Chapter 2

The Middle Dominating Graph of a Graph
ABSTRACT

Let \( G = (V, E) \) be a graph and \( A(G) \) is the collection of all minimal dominating sets of \( G \). The middle dominating graph of \( G \) is the graph denoted by \( M_d(G) \) with vertex set the disjoint union of \( V(G) \cup A(G) \) and \((u, v)\) is an edge if and only if \( u \cap v \neq \emptyset \) whenever \( u, v \in A(G) \) or \( u \in v \) whenever \( u \in V \) and \( v \in A(G) \). In this chapter, characterizations are given for graphs whose middle dominating graph is connected. Also we find the diameter, domatic number, vertex(edge) independence number and vertex(edge) connectivity of \( M_d(G) \).
2.1 Introduction

The graphs considered in this chapter are finite, undirected without loops or multiple edges.

Let $S$ be a finite set and let $F = S_1, S_2, \ldots, S_n$ be a partition of $S$. Then the intersection graph $\Omega(F)$ of $F$ is the graph whose vertices are the subsets in $F$ and in which two vertices $S_i$ and $S_j$ are adjacent if and only if $S_i \cap S_j \neq \emptyset$, $i \neq j$.

Let $G = (V, E)$ be a graph. A set $D \subseteq V$ is a dominating set of $G$ if every vertex in $V - D$ is adjacent to some vertex in $D$. A dominating set $D$ of $G$ is minimal if for any vertex $v \in D$, $D - \{v\}$ is not a dominating set of $G$. The domination number $\gamma(G)$ of $G$ is the minimum cardinality of a minimal dominating set of $G$. The upper domination number $\Gamma(G)$ of $G$ is the maximum cardinality of a minimal dominating set of $G$. For details on $\gamma(G)$, refer [27].

The minimal dominating graph $MD(G)$ of $G$ is an intersection graph on the minimal dominating sets of vertices of $G$ [41].

In [48], the concept of dominating graph $D(G)$ of $G$ was introduced and defined as $V(D(G)) = V(G) \cup \mathcal{S}(G)$, where $\mathcal{S}(G)$ is the set of all minimal dominating sets of $G$ with two vertices $u, v \in V(D(G))$ adjacent
if \( u \in V \) and \( v = D \) is a minimal dominating set containing \( u \).

The purpose of this chapter is to introduce a new class of intersection graph in the field of domination theory in graphs.

**Definition 2.1.** Let \( G = (V, E) \) be a graph and \( A(G) \) is the collection of all minimal dominating sets of \( G \). The *middle dominating graph* of \( G \) is the graph denoted by \( M_d(G) \) with vertex set the disjoint union of \( V(G) \cup A(G) \) and \( (u, v) \) is an edge if and only if \( u \cap v \neq \emptyset \) whenever \( u, v \in A(G) \) or \( u \in v \) whenever \( u \in V \) and \( v \in A(G) \).

In Figure 2.1, a graph \( G \) and its middle dominating graph \( M_d(G) \) are shown.

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![Figure 2.1](image-url)
2.2 Results

The followings will be useful in the proof of our results.

**Theorem 2.A** [41]. For any graph \( G \), \( MD(G) \) is complete if and only if \( G \) contains an isolated vertex.

**Theorem 2.B** [29]. For any nontrivial graph \( G \), \( \alpha_0 + \beta_0 = p = \alpha_1 + \beta_1 \).

**Theorem 2.C** [27]. If \( G \) is a graph without isolated vertices, then for any minimal dominating set \( D \), \( V - D \) is also a dominating set.

We start with a few preliminary results.

**Proposition 2.1.** \( Md(G) = K_{1,p} \) if and only if \( G = \overline{K}_p \).

**Proof.** Suppose \( G = \overline{K}_p \), then there exists exactly one minimal dominating set \( S \) containing all the vertices of \( G \). Hence the result follows from the definition of \( Md(G) \).

Conversely, assume \( Md(G) = K_{1,p} \) and \( G \neq \overline{K}_p \). Then there exists at least two disjoint minimal dominating sets \( S \) and \( S' \) which are independent in \( Md(G) \). Hence, \( \Delta(Md(G)) < p - 1 \), a contradiction. Hence \( G \) must be \( \overline{K}_p \).

**Proposition 2.2.** \( Md(G) = pK_2 \) if and only if \( G = K_p \).

**Proof.** Assume \( G = K_p \), then each vertex in \( K_p \) will form a minimal dominating set. Therefore by the definition of \( Md(G) \) each minimal dom-
inating set is adjacent to corresponding vertex of $G$ in $M_d(G)$ and form a $K_2$. Hence $M_d(G) \cong pK_2$.

