Chapter 1

Introduction
1.1 Introduction

The present thesis entitled On Some Advances in Theory of Graphs is in the field of Graph Theory, which is one of the ever growing branch of Mathematics. This growth is due to its applications in many fields like engineering, physical, social and biological sciences, linguistics, discrete optimization problems, combinatorial problems and classical algebraic problems. This thesis focuses mainly on the study of domination theory in graphs and different domination parameters in graph valued functions.

1.2 A brief history of Graph Theory

Graph-theoretical ideas date back to at least the 1730's, when Leonhard Euler published his paper on the problem of Seven Bridges of Königsberg. This puzzle asks whether there is a continuous walk that crosses each of the seven bridges of Königsberg only once and if so, whether a closed walk can be found. Furthermore, the large part of graph theory has been motivated by the study of games and recreational mathematics. Graphs are very convenient tools for representing the relationships among objects, which are represented by vertices. In their turn, relationships among
vertices are represented by connections. In general, any mathematical object involving points and connections among them can be called a graph. For a great diversity of problems such pictorial representations may lead to a solution. Examples of such applications include databases, physical networks, organic molecules, map colorings, signal-flow graphs, web graphs, tracing mazes as well as less tangible interactions occurring in social networks, ecosystems and in a flow of a computer program.

The graph models can be further classified into different categories. For instance, two atoms in an organic molecule may have multiple connections between them, an electronic circuit may use a model in which each edge represents a direction, or a computer program may consist of loop structures. Therefore, for these examples we need multigraphs, directed graphs or graphs that allow loops. Thus, graphs can serve as mathematical models to solve an appropriate graph-theoretic problem, and then interpret the solution in terms of the original problem.

In particular, the term graph was introduced by Sylvester [62] in a paper published in 1878 in Nature, where he draws an analogy between quantic invariants and co-variants of algebra and molecular diagrams.

Different mathematicians have re-discovered graph theory many times.
while dealing with problems of their areas of work. These problems developed into different areas of graph theory like graph coloring, domination, graph labelling, graph decomposition etc.

The study of this thesis mainly concentrate on the domination theory in graphs.

1.3 A brief history of domination theory in graphs

Domination is a rapidly developing area of research in graph theory. The concept of domination has existed for a long time and early discussions on the topic can be found in works of Ore [55] and Berge [17]. The summary of the literature shows the following wide-known problems, which are considered among the earliest applications for dominating sets.

Queens Problem:

This problem was mentioned by Ore in [55]. According to the rules of chess a queen can, in one move, advance any number of squares horizontally, diagonally, or vertically (assuming that no other chess figure is on its way). How to place a minimum number of queens on a chessboard so that each square is controlled by at least one queen?

See one of the solutions in Figure 1.1.
Using graph theory to model this problem, the Queen's graph is formed by representing each of the 64 (8 x 8) squares of the chessboard as a vertex of a graph $G$. Two vertices (squares) are adjacent in $G$ if each square can be reached by a queen on the other square in a single move.

Obviously, to solve the queens problem we are looking for the minimum number of queens that dominate all the squares of the chessboard, that is, domination number. (Note that many variations on this problem are formed by considering different chess pieces and/or different size chessboards). The next appearance of domination in the literature was also associated with game applications.

In 1958, domination was formalized as a theoretical area in graph theory by C. Berge [17]. He referred to the domination number as the...
co-efficient of external stability and denoted it $\beta(G)$.

In 1962, Ore [55] was the first to use the term "domination" for undirected graphs and he denoted the domination number by $\delta(G)$ and also he introduced the concepts of minimal and minimum dominating sets of vertices in a graph.

In 1977, Cockayne and Hedetniemi [21] was introduced the accepted notation $\gamma(G)$ to denote the domination number.

In 1990, Hedetniemi and Laskar [30] had a survey of domination articles containing about 400 entries. This bibliography has grown to cover 1200 entries at the end of 1997. Which clearly shows the growth of domination.

In 1998, the publication of the first large two volume textbooks on domination. The books "Fundamentals of domination in graphs [27]" and "Domination in graphs: Advanced topics [28]" edited by Haynes, Hedetniemi and Slater stand as a strong base in the study of domination theory till date.

Later, a chapter on domination was included in text books authored by G. Chartrand and Lesniak [19], G. Chartrand and P.Zhang [20].

