Chapter 3

Wiener Number of a Maximal Outerplanar Graphs
CHAPTER-3

WIENER NUMBER OF A MAXIMAL OUTER PLANAR GRAPH'S.

INTRODUCTION

In a class of trees, the star \( K_{1,n-1} \) has a minimum wiener number and a path \( P_n \) has a maximum wiener number. In both the cases, the minimum degree \( \delta(G) = 1 \). That is, the class of graphs \( G \) in which \( \delta(G) = 1 \). We can get the extreme class graph of graphs namely stars and paths with respect to the wiener number. In other words, for any graph \( G \) with \( \delta(G) = 1 \),

\[
W(K_{1,n-1}) \leq W(G) \leq W(P_n)
\]

holds, where \( n \) is the number of vertices in \( G \). So, in general, the question remain unanswered is that, what are the class of extremal graphs \( G \) with the minimum degree \( \delta(G) \geq 2 \). The natural way to proceed in this direction for the class of graphs with \( \delta(G) = 2 \), in general and in particular for the maximal outer planar graphs, since every maximal outer planar graphs contains at least one vertex of degree 2. Before going to consider the wiener number of a maximal outer planar graphs we define the maximal outer planar graphs and some existing results concerning with maximal outer planar graphs.

A graph \( G \) is said to be embedded in a surface \( \delta \), when it is drawn on \( \delta \) so that no two edges intersect. A graph \( G \) is planar if it can be embedded in a plane. A planar graph \( G \) is a outer planar, if it can be embedded in the plane so
that all its vertices lie on the same face. An outer planar graph $G$ is maximal outer planar, if no line can be added without losing outer planarity. It is not difficult to see that, a maximal outer planar graph is just a triangulation of a polygon.

2. EXISTING RESULTS

The following theorem and a corollary are useful in our discussion about Wiener number, which we state without proofs.

**THEOREM 3.2.1** [4] Let $G$ be a maximal outer planar graph with $n \geq 3$ vertices lying on the exterior face. Then $G$ has $n-2$ interior faces.

**COROLLARY 3.2.2** [4]. Every maximal outer planar graph $G$ with $n$ vertices, has

1. $2n-3$ edges,
2. at least three vertices of degree not exceeding 3
3. at least two vertices of degree 2
4. the vertex connectivity $K(G) = 2$.

A vertex $V$ in a graph $G$ is said to be a central vertex if $e(v) = \text{rad}(G)$. The center $C(G)$ of a graph $G$ is the set of all central vertices of $G$. The central sub graph $<C(G)>$ is the subgraph of $G$ induced by its center $C(G)$. The following theorem due to Proskurowski gives list of central sub graphs of a maximal outer planar graphs.
THEOREM 3.3 (Proskurowski [5]): If $G$ a maximal outer planar graphs, then its central sub graph $<C(G)>$ is isomorphic to one of the seven graphs of the Figure 4.1.

![Graphs G1 to G7](image)

Figure 4.1

3. RESULTS Using the above results and definitions we present here some results on Wiener number concerning the extremal class of maximal outer planar graphs.

LEMMA 3.3.1: If $H$ is a spanning subgraph of a graph $G$, then $\text{diam}(H) \geq \text{diam}(G)$.

PROOF: Let $u, v$ be any two vertices in $H$ (and hence in $G$). Then, $d_H(u, v) \geq d_G(u, v)$ holds, since $H$ is a spanning subgraph of $G$. If $u$ and
v are diametral vertices in G, then \( \text{diam} (G) = d_G (u, v) \leq d_H (u, v) \leq \text{diam} (H) \) holds; and hence the Lemma.

**PROPOSITION 3.3.2:** For any maximal outer planar graph G of order \( n \geq 4 \),

\[
2 \leq \text{diam} (G) \leq \left\lceil \frac{n}{2} \right\rceil
\]

where \( \{u\} \) denotes least positive integer not less than \( x \).