Conversely, suppose $M_d(G) = pK_2$ and $G \neq K_p$. Then there exists at least one minimal dominating set $S$, such that $|S| \geq 2$. Hence $P_3$ will be an induced subgraph of $M_d(G)$, a contradiction. ■

By the definition of $M_d(G)$, the following remark is immediate.

**Remark 2.1.** For any graph $G$, $MD(G)$ and $D(G)$ are edge disjoint subgraphs of $M_d(G)$.

**Theorem 2.1** For a $(p,q)$ graph $G$, the middle dominating graph is

(a) of the type $(2p,p)$ if and only if $G \cong K_p$,

(b) of the type $(p+1,p)$ if and only if $G \cong K_p$.

**Proof.** Let $G$ be any $(p,q)$ graph. We consider the following cases.

**Case 1.** Let $M_d(G) = (2p,p)$. Since $pK_2$ is the only graph which satisfies the condition (a). Hence $M_d(G) = pK_2$. By Proposition 2.2, $G = K_p$.

**Case 2.** Let $M_d(G) = (p+1,p)$. Since $K_{1,p}$ is the only graph which satisfies the condition (b). Therefore, $M_d(G) = K_{1,p}$. Hence by Proposition 2.1, $G = K_p$.

Conversely, We consider the following cases.

**Case 3.** Suppose $G = K_p$, then by Proposition 2.2, $M_d(G) = pK_2$. 27
Since $pK_2$ is $(2p,p)$ graph, therefore (a) holds.

Case 4. Suppose $G = \overline{K}_p$, then by Proposition 2.1, $M_d(G) = K_{1,p}$.

Since $K_{1,p}$ is a $(p+1,p)$ graph, therefore (b) holds. ■

Next, we give the necessary and sufficient condition on a graph $G$ such that the middle dominating graph $M_d(G)$ is connected.

**Theorem 2.2** For any graph $G$ with at least two vertices, $M_d(G)$ of $G$ is connected if and only if $\Delta(G) < p - 1$.

**Proof.** Let $\Delta(G) < p - 1$. Suppose there is no minimal dominating set containing both $u$ and $v$. Then there exists a vertex $w \in V(G)$, such that $w$ not adjacent to both $u$ and $v$. Let $D$ and $D'$ be two maximal independent sets containing $u,w$ and $v,w$ respectively. Since every maximal independent set is a minimal dominating set. Therefore $u$ and $v$ are connected in $M_d(G)$ by a path $uDwD'$. Thus $M_d(G)$ is connected. From it follows that for any two vertices $u,v \in V$ either there exists a minimal dominating set $D$ containing $u$ and $v$ or there exists two disjoint minimal dominating sets $D_1$ and $D_2$ containing $u$ and $v$ respectively. This implies that in $M_d(G)$, $u$ and $v$ are connected by a path of length at most four.

Conversely, suppose $M_d(G)$ is connected. On the contrary assume that,
Δ(G) = p - 1. Let u is a vertex of degree p - 1. Then D = \{u\} is a minimum dominating set of G. Since every minimum dominating set is a minimal dominating set and further G has at least two vertices, with u adjacent to every other vertex of G, G has no isolates. Thus Theorem 2.C, V - D contains a minimal dominating set D'. Since D ∩ D' = φ, in Md(G) there is no path joining u to any vertex of V - D. This implies that Md(G) is disconnected, a contradiction. Hence Δ(G) < p - 1.

Theorem 2.3 For any (p, q) graph G, K_p ⊆ Md(G) if and only if G contains an isolated vertex.

Proof. Suppose K_p ⊆ Md(G). Since by Remark 2.1, MD(G) and D(G) are edge disjoint subgraphs of Md(G) i.e., Md(G) = D(G) ∪ MD(G). Then either MD(G) or D(G) is complete. Since for any graph G, D(G) can not be complete. Hence MD(G) must be complete. Then by Theorem 2.A, G contains an isolated vertex.

Conversely, the result follows from Theorem 2.A and Remark 2.1.

Theorem 2.4 For any graph G, Md(G) is either connected or it has at least one component which is K_2.
Proof. We consider the following cases.

Case 1. If $\Delta(G) < p - 1$, then by Theorem 2.2, $M_d(G)$ is connected.

Case 2. If $\delta(G) = \Delta(G) = p - 1$, then $G = K_p$. By Proposition 2.2, it follows that $M_d(G) \cong pK_2$.