A recent book on domination by V.R.Kulli can be found in [38].
By the end of 2006, domination theory and its related parameters has grown to more than 10,000 entries in the different journals / periodicals / proceedings, etc.

Hence domination has emerged as one of the most studied area in graph theory and its allied branch in mathematics.

The main view, the rapid growth in the number of domination papers is attributed largely to three factors:

1. The diversity of applications to both real-world and other mathematical covering or location problems.

2. Wide variety of domination and its related parameters can be defined.

3. The NP-completeness of the basic domination problem, its close and natural relationships to other NP-complete problems and the subsequent interest in finding polynomials time solutions to domination problems in special class of graphs.

1.4 A brief history of graph valued functions

The operation of forming the graph valued functions of a graph is probably the most interesting operation by which one graph is obtained
from other. The concept of the line graph, first studied by Whitney [64], has surprisingly discovered independently by many graph theorists. One of the important variations of the line graph is the middle graph, which has been studied by Hamada and Yoshimura [26]. Sampathkumar and Chikkodimath [58] introduced the concept of semitotal-point graph. Later, Kulli et.al., [41-42] and [47-48], introduced the graph valued functions in domination theory of graphs. They are listed as below:

1. The minimal dominating graph \( MD(G) \).

2. The common minimal dominating graph \( CD(G) \).

3. The vertex minimal dominating graph \( M_vD(G) \).

4. The dominating graph \( D(G) \).

1.5 Outline of the present investigation

The contents of thesis are conveniently categorized into three parts. The first part consists of four chapters from the beginning in which the first chapter is introductory in nature. In the next three chapters, we introduced the new graph valued functions The Middle Dominating Graph of a Graph, Mediate Dominating Graph of a Graph and
\textbf{Edge Dominating Graph of a Graph} in the field of domination theory of graphs.

The second part of the thesis contains four chapters. Here we introduced four new domination parameters \textbf{Complete cototal domination number of a graph}, \textbf{Connected cototal domination number of a graph}, \textbf{Degree equitable connected domination in graphs} and \textbf{Degree equitable edge domination in graphs}.

The third and final part of the thesis contains two chapters which are completely devoted to study of \textbf{Connected} and \textbf{Entire} domination in graph valued function “semitotal-point graph $T_2(G)$” of a graph $G$.

A brief summary of each chapter is as follows:

\textbf{Chapter 1}, is of introductory in nature.

\textbf{In Chapter 2}, we define a new graph valued function, the middle dominating graph of a graph as follows.

Let $G = (V, E)$ be a graph and $A(G)$ is the collection of all minimal dominating sets of $G$. The middle dominating graph of $G$ is the graph denoted by $M_d(G)$ with vertex set the disjoint union of $V(G) \cup A(G)$ and $(u, v)$ is an edge if and only if $u \cap v \neq \emptyset$ whenever $u, v \in A(G)$ or $u \in v$ whenever $u \in V$ and $v \in A(G)$. In this chap-
In Chapter 3, we define another graph valued function, mediate dominating graph of a graph as follows.

The mediate dominating graph $D_m(G)$ of a graph $G$ is a graph with $V(D_m(G)) = V' = V(G) \cup S(G)$, where $S(G)$ is the set of all minimal dominating sets of $G$ with two vertices $u, v \in V'$ are adjacent if they are not adjacent in $G$ or $v = S$ is a minimal dominating set containing $u$. In this chapter, some necessary and sufficient conditions are given for $D_m(G)$ to be connected, eulerian, complete graph, tree and cycle respectively. It is also shown that a given graph $G$ is a mediate dominating graph $D_m(G)$ of some graph. Further, some bounds on domination number of $D_m(G)$ are obtained in terms of vertices and edges of $G$. Finally, we conclude this chapter by exploring an open problem.

Chapter 4, is the last chapter of the first section. Here we define one more graph valued function edge dominating graph of a graph as
follows.

The edge dominating graph $E_D(G)$ of a graph $G = (V, E)$ is a graph with $V(E_D(G)) = E(G) \cup S(G)$, where $S(G)$ is the set of all minimal edge dominating sets of $G$ with two vertices $u, v \in V(E_D(G))$ adjacent if $u \in E(G)$ and $v$ is a minimal edge dominating set of $G$ containing $u$. In this chapter, we establish the bounds on order, size and diameter of $E_D(G)$. Further, we find vertex(edge) connectivity of $E_D(G)$.

