Chapter 6

Connected Cototal Domination

Number of a Graph
A dominating set $D \subseteq V$ of a graph $G = (V, E)$ is said to be a connected cototal dominating set if $\langle D \rangle$ is connected and $\langle V - D \rangle \neq \emptyset$, contains no isolated vertices. A connected cototal dominating set is said to be minimal if no proper subset of $D$ is connected cototal dominating set. The connected cototal domination number $\gamma_{ccd}(G)$ of $G$ is the minimum cardinality of a minimal connected cototal dominating set of $G$. In this chapter, we begin an investigation of the connected cototal domination number and obtain some interesting results.
6.1 Introduction

All graphs considered in this chapter are simple, finite, connected and nontrivial.

Our aim in this chapter is to introduce a new domination parameter in the field of domination in theory of graphs.

**Definition 6.1.** A dominating set $D \subseteq V$ of a graph $G = (V, E)$ is said to be a connected cototal dominating set if $\langle D \rangle$ is connected and $\langle V - D \rangle \neq \phi$, contains no isolated vertices. A connected cototal dominating set is said to be minimal if no proper subset of $D$ is connected cototal dominating set. The connected cototal domination number $\gamma_{cc}(G)$ of $G$ is the minimum cardinality of a minimal connected cototal dominating set of $G$.

For illustration, consider a graph in Figure 6.1,

![Figure 6.1](image)
In Figure 6.1, \( V(G) = \{1, 2, 3, 4, 5, 6\} \).

The minimal connected dominating sets are \( D_1 = \{2, 6\} \) \( D_2 = \{2, 4\} \).
Therefore, \( \gamma_c(G) = |D_1| = |D_2| = 2 \).

The minimal cototal dominating sets are \( D_1 = \{2, 5\} \), \( D_2 = \{3, 6\} \) and \( D_3 = \{1, 4\} \). Therefore, \( \gamma_d(G) = |D_1| = |D_2| = |D_3| = 2 \).

The minimal connected cototal dominating sets are, \( D_1 = \{1, 2, 6\} \) and \( D_2 = \{2, 3, 4\} \). Therefore, \( \gamma_{cd}(G) = |D_1| = |D_2| = 3 \).

### 6.2 Results

The following observations are immediate.

**Observation 6.1.** \( \gamma(G) \leq \gamma_d(G) \) and \( \gamma_c(G) \leq \gamma_{cd}(G) \).

**Observation 6.2.** For simplicity, the minimal connected cototal dominating set is denoted by \( \gamma_{cd} \)-set.

**Observation 6.3.** In a graph \( G \), \( \gamma_{cd} \)-set contains every pendant vertex (if any) and its support vertex in \( G \).

**Observation 6.4.** Let \( D \) be a \( \gamma_{cd} \)-set of \( G \), then \( \langle D \rangle \) is a tree.

**Observation 6.5.** In a graph \( G \), \( \gamma_c \)-set contains no pendant vertex (if any) in \( G \).
6.3 Characterization of connected cototal dominating sets

Obviously, we ask the natural question regarding the existence of connected cototal dominating sets. Our first theorem gives the characterization of the existence of connected cototal dominating sets in a graph $G$.

**Theorem 6.1** A graph $G$ has a connected cototal dominating set if and only if it satisfies the following conditions.

(i) $|V(G)| \geq 3$

(ii) $G$ is not a tree

(iii) Let $u \in V(G)$ and $D$ be a $\gamma_{cd}$-set. Then $V - D \neq \{u\}$.

The following theorem gives the relationship between $\gamma_{cd}(G)$ and $\gamma_{cd}(H)$, where $H$ is a spanning subgraph of $G$.

**Theorem 6.2** For any graph $G$, $\gamma_{cd}(G) \leq \gamma_{cd}(H)$. Further, the equality holds if and only if $\gamma_{cd}(G) = p - 2$ and $H$ is unicyclic.

**Proof.** Let $D$ be a $\gamma_{cd}$-set of $G$ and $H$ be any spanning subgraph of $G$. Let $D'$ be the $\gamma_{cd}$-set of $H$. By Theorem 6.1, $H$ must contain at
least one cycle. Obviously, $|D| \leq |D'|$. Hence, $\gamma_{cd}(G) \leq \gamma_{cd}(H)$.