**PROOF:** For \( n \geq 4 \), if \( \text{diam} (G) = 1 \), then \( G = K_n \). But, for \( n \geq 4 \), G is not maximal outer planar graph and hence, \( \text{diam} (G) \geq 2 \) always holds.

Also, since, every maximal outer planar graph is a triangulation of a polygon, and hence a cycle \( C_n \) is a spanning subgraph of every maximal outer planar graph. Therefore, by the above Lemma, \( \text{diam} (G) \leq \text{diam} (C_n) \). But, \( \text{diam} (C_n) = \left\lceil \frac{n}{2} \right\rceil \) holds always. Hence \( \text{diam} (G) \leq \left\lceil \frac{n}{2} \right\rceil \) holds, and hence the result.

**PROPOSITION 3.3.3:** For any graph G with \( \text{diam} (G) \leq 2 \),

\[ W (G) = n^2-n-m, \]

Where \( n \) is the order and \( m \) is the size of a graph G.

**PROOF:** If \( \text{diam} (G) = 1 \), then, clearly \( G = K_n \). But, \( W (G) = W(K_n) = \frac{1}{2} n (n-1) = n^2-n-m \) holds, since \( m = \frac{1}{2} n (n-1) \).

Therefore the proposition holds, if \( \text{diam} (G) = 1 \).
Now, suppose that diam \( (G) = 2 \).

Let \( A = \{ u \in V(G) \mid e(u) = 1 \} \) and

\[
B = \{ u \in V(G) \mid e(u) = 2 \}
\]

and hence \( V(G) = A \cup B \) and \( A \cap B = \emptyset \). We consider two cases here.

**Case 1:** Let \( u \in A \).

Then, by the definition of \( A \), \( e(u) = 1 \), then \( deg(u) = n-1 \). Thus,

\[
d(u \mid G) = n-1 = 2n-2-deg(u)
\]

**Case 2:** Let \( u \in B \).

Then, \( e(u) = 2 \). Define sets \( B_i(u) \) as \( B_i(u) = \{ u \in V(G) \mid d(u, v) = i \} \), for \( i = 0, 1, 2 \). Then \( \left| \bigcup_{i=0}^{2} B_i(u) \right| = n \).

But \( d(u \mid G) = \sum_{v \in V(G)} d(u, v) = \sum_{v \in B_1(u)} d(u, v) + \sum_{v \in B_2(u)} d(u, v) \)

\[
= |B_1(u)| + 2 |B_2(u)|
\]

\[
= |B_1(u)| + |B_2(u)| + |B_2(u)|
\]

\[
= n-1 + |B_2(u)|, \text{ since } \left| \bigcup_{i=0}^{2} B_i(u) \right| = n,
\]

\[
= n-1 + (n-1-|B_1(u)|)
\]

\[
= 2n-2-|B_1(u)|
\]

\[
= 2n-2-deg(u).
\]
Thus, from cases 1 and 2, we have

\[ W(G) = \frac{1}{2} \sum_{u \in V(G)} d(u|G) \]

\[ = \frac{1}{2} \left[ \sum_{u \in A} d(u|G) + \sum_{u \in B} d(u|G) \right] \]

\[ = \frac{1}{2} \left[ (2n - 2 - \deg u)|A| + (2n - 2 - \deg u)|B| \right] \]

\[ = \frac{1}{2} \left[ (2n-2-\deg u)(|A| + |B|) \right] \]

\[ = \frac{1}{2} \left[ (2n-2)\sum_{u \in A} \deg u - \sum_{u \in B} \deg u \right] \]

\[ = \frac{1}{2} \left[ 2n(n-1) - \sum_{u \in V(G)} \deg u \right] \]

\[ = \frac{1}{2} [2n(n-1) - 2m] \]

\[ = n(n-1) - m \]

\[ = n^2 - n - m. \]

Thus for all graph G of order n, size m with \( \text{diam}(G) \leq 2 \), \( W(G) = n^2 - n - m. \) holds.