In Chapter 5, we define a new domination parameter, complete cototal domination number of a graph as follows.

Let $G = (V, E)$ be a graph. A dominating set $D \subseteq V$ is said to be complete cototal dominating set if every vertex in $V$ is adjacent to some vertex in $D$ and there exists a vertex $u \in D$ and $v \in V - D$ such that $uv \in E(G)$ and $N(v) - u = x \in V - D$. The complete cototal domination number $\gamma_{\omega\omega}(G)$ of $G$ is the minimum cardinality of a complete cototal dominating set of $G$. In this chapter, we initiate the study of complete cototal domination in graphs and present bounds and some exact values for $\gamma_{\omega\omega}(G)$. Also its relationship with other domination parameters are established and related two open
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problems are explored.

In Chapter 6, we study the another domination parameter, connected cototal domination number of a graph as follows.

A dominating set $D \subseteq V$ of a graph $G = (V, E)$ is said to be a connected cototal dominating set if $\langle D \rangle$ is connected and $\langle V - D \rangle \neq \emptyset$, contains no isolated vertices. A connected cototal dominating set is said to be minimal if no proper subset of $D$ is connected cototal dominating set. The connected cototal domination number $\gamma_{cd}(G)$ of $G$ is the minimum cardinality of a minimal connected cototal dominating set of $G$. In this chapter, we begin an investigation of the connected cototal domination number and obtain some interesting results.

In Chapter 7, we define another new domination parameter, degree equitable connected domination in graphs as follows.

A connected dominating set $D$ is to be an equitable connected dominating set if for every vertex $u \in V - D$ there exists a vertex $v \in D$ such that $uv \in E(G)$, $\langle V - D \rangle \neq \emptyset$ and $|deg(v) - deg(u)| \leq 1$. The minimum cardinality of such a connected dominating set is denoted by $\gamma^*_c(G)$ and is called the equitable connected domination number.
In this chapter, we obtain some bounds for $\gamma^c_e(G)$ and characterize the cubic graphs with $\gamma^c(G) = \gamma^c_e(G)$. Also Nordhaus-Gaddum type results are obtained.

Chapter 8, is the last chapter of the second part. Here we defined another new domination parameter degree equitable edge domination in graphs as follows.

Let $G = (V, E)$ be a graph. A subset $F$ of $E$ is called an equitable edge dominating set if for every edge $e \in E - F$ there exists an edge $e' \in F$ such that $e$ and $e'$ are adjacent edges in $G$ and $|\deg(e) - \deg(e')| \leq 1$, where $\deg(e)$ and $\deg(e')$ denotes the edge degree of $e$ and $e'$ respectively. The minimum cardinality of such an edge dominating set is called the equitable edge domination number $\gamma^e_e(G)$ of $G$. In this chapter, we obtained some bounds on $\gamma^e_e(G)$. Also Nordhaus-Gaddum type results are obtained.

Chapter 9, deals with connected domination in semitotal-point graph.

The semitotal-point graph $T_2(G) = H$ of $G$ is the graph whose vertex set is $V(G) \cup E(G)$, where two vertices are adjacent if and only if (i) they are adjacent vertices of $G$ or (ii) one is a vertex of $G$ and the other is an edge of $G$, incident with it.
A dominating set $D$ of a graph $H$ is a connected semitotal-point dominating set if the $\langle D \rangle$ is connected. The connected semitotal-point domination number $\gamma_{ctp}(G)$ of $G$ is the minimum cardinality of a connected semitotal-point dominating set of $G$. In this chapter, we study the connected domination in semitotal-point graphs and obtain many bounds of $\gamma_{ctp}(G)$ in terms of elements of $G$ but not the elements of $T_2(G)$. Also its relationship with other domination parameters are established.

In Chapter 10, we study another domination parameter called the entire domination $\varepsilon(G)$ in graphs. A dominating set $X$ of a graph $G$ is called an entire dominating set of $G$, if every element not in $X$ is either adjacent or incident to at least one element in $X$. The entire domination number $\varepsilon(G)$ of $G$ is the minimum cardinality of an entire dominating set of $G$.