Case 3. If $\delta(G) < \Delta(G) = p - 1$.

Let $u_1, u_2, \ldots, u_n$ be the vertices of degree $p - 1$ in $G$.

Let $H = G - \{u_1, u_2, \ldots, u_n\}$. Then clearly, $\Delta(H) < p - 1$. By Theorem 2.2, $M_d(G)$ is connected.

Since, $M_d(G) = M_d(H) \cup (\{u_1\} + u_1) \cup (\{u_2\} + u_2) \cup \cdots \cup (\{u_n\} + u_n)$.

Therefore it follows that at least one component of $M_d(G)$ is $K_2$. ■

In the next theorem we determine the diameter of $M_d(G)$.

Theorem 2.5 For any graph $G$ with $\Delta(G) < p - 1$, $diam(M_d(G)) \leq 4$.

Proof. Let $G$ be any graph with $\Delta(G) < p - 1$. We consider the following cases.

Case 1. Let $\Delta(G) < p - 1$. Then by Theorem 2.2, $M_d(G)$ is connected. Let $u$ and $v$ be any two nonadjacent vertices in $G$. Suppose there is no minimal dominating set containing both $u$ and $v$. Then there exists another vertex $w \in V$, which is nonadjacent to both $u$ and $v$. Let $D$ and $D'$ be maximal independent set containing $u, w$ and $v, w$ respectively.
Since every maximal independent set is a minimal dominating set. Thus \( u \) and \( v \) are connected in \( M_d(G) \) by a path \( uDwD'v \). Hence \( d_{M_d(G)}(u, v) = 4 \). Where \( d(u, v) \) is the distance between \( u \) and \( v \).

**Case 2.** Suppose \( u \in V \) and \( v \notin V \). Then \( v = D \) is a minimal dominating set of \( G \). If \( u \in D \), then \( d_{M_d(G)}(u, v) = 1 \). If \( u \notin D \), then there exists a vertex \( w \in D \) adjacent to \( u \) and hence
\[
d_{M_d(G)}(u, v) = d_{M_d(G)}(u, w) + d_{M_d(G)}(w, v) = 2.
\]

**Case 3.** Suppose \( u, v \in V \). Then \( u = D \) and \( v = D' \) are two minimal dominating sets of \( G \). If \( D \) and \( D' \) are disjoint, then every vertex \( w \in D \) is adjacent to some \( x \in D' \) and vice versa. This implies that,
\[
d_{M_d(G)}(u, v) = d_{M_d(G)}(u, w) + d_{M_d(G)}(w, x) + d_{M_d(G)}(x, v) = 3.
\]
If \( D \) and \( D' \) have a vertex in common then,
\[
d_{M_d(G)}(u, v) = d_{M_d(G)}(u, w) + d_{M_d(G)}(w, v) = 2.
\]
Thus, from all the above cases result follows.

**Definition 2.2.** The maximum number of classes of a partition of vertex set of \( G \) into dominating sets is called the **domatic number** of \( G \) and is denoted by \( d(G) \).

The next theorem gives the domatic number of \( M_d(G) \).
Theorem 2.6 For any graph $G$, $d(Md(G)) = 2$ if and only if $G = K_p$ or $\overline{K}_p$. Where $d(G)$ is the domatic number of $G$.

Proof. Let $G = K_p$ or $\overline{K}_p$. By Proposition 2.2 and Proposition 2.1, $Md(G) = pK_2$ or $K_{1,p}$ respectively.

Since $d(pK_2) = d(K_{1,p}) = 2$. Therefore $d(Md(G)) = 2$.

The converse is obvious. ■

In the next theorem we find $\alpha_0$ and $\beta_0$ of $Md(G)$.

Theorem 2.7 For any graph $G$,

(i) $\beta_0(Md(G)) = p$

(ii) $\alpha_0(Md(G)) = |S(G)|$.

Where $S(G)$ is the set of all minimal dominating sets of $G$.

Proof.

(i) Let $G$ be any graph of order $p$. By definition of $Md(G)$, each vertex $v_i$, $i = 1, \cdots, p$ of $G$ are independent in $Md(G)$. Hence these vertices together will form maximum independent set of $Md(G)$.

Hence (i) follows.

(ii) Follows from (i) and Theorem 2.B. ■

Next, theorem gives the relation between domatic number of $G$ and in-
dependence number of $M_d(G)$.

**Theorem 2.8** For any graph $G$, $d(G) = \beta_0(M_d(G))$, if and only if $G = K_p$.

**Proof.** Suppose $d(G) = \beta_0(M_d(G))$. By Theorem 2.7, $\beta_0(M_d(G)) = p$, this implies that $d(G) = p$. Hence $G = K_p$.

