equation $X^{2n} + Y^{2n} = Z^2$, does not possess integer solutions in which $xyz \neq 0$.

For $n = 2$, even though $x^2 + y^2 = z^2$ exhibits an infinitude of integer solutions, the equation $x^4 + y^4 = z^2$ does not have non-degenerate solution in which $x, y, z$ are integers. This result was proved by Fermat himself. Tanner and Wagstaff, [6] have studied this problem for $3 \leq n \leq 1,50,000$ and have observed that for this range of $n$ the equation

$$x^{2n} + y^{2n} = z^2 \quad (2.1.1)$$

had only a trivial solution. Further, Kausler [2] and Kapferer [1] have considered the equation for $n = 3$ and established that it had no non-trivial
solutions. Interestingly enough there are some results relating to $X^{2n} \pm Y^{2n} = Z^2$ by Smith, [5]. He studied $X^{10} \pm Y^{10} = Z^2$ and in the process he came up with certain results that link $x^{2n} + m x^n y^n + y^{2n} = z^2$, $n \geq 2$ and $m$ some odd composite number. The following conjectures are due to him.

Conjecture 2.1.1: There are no non-trivial solutions to $x^{2n} \pm y^{2n} = z^2$, for $n \geq 2$

In this chapter we formulate a result with regard to a homogenous equation for $n = 2$ and $m = 5$ and show that it does not have any integer solutions and further we also generalise it to an arbitrary $m$.

2.2 THE EQUATION $x^4 + 5x^2 y^2 + y^4 = z^2$:

While investigating the diophantine equations $x^{10} \pm y^{10} = z^2$, Smith formulated some results. The following equation is due to him.

$$x^4 + 3x^2 y^2 + y^4 = z^2$$

(2.2.1)

He observed that, it did not possess non-degenerate solutions. On the basis of his arguments, the following proposition is formulated and it is due to [7].

Proposition 2.2.1: The equation $x^4 + 5x^2 y^2 + y^4 = z^2$ does not admit integer solutions in $x, y, z$ for which $xyz \neq 0$
Proof: Suppose that the equation \( x^4 + 5x^2y^2 + y^4 = z^2 \) admits a solution in which \( x, y, z \) are all non-zero integers and \( z \) is minimal. Then for \( z > 0 \) it is not difficult to see that \( x, y, z \) are relatively prime.

This would imply that at least one among \( x, y \) is odd. Since the equation is symmetric we can assume \( y \) to be odd (the argument holds good for \( x \) also).

Multiplying the given equation by 4 and re-arranging the terms we observe that the equation reduces to an equivalent form which is given by

\[
(2x^2 + 5y^2)^2 - 4z^2 = 21y^4 \tag{2.2.2}
\]

If \( 21 \mid z \), then from (2.2.1) we observe that \( 21 \mid (2x^2 + 5y^2) \) and therefore, \( (2x^2 + 5y^2)^2 - 4z^2 \) is divisible by \( 21^2 \). This would imply \( 21 \mid y \) which is a contradiction to the fact that \( (y, z) = 1 \). Therefore, 21 does not divide \( z \).

Factorizing (2.2.2) we get

\[
(2x^2 + 5y^2 + 2z)(2x^2 + 5y^2 - 2z) = 21y^4 \tag{2.2.3}
\]

In (2.2.3), any common divisor of the two factors appearing on the left hand side should divide their difference and their product as well. That is, \( 4z \) and \( 21y^4 \) are relatively prime. However,

\[
(4z, 21y^4) = 1
\]

Since \( (y, z) = 1 \) and 21 does not divide \( z \).

Therefore,

\[
(2x^2 + 5y^2 + 2z, 2x^2 + 5y^2 - 2z) = 1 \tag{2.2.4}
\]

From (2.2.3) and (2.2.4), we have the following implications.
I. \(2x^2 + 5y^2 + 2z = 21a^4\), \(2x^2 + 5y^2 - 2z = b^4\)

II. \(2x^2 + 5y^2 + 2z = b^4\), \(2x^2 + 5y^2 - 2z = 21a^4\)

for some integers \(a, b\) such that \((a, b) = 1\).

Then

\[ab = y \quad (2.2.5)\]

Since \(y\) is odd and both \(a, b\) are relatively prime both of them are odd.