An entire dominating set $X$ of a graph $T_2(G)$ is an entire semitotal-point(ESP) dominating set if every element not in $X$ is either adjacent or incident to at least one element in $X$. An ESP domination number $\varepsilon_{tp}(G)$ of $G$ is the minimum cardinality of an ESP dominating set of $G$. In this chapter, many bounds on $\varepsilon_{tp}(G)$ are
obtained in terms of elements of \( G \). Also its relationship with other domination parameters are investigated.

**Basic Terminology and Definitions**

This preliminary section is included in order to make the thesis self-contained.

A graph \( G \) consists of a pair \( (V(G), E(G)) \), where \( V(G) \) is the nonempty finite set whose elements are called **vertices** and \( E(G) \) is a set of unordered pairs of distinct elements of \( V(G) \). The elements of \( E(G) \) are called **edges** of the graph \( G \). Graphs discussed in this thesis are **undirected** and **simple**. For graph theoretic terminology, we refer [27] and [29].

A graph with \( p \) vertices and \( q \) edges is called a \((p, q)\) graph. The \((1,0)\) graph is **trivial**. A graph with more than one vertex is a **non-trivial** graph. The **order** of a graph \( G \) is the number of vertices in \( G \) and it is denoted by \(|G|\). The **size** of a graph \( G \) is the number of edges in \( G \). When there is no possibility of confusion, we write \( V(G) = V \) and \( E(G) = E \). A graph \( H \) is said to be a **subgraph** of a graph \( G \) if \( V(H) \) is a subset of \( V(G) \) and \( E(H) \) is a subset of \( E(G) \). If \( H \) is a subgraph of a graph \( G \) and \( V(H) = V(G) \), then we say that \( H \) is
a spanning subgraph of $G$. The most important subgraph which we shall encounter are the induced subgraphs. For any subset $S$ of $V(G)$, the induced subgraph $\langle S \rangle$ is the maximal subgraph of $G$ with vertex set $S$. Two graphs $G$ and $H$ are isomorphic written as $G \cong H$ or sometimes $G = H$, if there exists a one-to-one correspondence between their vertex sets which preserves adjacency. The degree of a vertex $v$ is the number of edges of $G$ incident with $v$, and is denoted by $\text{deg}_G(v)$ or simply $\text{deg}(v)$. The edge degree of an edge $x = uv$ of a graph $G$ is the sum of the degrees of $u$ and $v$. A vertex(edge) of degree zero in $G$ is called an isolated vertex (edge). A vertex of degree one in $G$ is called a pendant vertex. A pendant edge is an edge incident to a pendant vertex. The minimum degree among all the vertices(edges) of $G$ is denoted by $\delta(G)$ ($\delta'(G)$) and $\Delta(G)$ ($\Delta'(G)$) denotes the maximum vertex(edge) degree of $G$. If $\delta(G) = \Delta(G)$, then the graph $G$ is said to be regular. A complete $(p, q)$ graph is a $p - 1$ regular graph having $\frac{p(p-1)}{2}$ edges and is denoted by $K_p$. A walk is defined as a finite alternating sequence of vertices and edges, beginning and ending with vertices, in which each edge is incident with the two vertices immediately preceding and following it. A trail is a walk in which all the edges
are distinct, and it is a path if all the vertices are distinct. A path with
$p$ vertices is denoted by $P_p$. A closed path is called a cycle. A cycle
with $p$ vertices is denoted by $C_p$. The length of a cycle or a path is
the number of occurrences of edges in it. This term is undefined if $G$
has no cycles. For $p \geq 4$, the wheel $W_p$ is defined to be the graph
$K_1 + C_{p-1}$. A clique of a graph $G$ is maximal complete subgraph. A
graph is said to be connected if every pair of its vertices are joined by
a path. A graph which is not connected is said to be disconnected. A
graph whose edge set is empty is called a totally disconnected graph.
A nonseparable graph is connected, nontrivial and has no cutvertices.
A block of a graph is a maximal nonseparable subgraph. If $G$ is non-
separable, then $G$ itself is often called a block. The distance $d(u,v)$
between two vertices $u$ and $v$ in $G$ is the length of a shortest path join-
ing them if any; otherwise $d(u,v) = \infty$. A shortest $u-v$ path is called
a geodesic. The diameter of a connected graph $G$ is the length of any
longest geodesic and is denoted as $diam(G)$. A graph $G$ is said to be
bipartite graph or bigraph if its vertex set $V(G)$ can be partitioned
into two subsets $V_1$ and $V_2$ such that every edge of $G$ joins a vertex of
$V_1$ with a vertex of $V_2$. If $G$ contains every edge joining $V_1$ and $V_2$,
then $G$ is a complete bipartite graph. If $V_1$ and $V_2$ have $m$ and $n$ vertices, we write $G = K_{m,n}$. A star is a complete bipartite graph $K_{1,n}$. A graph with cycles is called cyclic graph, otherwise acyclic. A tree is a connected acyclic graph. Any graph without cycles is called forest. If $G$ is a simple graph with vertex set $V(G)$, then its complement denoted by $\overline{G}$ is the simple graph with vertex set $V(G)$ in which two vertices are adjacent if and only if they are not adjacent in $G$. A set $S \subseteq V$ of vertices which covers all the edges of a graph $G$ is called a vertex cover of $G$. A set of vertices in $G$ is an independent set if no two of them are adjacent. The vertex connectivity $\kappa(G)$ (edge connectivity $\lambda(G)$) of a graph $G$ is the minimum number of vertices(edges) whose removal results in a disconnected or trivial graph. The neighborhood of a vertex $u$ in $V$ is the set $N(u)$ consisting of all vertices $v$ which are adjacent with $u$. The closed neighborhood is $N[u] = N(u) \cup \{u\}$. Let $S$ be a set of vertices and let $u \in S$. A vertex $v$ is a private neighbor of $u$ with respect to $S$ if $N[v] \cap S = \{u\}$. The private neighbor set of $u$ with respect to $S$ is the set $pn[u,S] = \{v : N[v] \cap S = \{u\}\}$. If $u \in pn[u,S]$ and $u$ is an isolated vertex in $\langle S \rangle$, then $u$ is called its own private neighbor.
The union of two graphs $G_1$ and $G_2$ denoted by $G_1 \cup G_2$ is the graph $G$ with vertex set $V(G) = V(G_1) \cup V(G_2)$ and the edge set $E(G) = E(G_1) \cup E(G_2)$.