**COROLLARY 3.3.4:** For any maximal outer planar graph G of order n, with \( \text{diam}(G) = 2 \).

\[ W(G) = n^2 - 3n + m. \]
PROOF: The proof follows from the proposition 4.3.3 and the corollary 4.2.2, since, \( m = 2n - 3 \), for any maximal outer planar graph of order \( n \).

For any tree \( T \), as we know \( W(T) > W(K_{1,n}) = n^2 + 1 \). In case of maximal outer planar graphs, the question is what is the maximal outer planar graph for which the wiener number is the smallest one. In the following theorem, we prove that the wiener number of a maximal outer planar graphs of diameter two has the lowest wiener number among all maximal outer planar graphs.

**THEOREM 3.3.5:** For any maximal outer planar graph \( G \) of order \( n \),

\[
W(G) \geq n^2 - 3n + m.
\]

**PROOF:** Let \( G_1 \) be a maximal outer planar graph of order \( n \) with \( \text{diam}(G) = 2 \). Then by the above corollary 4.3.4, we have \( W(G_1) = n^2 - 3n + 3 \). To prove the theorem, it is sufficient to prove that

\[
W(G_1) \leq W(G),
\]

For any maximal outer planar graph \( G \) of order \( n \) we prove this inequality by induction on \( n \). It is obvious that the result is true for small values of \( n \). Now, assume that the result is true for all maximal outer planar graphs of orders \( < n \).

Let \( G \) be a maximal outer planar graph of order \( n \) with the labellings \( v_1, v_2, \ldots, v_{n-1}, v_n \) and \( v_n \) being the vertex of degree two, as every maximal outer planar graph contains one such vertex. By the property of maximal outer planar
graph, \( G' = G-v_n \) is again a maximal outer planar graph. Therefore, by inductive hypothesis,

\[ W(G') \geq W(G'_i) \]

where \( G'_i = G_1 . u_n \), where \( u_n \) is a vertex of degree two in \( G_1 \) and \( \text{diam} (G'_i) = 2 \).

But then,

\[ W(G) = W(G'_i) + d(v_n \mid G) \geq W(G') + 2 + 2(n-3) \]
as \( u_n \) is of degree 2 and \((n-3)\) vertices are at a distance \( \geq 2 \).

\[ = W(G') + 2n-4 \geq W(G'_i) + 2n-4, \text{ by inductive hypothesis.} \]

\[ = (n-1)^2 - 3(n-1) + 3 + 2n-4 \]

\[ = n^2 - 3n + 3 = W(G_i). \text{ This completes the proof.} \]

The following theorem gives the lower bound for the wiener number \( W(G) \) of a graph \( G \) with \( \text{diam} (G) = 3 \), in terms of its order and size.

**THEOREM 3.3.6:** For any graph \( G \) of order \( n \), size \( m \) with \( \text{diam} (G) = 3 \),

\[ W(G) \geq n^2 - n + 1 - m. \]

**PROOF:** Suppose \( G \) be a graph of diameter 3, with \( n \) vertices and \( m \) edges.

Then every vertex has an eccentricity two or three. So, let

\[ A = \{ u \in V(G) \mid e(u) = 2 \} \]
And \( B = \{ u \in V(G) \mid e(u) = 3 \} \).

But, we know that

\[
W(G) = \frac{1}{2} \sum_{u \in V(G)} d(u|G)
\]

\[
= \frac{1}{2} \left\{ \sum_{u \in A} d(u|G) + \sum_{u \in B} d(u|G) \right\}
--- (1)
\]

Here, we consider two cases to find \( d(u|G) \), for any arbitrary vertex \( u \) in \( G \) depending on whether \( u \in A \) or \( u \in B \).