In either case adding the equations I and II we notice that

\[4x^2 + 10y^2 = 21a^4 + b^4\]

which is the same as

\[4x^2 = 21a^4 + b^4 - 10y^2 \quad (2.2.6)\]

Eliminating \(y^2\) from (2.2.6) by using (2.2.5) we obtain

\[4x^2 = 21a^4 - 10a^2b^2 + b^4 \quad (2.2.7)\]

Now the following cases arise.

Case 1: If \(b < a\) then (2.2.7) factors into

\[4x^2 = (7a^2 - b^2)(3a^2 - b^2) \quad (2.2.8)\]

Further

\[(7a^2 - b^2) - (3a^2 - b^2) = 4a^2 \quad (2.2.9)\]

Therefore, the gcd of \((7a - b^2)\) and \((3a^2 - b^2)\) divides \((4x^2, 4a^2) = 4 (x^2, a^2) = 4\)

Since \(a \mid y\) and \((x, y) = 1\),

hence \((x^2, a^2) = 1\)

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But \( y = ab \) is odd and so \( a \) and \( b \) and further \( 7a^2 - b^2 \) and \( 3a^2 - b^2 \) are even and congruent to 2 (mod 4).

Therefore

\[
(7a^2 - b^2, 3a^2 - b^2) = 2
\]

and (2.2.9) gives

\[
7a^2 - b^2 = 2c^2, 3a^2 - b^2 = 2d^2 \tag{2.2.10}
\]

Where again \((c, d) = 1\) and \(cd = x\). Since \(a\) and \(b\) are odd, \(a^2 = b^2 \equiv 1 \pmod{8}\)

\[
3a^2 - b^2 = 2d^2 \text{ in (2.2.10)} \implies d^2 \equiv 1 \pmod{8}
\]

so \(d\) must be odd and \(7a^2 - b^2 = 2c^2\) gives \(c^2 \equiv 3 \pmod{8}\), a contradiction.

Case 2:

If \(a < b\) then (2.2.7) factors into

\[
4x^2 = b^2 - 7a^2 \text{ and } b^2 - 3a^2 = 2e^2
\]

for some integers \(e, f\) such that \((e, f) = 1\) and \(ef = x\)

Now, reversing the arguments as done in earlier in case 1, we observe that case 2 also leads to a contradiction. Consequently the proposition follows:

1. Since \((a, b) = 1\) the case of \(a = b\) does not arise

2. By adding the equations I and II would result in an equation of the type,

\[
x^4 + 5x^2y^2 + y^4 = z^2
\]

independent of \(z\) and fall in line with the ones already discussed.
3. The equation can be generalized to $x^4 + kx^2y^2 + y^4 = z^2$. This is being considered in the next section.

2.3 THE EQUATION $x^4 + kx^2y^2 + y^4 = pz^2$

Smith [5] noted the equation $x^4 + 3x^2y^2 + y^4 = 5z^2$ (LEMMA 1.2) is such that $x$, $y$ and $z$ are all odd. This result enabled him to study $x^{10} ± y^{10} = z^2$. Here this equation has been generalized in the following proposition.

**Proposition 2.3.1**: Let $x$, $y$ and $z$ be pair wise relatively prime integers such that $x^4 + kx^2y^2 + y^4 = pz^2$, where $k$ is some odd positive integer and $p$ some prime of the form $4m + 1$ then, $x$, $y$ and $z$ are all odd.

Proof: If $z$ is even then $x$ and $y$ are odd, since $(x, z) = 1 = (y, z)$.

But then $x^4 + kx^2y^2 + y^4 = pz^2$ is odd a contradiction. Hence $z$ is odd.

Let at least one of $x$, $y$ be odd. Since the equation in symmetric in $x$ and $y$ let us suppose that $x$ is odd. Then multiplying the given equation by $4$ and rearranging the terms we see that

$$(2x^2 + ky^2) - (k^2 - 4)y^4 = 4pz^2$$

(2.3.1)
Suppose y is even. Say y = 2t for some non-zero integer t. Now, k^2 - 4 does not divide y. For, if k^2 - 4 | y then the equation in the Lemma shows that k^2 - 4 | x as well contradicting (x, y) = 1. Thus, k^2 - 4 does not divide t.