The join of two graphs is denoted by $G_1 + G_2$ and is the union of $G_1$ and $G_2$ as well as all edges $uv$ with $u \in V(G_1)$ and $v \in V(G_2)$.

The cartesian product of the graphs $G_1$ and $G_2$ denoted by $G_1 \times G_2$ is the graph with vertex set $V(G_1) \times V(G_2)$, two vertices $(u_1, u_2)$ and $(v_1, v_2)$ being adjacent in $G_1 \times G_2$ if and only if either $u_1 = v_1$ and $u_2 v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1 v_1 \in E(G_1)$.

The $G_2$ corona of $G_1$ is the graph $G_1 \circ G_2$ formed from one copy of $G_1$ and $V(G_1)$ copies of $G_2$ where the $i^{th}$ vertex of $G_1$ is adjacent to every vertex in the $i^{th}$ copy of $G_2$.

We now define domination and its related parameters which have been studied by different mathematicians. These parameters are of great practical interest because of the applications of domination theory in different fields.

A set $D$ of vertices in a graph $G$ is a dominating set if every vertex in $V - D$ is adjacent to some vertex in $D$. The domination number $\gamma(G)$ of $G$ is the minimum cardinality of a dominating set of $G$. A
minimum dominating set of a graph \( G \) is called a \( \gamma \)-set of \( G \).

- A dominating set \( D \) is a connected dominating set if \( \langle D \rangle \) is connected.

- A dominating set \( D \) is independent dominating set if \( \langle D \rangle \) is independent.

- A dominating set \( D \) is called a total dominating set if there are no isolates in \( \langle D \rangle \).

- A dominating set \( D \) is called a split dominating set if \( \langle V - D \rangle \) is disconnected.

- A dominating set \( D \) is called a nonsplit dominating set if \( \langle V - D \rangle \) is connected.

- A dominating set \( D \) is called a cototal dominating set if \( \langle V - D \rangle \) contains no isolated vertices.

- A dominating set \( D \) is called a maximal dominating set if \( \langle V - D \rangle \) is not a dominating set.

The minimum cardinality taken over all connected \ independent \ total \ split \ nonsplit \ cototal \ maximal gives the respective domi-
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The other terminologies and different domination parameters are defined as and when the need arises.

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