For equality, suppose $\gamma_{cd}(G) = 2$ and $H$ is unicyclic. Let $v_i v_j v_k v_i$ be a cycle in $H$. Since $\gamma_{cd}(G) = p - 2$, therefore $D = \{v_1 v_2, \ldots, v_{p-2}\}$ is a minimal connected dominating set of $H$, such that $V - D = \{v_i, v_j\}$ and the induced subgraph of $(V - D)$ will form $K_2$. Hence $(V - D)$ contains no isolated vertex. Therefore,

$$\gamma_{cd}(H) = |D|$$
$$= |V| - |\{v_i, v_j\}|$$
$$= p - 2$$
$$= \gamma_{cd}(G).$$

The converse is obvious.

In the next theorem, we calculate the $\gamma_{cd}(G)$ of some standard class of graphs.

**Theorem 6.3**

(i) For any cycle $C_p$; $p \geq 3$, $\gamma_{cd}(C_p) = p - 2$.

(ii) For any wheel $W_p$; $p \geq 4$, $\gamma_{cd}(W_p) = 1$.

(iii) For any complete graph $K_p$; $p \geq 3$, $\gamma_{cd}(K_p) = 1$.

(iv) For any graph $H = G + K_1$, $\gamma_{cd}(H) = 1$. 

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(v) For any complete bipartite graph $K_{m,n}$; $2 \leq m \leq n$, $\gamma_{cd}(K_{m,n}) = 2$.

(vi) For any grid graph $P_2 \times P_k$; $k \geq 2$, $\gamma_{cd}(P_2 \times P_k) = 2\left\lceil \frac{k}{3} \right\rceil$.

(vii) For any grid graph $C_3 \times C_k$; $k \geq 3$, $\gamma_{cd}(C_3 \times P_k) = 3\left\lceil \frac{k}{3} \right\rceil$.

The following result is immediate.

**Theorem 6.4** For any graph $G$, 
$$1 \leq \gamma_{cd}(G) \leq p - 2.$$ 
Further, the equality of lower bound is attained if and only if $\Delta(G) = p - 1$ and the equality of an upper bound holds if $G = C_p$; $p \geq 3$ or unicyclic.

**Proof.** Let $G$ be any nontrivial connected graph of order at least three. Suppose $\gamma_{cd}(G) = p - 1$. Let $D$ be a minimal connected cototal dominating set of $G$. Then $\gamma_{cd}(G) = |D| = p - 1$ and $(V - D) = \{v_1\}$ is an isolated vertex, a contradiction. Hence $\gamma_{cd}(G) \leq |D| - 1 = p - 2$.

For the equality of lower bound, suppose $\delta(G) \geq 2$ and $\Delta(G) = p - 1$. Let $v$ be a vertex of maximum degree. Then $D = \{v\}$ and such that $(V - D)$ has no isolated vertex. Therefore $D$ is a minimal connected cototal dominating set of $G$. Hence $\gamma_{cd}(G) = |D| = |\{v\}| = 1$.

Equality of an upper bound can be easily verified.

Converse is easy to follow. ■

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To prove the next theorem we need the following result.

**Theorem 6.A** [27] For any graph $G$,

$$\left\lceil \frac{p}{1+\Delta(G)} \right\rceil \leq \gamma(G).$$

**Theorem 6.5** For any graph $G$,

$$\left\lceil \frac{p}{1+\Delta(G)} \right\rceil \leq \gamma_{cd}(G) \leq 2q - p.$$ 

Further, the equality of a lower bound is attained if $\Delta(G) = p - 1$ and $\delta(G) \geq 2$ and equality of an upper bound is attained if $\gamma_{cd}(G) = p - 2$.

**Proof.** The lower bound follows from Theorem 6.A and Observation (i). Further, if $\Delta(G) = p - 1$ and $\delta(G) \geq 2$, then equality of lower bound can be easily verified.

Now, for the upper bound, we have by Theorem 6.4,

$$\gamma_{cd}(G) \leq p - 2$$

$$\leq 2(p - 1) - p$$

$$\leq 2q - p.$$ 

If $\gamma_{cd}(G) = p - 2$, then equality of an upper bound can be easily verified.