Conversely, if $G = K_p$, then by Theorem 2.7, the result follows. ■

**Theorem 2.9** $M_d(G) \cong G \cup K_2$ if and only if $G = K_{1,p}$.

**Proof.** Let $G = K_{1,p}$. Let $v_i$ be the vertex of degree $p - 1$. Then by Theorem 2.2, $M_d(G)$ has at least two components. Since $K_{1,p}$ has exactly two minimal dominating sets. Say $D_1$ and $D_2$. Then $D_1 = \{v_i\}$ and $D_2 = V - \{v_i\}$. Then $v_i$ is adjacent to the corresponding vertex of $G$ in $M_d(G)$ and the remaining vertices are adjacent to $D_2$. So the resulting graph will be $G \cup K_2$. Therefore $M_d(G) \cong G \cup K_2$.

Conversely, suppose $G$ be any graph of order $p$. Suppose $M_d(G) \cong G \cup K_2$, then by Theorem 2.2, $\Delta(G) = p - 1$. We consider the following cases.

**Case 1.** Suppose $\delta(G) = \Delta(G) = p - 1$, then $G = K_p$. By Proposition 2.2, $M_d(K_p) = pK_2$. 

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Case 2. \( \delta(G) < \Delta(G) = p - 1 \), then by Theorem 2.4, \( M_d(G) \) is either connected or has at least one component is \( K_2 \).

From all the above cases it follows that \( G \) must be \( K_{1,p} \).

Before going to next result we need the following definition.

**Definition 2.3.** For a given graph \( G \), the end edge graph \( G^+ \) is the graph obtained from \( G \) by adjoining a new edge \( u_iu'_i \) at each vertex \( u_i \) of \( G \) in such a way that this edge has exactly one vertex \( u_i \) in common with \( G \).

**Theorem 2.10** If \( G = K_p \cup K_1 \), \( p \geq 3 \) then \( K^+_p \) and \( K_{1,p} \) are edge-disjoint subgraphs of \( M_d(G) \).

**Proof.** Let \( G = K_p \cup K_1 \). Let \( v_i \) be an isolated vertex in \( G \). Then \( \text{deg}(v_i) \) in \( M_d(G) \) is \( p - 1 \). Also by Theorem 2.4, it follows that \( K_{p+1} \) is a subgraph of \( M_d(G) \). So clearly \( K^+_p \) and \( K_{1,p} \) are edge disjoint subgraphs of \( M_d(G) \).

In the next result we find the vertex connectivity of \( M_d(G) \).

**Theorem 2.11** For any graph \( G \),

\[ \kappa(M_d(G)) = \min \{ \min_{i \leq p} \text{deg}_{M_d(G)}(v_i), \min_{i \leq n} |S_j| \} \]

Where \( S_j \)'s are the minimal dominating sets of \( G \)
Proof. Let $G$ be a $(p,q)$ graph. We consider the following cases.

Case 1. Let $x \in v_i$ for some $i$, having minimum degree among all $v_i's$ in $M_d(G)$. If the degree of $x$ is less than any vertex in $M_d(G)$, then by deleting those vertices of $M_d(G)$ which are adjacent with $x$, results in a disconnected graph.

Case 2. If $y \in S_j$ for some $j$, having minimum degree among all vertices of $S_j's$. If degree of $y$ is less than any other vertices in $M_d(G)$. Then by deleting those vertices which are adjacent with $y$, results in a disconnected graph.

In the next result we find the edge connectivity of $M_d(G)$.

Theorem 2.12 For any graph $G$,

$$\lambda(M_d(G)) = \min \{ \min_{i \leq p} (\deg_{M_d(G)}(v_i)), \min_{i \leq S \in S} |S| \}$$

Where $S_j's$ are the minimal dominating sets of $G$

Proof. Let $G$ be a graph with $p$ vertices and $q$ edges. We consider the following cases.

Case 1. If $x \in v_i$ for some $i$, having minimum degree among all $v_i's$ in $M_d(G)$. If the degree of $x$ is less than any vertex in $M_d(G)$. Then by deleting those edges of $M_d(G)$ which are incident with $x$, results in a disconnected graph.
Case 2. If $y \in S_j$ for some $j$, having minimum degree among all vertices of $S_j's$. If degree of $y$ is less than any other vertices in $M_d(G)$. Then by deleting those edges which are incident with $y$, results in a disconnected graph.