**Case 1:** Let \( u \in A \).

Define a sets \( A_1(u) \) as

\[
A_1(u) = \{ u \in V(G) \mid d(u, v) = i \}
\]

For \( i = 0, 1, 2 \). Then,

\[
d(u|G) = \sum_{u \in V(G)} d(u, v)
\]

\[
= \sum_{u \in A_1(u)} d(u, v) + \sum_{u \in A_2(u)} d(u, v)
\]

\[
= |A_1(u)| + 2|A_2(u)|
\]

\[
= |A_1(u)| + |A_2(u)| + |A_2(u)|
\]

\[
= n - 1 + (n - 1 - |A_1(u)|)
\]

\[
= 2n - 2 - |A_1(u)|
\]

\[
= 2n - 2 - \text{deg}u
\]
Case 2: Let $u \in B$.

As in case 1, define the sets $B_i(u)$ as $B_i(u) = \{u \in V(G) | d(u,v) = i\}$ for $i=0, 1, 2, 3$.

\[
\therefore d(u|G) = \sum_{u \in V(G)} d(u, v) = \sum_{v \in B_1(u)} d(u, v) + \sum_{v \in B_2(u)} d(u, v) + \sum_{v \in B_3(u)} d(u, v) = |B_1(u)| + 2|B_2(u)| + 3|B_3(u)|
\]

\[
= |B_1(u)| + |B_2(u)| + |B_3(u)| + |B_2(u)| + 2|B_3(u)| = n - 1 + |B_2(u)| + |B_3(u)| + |B_3(u)| = 2n - 2 - \deg u + |B_3(u)| \quad \text{--- (2)}
\]

As $u \in B$, and hence $e(u) = 3$, so there exists at least one vertex $v \neq u$ such that $d(u, v) = 3$. Therefore, $B_3(u) \neq \emptyset$, hence $|B_3(u)| \geq 1$. Therefore, (2), becomes,

\[
d(u|G) = 2n - 2 - \deg u + |B_3(u)| = 2n - 2 - \deg u + 1
\]

Therefore, from cases 1 and 2, the equation 1 reduces to

\[
W(G) \geq \frac{1}{2} \left\{ \sum_{u \in A} (2n - 2 - \deg u) + \sum_{u \in B} 2n - 1 - \deg u \right\}
\]
\[
= \frac{1}{2} \left( (2n-2 \text{deg}_u) |A| + (2n-1 \text{deg}_u) |B| \right)
\]

\[
= \frac{1}{2} \left\{ 2n (|A| |B|) - 2 |A| - |B| + \sum_{u \in A} \text{deg}_u - \sum_{u \in B} \text{deg}_u \right\}
\]

\[
= \frac{1}{2} \left\{ 2n |A| - 2 |A| + |B| - \sum_{u \in V(G)} \text{deg}_u \right\}
\]

\[
= \frac{1}{2} \left\{ 2n^2 - 2n + |B| - 2m \right\}
\]

\[
= n^2 - n - m + \frac{1}{2} |B|
\]

\[
\geq n^2 - n - m + 1, \text{ as } |B| \geq 2.
\]

\[
\therefore W(G) \geq n^2 - n + 1 - m.
\]

This completes the proof.

**Note 1:** The equality in the above theorem holds, if and only if, \( G \) contains exactly two diametral vertices. This follows from the inequality derived in the above theorem.

**COROLLARY 3.3.7:** If \( G \) is a maximal outer planar graph of order \( n \) with \( \text{diam}(G) = 3 \), then

\[
W(G) \geq n^2 - 3n + 4.
\]

Further, the equality holds, if and only if \( G \) is one of the following graphs of the Figure 2.
PROOF: The inequality follows from the above theorem by taking $m = 2n - 3$.