**Theorem 6.6** Let $G_1$ and $G_2$ be two connected graphs with $\delta(G_1) \geq 2$ and $\delta(G_2) \geq 2$. Then $\gamma_{cd}(G_1 \circ G_2) = |V(G_1)| + |V(G_2)|\left(\gamma_{cd}(G_2) - 1\right)$. 

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Proof. Let $G_1$ and $G_2$ be any two nontrivial connected graphs of order at least two. Let us construct a minimal connected cototal dominating set $D$, such that, $|D| = |V(G_1)| + |V(G_1)|(\gamma_{cd}(G_2) - 1)$. Let $X$ be the spanning subgraph of $G_1$, then clearly $X \subseteq D$. By Theorem 6.4, $\gamma_{cd}(G_1) \leq p - 2$ and $\gamma_{cd}(G_2) \leq p' - 2$, where $p'$ is the $|V(G_2)|$. By the definition of corona of a graph, each vertex of $G_1$ is attached with a vertex of $G_2$. Therefore $|D| = |X| + |X|(p - 3)$ is a minimal connected cototal dominating set. Therefore,

$$\gamma_{cd}(G_1 \circ G_2) = |D| = |X| + |X|(p - 3) = |V(G_1)| + |V(G_1)|(p - 2 - 1) = |V(G_1)| + |V(G_1)|(\gamma_{cd}(G_2) - 1).$$

Corollary 6.1. Let $G_1$ be any graph and $G_2$ any complete graph of order at least three. Then $\gamma_{cd}(G_1 \circ G_2) = |V(G_1)|$.

6.4 Particular Values of $\gamma_{cd}(G)$

Theorem 6.7 Let $G$ be any nontrivial connected graph of order at least three. Then $\gamma_{cd}(G) = 1$ if and only if $\delta(G) \geq 2$ and $\gamma(G) = 1$. 
Proof. Let $G$ be any graph of order at least three with $\gamma_{cd}(G) = 1$. We consider the following cases.

Case 1. Suppose $\delta(G) = 1$ and $\gamma(G) = 1$. Let $\{v_i\}; \ 1 \leq i \leq p - 1$ be the set of vertices of degree $p - 1$ and $\{u_j\}$ be the set of its neighbors. Then by the definition of connected cototal dominating set, $\gamma_{cd}(G) = |\{v_i\} \cup \{u_j\}| \geq 2$, a contradiction.

Case 2. Suppose $\delta(G) = 2$ and $\gamma(G) \geq 2$, then there exist a vertex $u$ of maximum degree less than or equal to $p - 2$. Let $v$ be a vertex of minimum degree. Then by the definition of connected cototal dominating set, $X = v_0v_1\ldots u$ is a path in $G$. If $(V - X)$ has no isolated vertices, then $|X|$ is a connected cototal dominating set. If $X$ is minimal then obviously, $|X| \geq 2$.

Hence $\gamma_{cd}(G) = |X| \geq 2$, a contradiction.

Conversely, suppose $\delta(G) \geq 2$ and $\gamma(G) = 1$, then there exist a vertex $u$ of degree $p - 1$. Since $\delta(G) \geq 2$, therefore $(V(G) - \{u\})$ contains no isolated vertices. Therefore, $\{u\}$ is a minimal connected cototal dominating set. Hence $\gamma_{cd}(G) = |\{u\}| = 1$.

Theorem 6.8 Let $G$ be any graph with at least three vertices. Then $\gamma_{cd}(G) = 2$ if and only if there exist at least two adjacent vertices of
degree $p - 2$ and $(V - D)$ has no isolated vertices.

**Proof.** Let $G$ be any graph of order at least three with $\gamma_{cd}(G) = 2$. Suppose $G$ does not contain two adjacent vertices of degree $p - 2$, then we consider the following cases.

**Case 1.** Suppose there exist a vertex of degree $p - 1$. Then by Theorem 6.7, $\gamma_{cd}(G) = 1$.

**Case 2.** Suppose there exist exactly one vertex of degree $p - 2$. Then we consider the following subcases.

**Subcase 2.1.** If $\delta(G) = u$ and $\Delta(G) = p - 2 = v$, then there exist a vertex $x$ which is nonadjacent to $v$ but adjacent to a vertex $y$ such that $\deg(y) \geq 2$. Therefore $D = \{u, v, y, x\}$ and $(V - D)$ contains no isolated vertices. Therefore $D$ is a minimal connected cototal dominating set of $G$. Therefore,

$$\gamma_{cd}(G) = |D|$$

$$= |\{u, v, y, x\}|$$

$$= 4,$$ a contradiction.