Therefore, either

1. \( t^2 \equiv 1 \mod (k^2 - 4) \) \hspace{1cm} (2.3.2)
2. \( t^2 \equiv -1 \mod (k^2 - 4) \)

Thus at least one of \( t^2 + z \), \( t^2 - z \) is not divisible by \( k^2 - 4 \).

Case I: Suppose \( k^2 - 4 \) does not divide \( t^2 - z \)

Now, taking \( (k^2 - 4) \) to be the right hand side, we may express (2.3.1) as

\[
(k^2 - 4) y^4 + 4p z^2 = (a y^2 + bz)^2 + (c y^2 - dz)^2
\]

where a, b, c, d are some non-zero pair wise relatively prime integers satisfying certain conditions which would be given in a moment of time. Expanding the right hand side expression and comparing the coefficients of the like terms we notice that,

\[
a^2 + c^2 = k^2 - 4 \hspace{1cm} (i) \\
b^2 + d^2 = 4p \hspace{1cm} (ii) \\
ab - cd = 0 \hspace{1cm} (iii)
\]

Then (2.3.1) can be written as

\[
(2x^2 + ky^2)^2 = (ay^2 + bz)^2 + (cy^2 - dz)^2 \hspace{1cm} (2.3.4)
\]

Now,

Put \( y = 2t \) in (2.3.4), Then we obtain
\[(2x^2 + 4kt^2)^2 = (4at^2 + bz)^2 + (4ct^2 - dz)^2 \quad (2.3.5)\]

Next our claim is \((4at^2 + bz, 4ct^2 - dz) = 1\)

For this, observe that
\[
\begin{align*}
  b(4at^2 + bz) - d(4ct^2 - dz) &= 4abt^2 + b^2z - 4cdt^2 + d^2z \\
  &= 4kt^2 + b^2z - 4kt^2 + d^2z
\end{align*}
\]
which in view of (2.3.3) becomes 4pz.

Hence,
\[
b(4at^2 + bz) - d(4ct^2 - dz) = 4pz \quad (2.3.6)
\]

Similarly
\[
a(4at^2 + bz) + c(4ct^2 - dz) = 4(a^2 + c^2)t^2 \quad (2.3.7)
\]

Again in view of (2.3.3) this reduces to
\[
a(4at^2 + bz) + c(4ct^2 - dz) = 4(k^2 - 4)t^2
\]

Thus
\[
(4at^2 + bz, 4ct^2 - dz) | (4pz, 4(k^2 - 4)t^2)
\]

Since z is odd \((z, t) = 1\)

The latter holds because \(t | y\) and \((z, y) = 1\). Therefore
\[
(4at^2 + bz, 4ct^2 - dz) = 1 \text{ or } 4
\]

But 4 does not divide \((t^2 - 2)\) \(\text{ (our assumption) }\). So
\[
(4t^2 + z, 2t^2 - 2z) = 1
\]

From (2.3.5), then one of \((4ct^2 - dz, 4at^2 + bz, 2x^2 + 4kt^2)\) or \((dz - 4ct^2, 4at^2 + bz, 2x^2 + 4kt^2)\)

is a primitive Pythagorean triple. In either case, there exists \(p\) and \(q\) such that
\[
2x^2 + 4kt^2 = p^2 + q^2 \text{ with } x \text{ odd.}
\]
Then $t$ is even; else

$$2x^2 + 4kt^2 = 3 = p^2 + q^2 \pmod{4}$$

which is impossible. On the other hand, since $x$ and $z$ are odd

$$(2x^2 + 4kt^2)^2 = (4at^2 + bz)^2 \equiv 1 \pmod{8}$$

Hence, (2.3.5) implies $4 \mid (dz - 4ct^2)$

But $z$ is odd, so therefore $t$ is odd and thus we have a contradiction.

Case 2: Suppose $5$ does not divide $t^2 + z$, then we can rewrite the equation (2.3.1) as done earlier and argue on similar lines by taking $y = 2t$. This would also lead to a conclusion that $y$ is odd. This result has a bearing in studying diophantine equations $x^{2n} \pm y^{2n} = z^n$, for suitable $n$.

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