To prove the second part of the corollary, if $G$ is one of the graphs of Figure 2, it is not difficult to see that $W(G_1) = 22$ and $W(G_2) = 32$. In either case, $W(G) = n^2 - 3n + 4$ holds. Conversely suppose $G$ is a maximal outer planar graph of order $n$ with $\text{diam}(G) = 3$ satisfying $W(G) = n^2 - 3n + 4$. Then by the theorem 4.3.6 and a note above, $G$ contains exactly two vertices of eccentricity three. Hence, $G$ has $n-2$ vertices of eccentricity two. Hence, these $n-2$ vertices are the vertices of a central subgraph of a graph $G$. As $G$ is a maximal outer planar graph, so, by the theorem 4.2.3, only graphs of the Figure 1, are the central subgraphs of $G$. So, the possible number of vertices in a central subgraphs of $G$ are 1, 2, 3, 4, 5 and 6. Thus the possible values of $n$ are 3, 4, 5, 6, 7 and 8. But the maximal outer planar graph with diameter three contains at least six vertices. Thus the possible central sub graphs of $G$ with $W(G) = n^2 - 3n + 4$ are of the Figure 3.
Now, we claim that $H_3$ is not an induced sub graph of $G$. For if $H_3$ is an induced sub graph of $G$, the following graphs $F_j$ of the Figure 4 are the possible maximal outer planar graphs of diameter three containing $H_3$ as an induced sub graph.

![Figure 3](image)

![Figure 4](image)

But, it is not difficult to see that $W(F_1) = 48$, $W(F_2) = 47$, and $W(F_3) = 46$. Thus, $W(F_i) > 44$, for $i = 1, 2, 3$; a contradiction to the assumption that $W(G)$
\[ n^2 - 3n + 4. \] Hence, the only left alternatives are the graphs \( G_1 \) and \( G_2 \) of the Figure 2, where \( G_1 \) contains \( H_1 \) as a central sub graph and \( G_2 \) contains both \( H_2 \) and \( H_4 \) as a central sub graphs. This completes the proof.

This is all about the lower bounds of a wiener number for the maximal outer planar graphs. Now, we consider the upper bound for \( W(G) \), when \( G \) is a maximal outer planar graph. Towards this end, let us consider the two special class of graphs \( F \) and \( H \) as in the Figure 4 along with the labellings:

![Figure 5](image)

Clearly, these are the extremal class of graphs with maximum diameter, that is \( \text{diam}(G) = \left\lceil \frac{n}{2} \right\rceil \), where \( n = 2k \) or \( n = 2k+1 \) according as \( G = H \) or \( G = F \).

Now, consider the graph \( H \). By expressing the wiener number of \( H \) in a distance matrix form as below:

\[
W(H) = \frac{1}{2} \sum_{i,j} d_{ij}
\]

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\[ \sum_{w \in \text{V(H)}} d(v_1, w) = d(v_1, v_2) + d(v_1, v_3) + \ldots + d(v_1, v_k) + d(v_1, u_1) + d(v_1, u_2) + \ldots + d(v_1, u_k) \]

Since \( d(v_1, v_i) = d(v_1, u_{i-1}) \) for \( i = 2, 3, \ldots, k \).

\[ = 2 \{ d(v_1, v_2) + d(v_1, v_3) + \ldots + d(v_1, v_k) \} \]

\[ = 2 \{ 1 + 2 + 3 + \ldots + (k-1) \} + k \]
Therefore

\[ W(H) = W(H - v_1) + \frac{n^2}{4} \quad --- (1) \]

Similarly, for the graph \( F \), we can prove that

\[ W(F) = W(F - v) + d(v \mid F) \]

\[ = W(F - v) + \frac{n^2 - 1}{4} \quad --- (2) \]

With the help of the equations (1) and (2), we prove the following theorem.

**THEOREM 3.3.8:** For any maximal outer planar graph \( G \) of order \( n \),

\[ W(G) \leq \begin{cases} W(H), & \text{if } n \text{ is even} \\ W(F), & \text{if } n \text{ is odd} \end{cases} \]

To prove this Theorem, we need to prove the following Lemma.