**Subcase 2.2.** If $\delta(G) = 2$ and $\Delta(G) = p - 2$. Let $u$ be a vertex of minimum degree, which is adjacent to $v$ and $x$, such that $\deg(v) = p - 2$ and $\deg(x) = 2$. Suppose there exist a vertex $z$ which is not adjacent
to $x$ then by Subcase 2.1, $\gamma_{cd}(G) \geq 3$, a contradiction. Therefore, either $D_1 = \{v, x\}$ or $D_2 = \{u, v\}$ is a minimal connected cototal dominating set of $G$. If $D_1$ is a minimal connected cototal dominating set, then $\langle V - D_1 \rangle = \{u\}$ contains an isolated vertex, a contradiction. If $D_2$ is a minimal connected cototal dominating set, then $\langle V - D_2 \rangle = \{x\}$ contains an isolated vertex, a contradiction. Let $\langle V - D_3 \rangle = \{u, v, x\}$ be a connected dominating set of $G$. If $\langle V(G) - D_3 \rangle$ does not contain an isolated vertex, then $D_3$ is a minimal connected cototal dominating set of $G$. Therefore,

$$\gamma_{cd}(G) = |D_3|$$

$$= |\{u, v, x\}|$$

$$= 3, \text{ a contradiction.}$$

**Case 3.** Suppose there exist two nonadjacent vertices $u$ and $v$ such that $\deg(u) = \deg(v) = p - 2$, then there exists a vertex $x$ such that $x$ is nonadjacent to both $u$ and $v$. Now to dominate $x$, we consider a vertex $z$ such that $z \in N[x]$ and which is adjacent to both $u$ and $v$. Then clearly, $\gamma_{cd}(G) \geq 3$, a contradiction.

Conversely, let $u$ and $v$ be any two adjacent vertices in $G$, such that $\deg(u) = \deg(v) = p - 2$. Let $D = \{u, v\}$. Since $\delta(G) \geq 2$, therefore
(V − D) contains no isolated vertices. Then clearly $D$ is a minimal connected cototal dominating set of $G$. Hence,

$$
\gamma_{cd}(G) = |D| = |\{u, v\}| = 2.
$$

**Theorem 6.9** Let $G$ be a graph of order at least four with $\delta(G) \geq 2$ and $diam(G) = 2$. Then $2 \leq \gamma_{cd}(G) \leq 3$.

**Proof.** If $G$ satisfies the hypothesis of the theorem, then clearly $\gamma_{cd}(G) \geq 2$.

For upper bound, let $\gamma_{cd}(G) \leq 3$ and $diam(G) \neq 2$. Let $D$ be a minimal connected cototal dominating set of $G$.

We consider the following cases.

**Case 1.** If $diam(G) = 1$ then $G = K_p$, by Theorem 6.3, $\gamma_{cd}(G) = 1$, a contradiction.

**Case 2.** If $diam(G) \geq 3$ then clearly $|D| \geq 4$.

Hence $\gamma_{cd}(G) = |D| \geq 4$, a contradiction. Hence $diam(G) = 2$. ■
6.5 Comparison of $\gamma_{\text{cd}}(G)$ with other domination parameters

In the following theorem we give the relationship between $\gamma_{\text{cd}}(G)$ and $\gamma_{\text{ns}}(G)$.

**Theorem 6.10** For any minimal connected cototal dominating set $D$ of $G$, $V - D$ is a nonsplit dominating set of $G$ if and only if for each $v \in D$, the following conditions hold.

(i) there exist a cycle in $G$ containing $v$

(ii) for some cycle $C_p$, $\langle D \rangle$ contains a path $P_{p-1} < C_p$ with $v$ as a pendant vertex of $P_{p-1}$

**Proof.** First we prove the necessity.

Suppose $V - D$ is a dominating set of $G$ and let some vertex $v \in D$. Suppose one of the given condition is not satisfied, then $N[v] \subseteq D$. This implies that $v$ is nonadjacent to any vertex of $V - D$, a contradiction to the hypothesis that $V - D$ is a dominating set. Hence in $\langle V - D \rangle$ there exist a cycle containing $v$, which proves (i).