**LEMMA 3.3.9:** For any maximal outer planar graph \( G \) of order \( n \),

\[ W(G) \leq W(G - v) + k(n - 2 - k) \]

where \( v \) is the vertex of degree two and \( k = e_G(v) - \text{the eccentricity of } v \text{ in } G \).
PROOF: Let G be the maximal outer planar graph of order n. Since G contains at least two vertices of degree two and let \( v \) be the one such vertex in G and let 
\[ k = e_0 (v) \]
Define a sets \( A_i (v) \) as 
\[ A_i (v) = \{ u \in \nu (G) \mid d (v, u) = i \}, \text{ for } i = 0, 1, 2, ..., k. \]

Then, the vertices of G can be partitioned into the sets \( A_0 (v), A_1 (v), A_2 (v), ..., A_k (v) \) and let \( | A_i (v) | = x_i \). Clearly, \( x_i \geq 2 \), for \( i \geq 1 \), since G is a maximal outer planar graph, otherwise G would contain a cut vertex. Then clearly,
\[
\sum_{u \in \nu (G)} d(v, u) = \sum_{u \in A_0 (v)} d(v, u) + \sum_{u \in A_1 (v)} d(v, u) + \sum_{u \in A_2 (v)} d(v, u) + ... + \sum_{u \in A_k (v)} d(v, u)
\]
\[
= | A_1 (v) | + 2 | A_2 (v) | + 3 | A_3 (v) | + ... + k | A_k (v) |
\]
\[
= x_1 + 2x_2 + 3x_3 + ... + kx_k \quad --- (3)
\]

Using the relations \( x_1 + x_2 + x_3 + x_4 + ... + x_k = n-1 \), and \( x_i \geq 2 \)

Therefore,
\[
x_1 + x_2 + x_3 + x_4 + ... + x_k = n-1
\]
\[
x_2 + x_3 + x_4 + ... + x_k = n-1 - x_1 \leq n - 3
\]
\[
x_3 + x_4 + ... + x_k = n-1 - (x_1 + x_2) \leq n - 5
\]

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\[ x_k = n - 1 - (x_1 + x_2 + \ldots + x_{k-1}) \]
\[ \leq n - 1 - (2(k-1)) \]
\[ = n - 2k + 1 \]

Adding these inequalities we have, \( x_1 + 2x_2 + 3x_3 + \ldots, kx_k \leq (n-k-2) \).

Thus, (3) gives us that
\[ d(v \mid G) \leq k(n-k-2). \]

But then
\[ W(G) = W(G-v) + d(v \mid G) \]
\[ \leq W(G-v) + k(n-k-2) \]

holds.

**PROOF OF THE THEOREM:**

Let \( G \) be any maximal outer planar graph of even order \( n \). We prove the Theorem by induction on \( n \). It is not difficult to see that the result is true for small values of \( n \), that is, Viz, for \( n = 4 \), and \( n = 6 \). Assume that the result is true for all maximal outer planar graphs \( G \) of order less than \( n \). Let \( G \) be a maximal outer planar graph of even order \( n \) and \( u \) be a vertex of degree two with \( e_G(u) = k \) and \( u \) be the vertex of degree two in \( H \). Then by the Lemma 4.3.9, we have
\[ W(G) = W(G-v) + d(v | G) \]
\[ \leq W(G-v) + k(n-k-2) \]
\[ \leq W(H-u) + k(n-k-2) \text{ by inductive hypothesis} \]
\[ \leq W(H-u) + \frac{(\frac{n}{2} - 1)^2}{4}, \text{ as } k(n-k-2) \text{ is maximum only when } k = \frac{n}{2} - 1. \]
\[ \leq W(H-u) + \frac{n^2}{4} \]
\[ = W(H) \text{ by equation 1.} \]

Similarly we can prove that

\[ W(G) \leq W(F) \]

If \( n \) is odd.

This completes the proof of the theorem
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