Similarly we can prove (ii).

Sufficiency is obvious. \[\qed\]
Next theorem gives the relationship between $\gamma_{cd}(G)$ and $\gamma_c(G)$.

**Theorem 6.11** A connected dominating set $D$ is a connected cototal dominating set if and only if $(V - D)$ has no isolated vertices.

**Proof.** Follows from the definition of connected cototal dominating set of $G$. ■

The following theorem gives the relationship between $\gamma_{cd}(G)$ and $\gamma_t(G)$.

**Theorem 6.12** Let $G$ be any graph and $D$ be a minimal connected cototal dominating set of $G$. Then $V - D$ is a total dominating set if and only if $G$ satisfies the following conditions.

(i) $\delta(G) \geq 2$

(ii) $N(v) \cap (V - D) \neq \emptyset$ for all $v \in D$.

**Proof.** Let $D$ be a minimal connected cototal dominating set of $G$ for which $V - D$ is a total dominating set. We consider the following cases.

**Case 1.** Let $v$ be a vertex of minimum degree. Suppose $\delta(G) = 1$. Then $u \in D$. Since $V - D$ is a total dominating set, therefore $u$ must be adjacent to at least one vertex of $V - D$, a contradiction.

**Case 2.** Since every vertex of $G$ is adjacent to at least one vertex of $V - D$, therefore $N(v) \cap (V - D) \neq \emptyset$ for all $v \in D$. 

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Conversely, suppose the given conditions are satisfied. Then obviously every vertex in $V$ is adjacent to some vertex in $V - D$ and hence $V - D$ is a total dominating set of $G$.

In the next theorem we characterize the graphs which have equal connected domination and connected cototal domination number.

**Theorem 6.13** For any graph $G$, $\gamma_c(G) = \gamma_{cd}(G)$ if and only if $G$ satisfies the following conditions.

(i) $\delta(G) \geq 2$

(ii) $G$ contains $C_3$ as an induced subgraph having the vertex set $\{x, y, z\}$, and

(a) If $\text{deg}(x) = 2$ then $\text{deg}(y) = 2$ and $\text{deg}(z) \geq 3$

(b) If $\text{deg}(x) \geq 3$ then $\text{deg}(y) \geq 2$ and $\text{deg}(z) \geq 2$

**Proof.** Let $G$ be any graph of order at least three with $\gamma_c(G) = \gamma_{cd}(G)$.

Let $D$ be a minimal connected cototal dominating set of $G$. Then we consider the following cases.

**Case 1.** Suppose $\delta(G) = 1$ and $G$ satisfies the condition (ii). Let $v$ be a vertex of minimum degree, such that $\text{deg}(v) = 1$. By Observation (6.1), $\{6.2\} \subseteq D$. Also by Observation (ii), $v \notin D'$, where $D'$ is a minimal connected dominating set of $G$. Hence $|D| + 1 = |D'|$. Therefore,
\[ \gamma_{cd}(G) = |D| \]
\[ = |D'| - 1 \]
\[ = \gamma_c(G) - 1, \text{ a contradiction.} \]

**Case 2.** If \( \delta(G) = 1 \) and does not satisfy the condition (ii). Then by Case 1, \( \gamma_{cd}(G) > \gamma_c(G) \), a contradiction.

Conversely, suppose \( G \) satisfies the conditions (i) and (ii). Then one can easily observe that \( \gamma_c(G) = \gamma_{cd}(G). \)

Next, we obtain Nordhaus-Gaddum type results for \( \gamma_{cd}(G) \).

**Theorem 6.14** Let \( G \) be any graph such that both \( G \) and \( \overline{G} \) are connected, then

\[ (i) \ \gamma_{cd}(G) + \gamma_{cd}(\overline{G}) \leq 2(p - 1) \]
\[ (ii) \ \gamma_{cd}(G) \cdot \gamma_{cd}(\overline{G}) \leq (p - 1)^2. \]